
ORA ET LABORA
ARTHUR EDWIN KENNELLY

STATICS

MINCHIN

London

HENRY FROWDE

OXFORD UNIVERSITY PRESS WAREHOUSE

AMEN CORNER, E.C.

Clarendon Press Series

A

TREATISE ON STATICS

WITH

APPLICATIONS TO PHYSICS

BY

GEORGE M. MINCHIN, M.A.

PROFESSOR OF APPLIED MATHEMATICS

IN THE ROYAL INDIAN ENGINEERING COLLEGE, COOPER'S HILL

VOL. II

FOURTH EDITION

CORRECTED AND ENLARGED

Oxford

AT THE CLARENDON PRESS

M.DCCC.LXXXIX

PREFACE

TO THE SECOND VOLUME.

THE subject-matter of this second volume differing very greatly from that of the first, a few words with regard to the manner in which I have treated it seem to be necessary.

The reader will observe that in the Chapter dealing with Virtual Work, I have ventured to reject the term 'Generalised Component of Force,' and to replace it by the term 'Work Coefficient,' the former term being, to my mind, open to the objection of conveying an erroneous idea with regard to the nature of the thing defined.

In the Chapter on Attractions the great object which I have constantly kept in view has been the fixing of a *definiteness of idea* in the mind of the student with regard to the various physical magnitudes which are represented by symbols in our equations. To this end, I have explicitly adopted the C. G. S. system, and I have introduced a sufficient number of numerical illustrations in which Forces and Potentials are definitely presented as so many Dynes and so many Ergs per gramme. The C. G. S. system stands pre-eminent for its extreme simplicity; and when once the student of Mathematical Physics learns how to make a real working use of its units—to recognise, without mental effort and as a mere matter of course, that his symbol, ρ, for volume-density always means so many grammes per cubic centimètre; that his symbol, X, for force-intensity means so many dynes per gramme; and so on—he will never experience any difficulty in altering the values of fundamental numerical constants to suit the units of mass, time, and length which are adopted in any other system. In the calculation of Attractions—and especially in the domains of Electricity and Magnetism—the ever present notion of a *concrete reality* corresponding to every algebraic symbol is of immense importance. Indeed, without this definiteness of idea, no knowledge of the slightest value can exist.

The result of perpetually fixing the mind on mere symbols and repelling the natural realities for which they stand is to

produce in the mind a crystallisation of ignorance; and it is to prevent this that I have so persistently kept before the student the gramme, the dyne, the erg, &c.

Hence in this Chapter Poisson's equation always appears in the form $\nabla^2 V = -4\pi\gamma\rho$, in which γ is the C.G.S. constant of gravitation—viz. about $\frac{1 \text{ dyne}}{1543 \times 10^4}$—and a familiarity with its value gives the student a useful idea with regard to the nature of gravitation.

In this Chapter I have also ventured to introduce the term 'Laplacian' with reference to those remarkable coefficients which occur in the development of the reciprocal of the distance between two points. The general term 'Spherical Harmonic' is, of course, retained; but it seems to me that the name of Laplace ought to be *explicitly* connected with the branch of Mathematical Physics which he did so much to develop, and which has now become of such great importance. The pure mathematicians having their 'Jacobians,' 'Hessians,' 'Cayleyans,' &c., the term 'Laplacian' is surely justified.

GEORGE M. MINCHIN.

R. I. E. College, Cooper's Hill,
February, 1886.

A new edition of the Second Volume was called for in the summer of 1888. It is to a very great extent a reprint of the previous edition; but it treats much more fully of Conical Angles, contains new Articles on Line- and Surface-Integrals and Magnetic Shells; and, finally, an improvement in the method of treating some questions of Strain and Stress, for which I am indebted to Professor Williamson, F.T.C.D. It is satisfactory to know that the introduction of the *gravitation constant* has met with high approval, and has found explicit recognition in some recent papers by able writers.

G. M. M.

Cooper's Hill, *October*, 1888.

TABLE OF CONTENTS.

ERRATA.

IN VOL. I. (3rd Ed.)

P. 30, line 6, *for* m *read* n.

P. 59, ex. 19, *for* $Q \tan \alpha$ *read* $Q \cot \alpha$.

P. 67, line 18, *for* $(-\lambda)$ *read* $(\theta-\lambda)$.

P. 91, line 6, *for* 74 *read* 75.

In the example in p. 115 the value of G should be 3;6 kilogramme-decimètres. Alter the result accordingly.

P. 130, line 12, *for* $\beta\Sigma Y$ *read* $\beta\Sigma X$.

P. 135, line 12, *for* β_2 *read* β_3.

P. 160, line 12, *for* matter *read* manner.

P. 170, line 20, *for* $\omega-\omega$ *read* $\omega-\omega'$.

IN VOL. II.

P. 299, line 9, *for* $\dfrac{a^3-a'^3}{D}$ *read* $\dfrac{c^3-a'^3}{c}$.

The following corrections in Vol. II. were kindly communicated by Professor Hoover, Ohio University, after the sheets had gone to press:

P. 271, in Ex. 12, *omit* the term 1 within the brackets $\{\ \}$.

P. 281, line 19, *for* $\sin\theta$ *read* $\cos\theta$.

P. 298, line 3, *for* $\sec P'PO$ *read* $\sec P'QO$.

P. 299, line 25, *for* C *read* C'.

P. 326, eq. (9), *for* $d\mu$ *read* $\mu d\mu$.

P. 357, eq. (β), *for* $d\mu$ *read* $d\mu'$.

P. 366, line 19, *for* $-\dfrac{1}{b^2}$ *read* $\dfrac{1}{b^2}$.

P. 370, lines 3 and 5, *for* Y_0 *read* aY_0.

P. 374, line 18, interchange K and E.

STATICS.

CHAPTER XIII.

NON-COPLANAR FORCES.

ARTICLE 198.] **Resultant of any Number of Forces applied to a Material Particle.** Let a force P, represented in magnitude and direction by OO' (Fig. 228), act on a particle at O; let Ox, Oy, and Oz, be any three rectangular axes drawn through O; and let the angles, $O'Ox$, $O'Oy$, and $O'Oz$, which the direction of P makes with the axes of reference be denoted by α, β, and γ, respectively. From O' let fall perpendiculars, $O'F$, $O'H$, $O'D$, on the planes, yz, zx, and xy, respectively, and let the parallelopiped be completed as in the figure. Then the force OO' may be replaced by the forces OD and OC, by the parallelogram of forces; and OD can again be replaced by OA and OB. Hence the force P is equivalent to the three forces

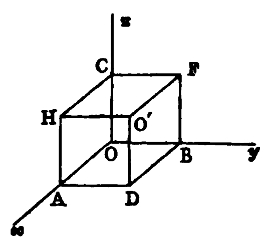

Fig. 228.

$$P\cos\alpha \text{ along } Ox,$$
$$P\cos\beta \quad „ \quad Oy,$$
and
$$P\cos\gamma \quad „ \quad Oz.$$

The converse proposition is also evidently true—namely, that any three forces, OA, OB, OC, along Ox, Oy, Oz (whether these are mutually rectangular directions or not), give a resultant represented in magnitude and direction by the diagonal, OO', of the parallelopiped determined by the forces.

If several forces, $P_1, P_2, \ldots P_n$, act at O and make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, ... $(\alpha_n, \beta_n, \gamma_n)$ with the axes, let each of them be replaced by its three components along Ox, Oy, Oz;

and if ΣX, ΣY, ΣZ denote the sums of the components along the axes, we shall have

$$\left.\begin{aligned}\Sigma X &= P_1 \cos a_1 + P_2 \cos a_2 + \ldots + P_n \cos a_n,\\ \Sigma Y &= P_1 \cos\beta_1 + P_2 \cos\beta_2 + \ldots + P_n \cos\beta_n,\\ \Sigma Z &= P_1 \cos\gamma_1 + P_2 \cos\gamma_2 + \ldots + P_n \cos\gamma_n,\end{aligned}\right\} \quad (1)$$

and the whole system of forces will be replaced by the three forces, ΣX, ΣY, and ΣZ along the axes of x, y, and z. But the resultant of three forces in these directions is the diagonal of the parallelopiped determined by them. Hence, R being the magnitude of this resultant,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}, \quad (2)$$

and if θ, ϕ, ψ, be the direction-angles of R,

$$\cos\theta = \frac{\Sigma X}{R}, \quad \cos\phi = \frac{\Sigma Y}{R}, \quad \cos\psi = \frac{\Sigma Z}{R}. \quad (3)$$

199.] **Graphic Representations of the Resultant.** Since the resultant of any two forces, OA and OB, acting at O is obtained by drawing from A a line, Ab, parallel and equal to OB, and joining O to b, it follows that if a particle is acted on by n forces, OA_1, OA_2, OA_3, ... OA_n, the resultant is obtained in magnitude and direction by drawing $A_1 a_2$ parallel and equal to OA_2, $a_2 a_3$ parallel and equal to OA_3, ... $a_{n-1} a_n$ parallel and equal to OA_n, and joining O to a_n; or, in other words, the side Oa_n which closes the polygon $OA_1 a_2 a_3 \ldots a_n$ represents the resultant in magnitude and direction. When the sides of the polygon are not all coplanar, the figure is called a *gauche polygon*. Thus the second graphic representation of the resultant of a system of coplanar forces, which has been given in p. 19, vol. i, is equally applicable to non-coplanar forces. Hence, of course, it follows that a particle acted on by any set of forces which are parallel and proportional to the sides of a gauche polygon taken in order is at rest.

Again, since by the parallelogram of forces, the resultant of OA_1 and OA_2 is $2 . Og_1$, where g_1 is the middle point of $A_1 A_2$; and since the resultant of $2\,Og_1$ and OA_3 is $3\,Og_2$, where g_2 is determined exactly as in Art. 23, it follows that Leibnitz's graphic representation of the resultant is applicable to non-coplanar forces.

This result follows also analytically; for if (x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n) be the co-ordinates of the extremities

$A_1, A_2, \dots A_n$ of the forces acting on the particle, it is clear that

$$\Sigma X = x_1 + x_2 + \dots + x_n = \Sigma x = n.\bar{x},$$
$$\Sigma Y = y_1 + y_2 + \dots + y_n = \Sigma y = n.\bar{y},$$
$$\Sigma Z = z_1 + z_2 + \dots + z_n \quad \Sigma z = n.\bar{z};$$

where $\bar{x}, \bar{y}, \bar{z}$ are the co-ordinates of G, the centre of mass of equal masses placed at the extremities of the forces. Hence by equations (1) of Art. 198,

$$R = n.OG,$$

and
$$\cos\theta = \frac{\bar{x}}{OG}, \quad \cos\phi = \frac{\bar{y}}{OG}, \quad \cos\psi = \frac{\bar{z}}{OG},$$

which show that the resultant is represented in magnitude and direction by $n.OG$.

200.] **Transformation of Couples.** To what has been given in Chapter V on the transformation of couples it is necessary to add a few propositions relating to couples in different planes.

(a) A couple acting on a rigid body may be transferred to any plane parallel to its own.

Let AB (Fig. 229) be the arm of a couple (P, P) and let $A'B'$ be *any* line parallel and equal to AB. At A' introduce two equal and opposite forces, P and P', parallel to AP, and at B introduce the same forces. The introduction of these forces will not disturb the state of the body. Draw AB' and $A'B$, which will bisect each other at O. Then the force P at A and the force P' at B' will give a resultant equal to $2P$ at O; and P at B and P' at A' will give a resultant equal and opposite to this at the same point. Hence there remain the forces P at A' and P at B'; that is, the couple (P, P) with arm AB has been moved to any plane parallel to its own.

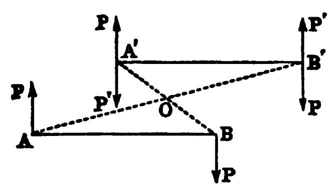

Fig. 229.

From Chapter V it is now clear that the only essential properties of a couple are (1) the constancy of its moment and (2) the direction of its plane; or, in other words, *the constancy of the magnitude and direction of its axis*; the actual *position* of the axis in space is of no consequence, but only its *direction*; two couples whose axes are of equal length and in the same direction are absolutely identical.

Hence the axis of a couple is what is called a *vector*, or directed line of constant magnitude—but not localised—and we shall always, as in the representation of forces, suppose the axis to be marked by an arrow-head.

(β) *Convention with regard to the sense of the axis of a couple.* The following convention for representing the magnitude and sense of the moment of a couple by means of an axis is adopted by common consent for the purpose of enabling us to compound and resolve couples in any planes:—Hold a watch with its plane parallel to the plane of the couple. Then, according as the motion of the hands is contrary to, or along with, the sense in which the couple tends to produce rotation, draw the axis of the couple through the *face* or through the *back* of the watch.

(γ) Two couples result in a single couple whose axis is found from the axes of the component couples by the parallelogram law.

Let the planes of the couples intersect in the line AB (Fig. 230) and the arm of each be made AB, by moving each couple in its own plane, and then suitably altering the forces of each couple (Art. 79, Chap. V). Let P, P be forces of one couple, and Q, Q those of the other. At B draw * Bp perpendicular to the plane $PABP$ and proportional to the moment of the couple (P, P). We may evidently take $Bp = P$,

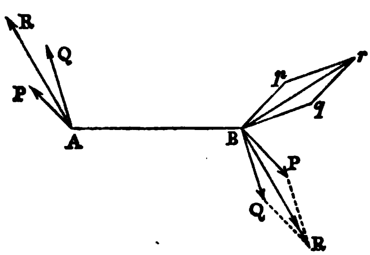

Fig. 230.

since the couples have a common arm. Draw Bq perpendicular to the plane $QABQ$ and equal to Q.

Now evidently the forces P and Q at B compound a resultant, R, equal and parallel to the resultant of P and Q at A. Hence the two couples compound a single couple.

Again, draw Br perpendicular to the plane $RABR$ and equal to R. Bp, Bq, and Br are then the axes of the couples (P, P), (Q, Q), and (R, R). But it is manifest that the figure $Bprq$ is

* According to the convention (β) the couples in this figure are both negative, and the axes Bp and Bq should be drawn downwards. This inaccuracy in the figure was detected too late for correction.

merely the figure $BPRQ$ turned round in its own plane through a right angle. Hence Br is the diagonal of the parallelogram determined by the axes of the component couples.

Conversely, any couple may be resolved into two couples whose axes are determined from the axis of the given couple by the parallelogram law; and, as in the case of forces acting at a point, any couple may be resolved into three couples whose axes are determined from the axis of the given couple by the parallelopiped law. All this follows as in Art. 198.

It is well to remark that the axis of a couple represents the moment of the forces of the couple round any line in space parallel to the axis.

(δ) To find the resultant of any number of couples acting in any planes on a rigid body.

Let the axes of the couples be all drawn, each in its proper sense according to the rule (β), at the same point, O (Fig. 228), and let each axis be resolved into three components along rectangular axes Ox, Oy, Oz, drawn through O. Let $L =$ the sum of the axes in the direction Ox; then L is the axis of the component of the resultant couple in the plane yz. Let M and N be the sums of the axes in the directions Oy and Oz, respectively. Then, if G is the resultant axis,

$$G = \sqrt{L^2+M^2+N^2}, \tag{1}$$

and if λ, μ, ν are the direction angles of G,

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}, \tag{2}$$

equations which are exactly analogous to (2) and (3) of Art. 198.

The axes of couples are, therefore, compounded and resolved in the same manner as forces. There is this difference between forces and couples, that, while any number of couples in any planes whatever always result in a single couple, any number of forces cannot, in general, be replaced by a single force, and this difference results from the *vectorial* nature of the axis of a couple.

(ϵ) A force and a couple acting on a rigid body cannot produce equilibrium.

For, let the couple be so transferred that one of its forces, P, acts at a point on the line of action of the force, R. Then R

A couple is vectorial
a force is a localised vector

and P at this point compound a single force which, in general, does not intersect the other force of the couple. Therefore, &c.

A force and a couple acting in the same plane are, of course, equivalent to a single force.

201.] **Virtual Work of a Couple.** Let a couple act on a rigid body which receives, or is imagined to receive, any small displacement whatever. It is required to find the work done by the couple in the displacement.

It will be shown subsequently that any displacement of the body may be produced by a motion of translation which is the same for all its points, accompanied by a motion of rotation round an axis through an angle which is the same for all its points.

Now since the forces of the couple are equal and in opposite senses, it is obvious that the sum of their works in any motion of translation is zero.

Again, resolve the motion of rotation into one round an axis perpendicular to the plane of the given couple, and one round an axis in the plane of the couple. It is obvious that the latter displacement will not be productive of work done by the couple, since the forces constituting it may be supposed to act at the points in which they intersect the axis of this component rotation.

There remains only the rotation round an axis perpendicular to the plane of the couple. Suppose O (Fig. 88, Art. 79) to be the point in which this axis intersects the plane of the couple, and let $\delta\theta$ be the angular rotation round the axis, *measured in the sense of the rotation of the couple.* Then the displacements of m and n are $Om \times \delta\theta$ and $On \times \delta\theta$, respectively, so that the work done by the forces is $P\,(Om\,.\,\delta\theta + On\,.\,\delta\theta)$, i.e.,

$$P\,.\,h\,.\,\delta\theta, \text{ or } G\,.\,\delta\theta,$$

where $G\ (= P.\,h) =$ the moment of the couple.

202.] **Theorem.** *A force acting on a rigid body in a given right line can always be replaced by an equal force acting at any chosen point together with a couple.*

Let a force P (Fig. 231) act at a point A, and let O be the chosen point. At O introduce two forces, P and P', opposite to each other and each equal and parallel to P. Then P at A and P' at O may be taken to constitute a couple whose

moment is Pp, p being the perpendicular from O on the line of action of P at A. There remains, then, the force P at O; and this force together with the couple may replace P at A.

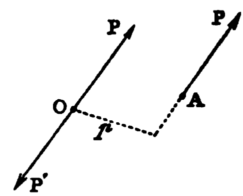

Fig. 231.

Let the axis of this couple be drawn at O; let x, y, z be the co-ordinates of A with respect to a rectangular system of axes drawn through O; and let α, β, γ be the angles which the direction of P makes with the axes of x, y, z, respectively.

The direction-cosines of OA are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, where $OA = r$, and it is easy to prove that the direction-cosines of the axis of the couple (which is at once at right angles to OA and to P) are

$$\frac{y\cos\gamma - z\cos\beta}{p}, \quad \frac{z\cos\alpha - x\cos\gamma}{p}, \quad \frac{x\cos\beta - y\cos\alpha}{p}.$$

Hence, the axis of the couple being equal to Pp, the projections of the axis on the axes of x, y, and z are

$$P(y\cos\gamma - z\cos\beta), \quad P(z\cos\alpha - x\cos\gamma), \quad P(x\cos\beta - y\cos\alpha);$$

but it is clear from (γ), Art. 200, that these are the axes of the component couples in the planes yz, zx, and xy, into which the couple Pp can be resolved. Putting $P\cos\alpha = X$, $P\cos\beta = Y$, $P\cos\gamma = Z$, we see that the three couples are

$$Zy - Yz, \quad Xz - Zx, \quad Yx - Xy. \tag{1}$$

The force P at O may also be replaced by its three components,

$$X, Y, Z. \tag{2}$$

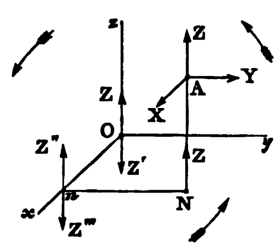

Fig. 232.

There is another way in which the reduction is sometimes effected.

Let P at A be resolved into its three components, X, Y, Z, let the line of Z meet the plane (xy) in N, and let Z at A be transferred to N. Let fall Nn perpendicular to Ox; at n introduce two opposite forces Z'' and Z''', each equal and parallel to Z; and at O introduce two opposite forces, Z and Z', each equal and parallel to Z. Now the senses of positive rotation in the planes xy, yz, zx being those indicated by the arrows, the forces Z at N and Z''' at n form a couple whose moment is

$$Zy \text{ parallel to plane } yz;$$

and the forces Z' at O and Z'' at n form a couple whose moment is

$$-Zx \text{ parallel to the plane } zx;$$

and in addition to these there is the force Z at O.

Similarly, the force X at A can be replaced by X at O together with two couples, Xz and $-Xy$, parallel to the planes zx and xy, respectively; and the force Y at A can be replaced by Y at O together with the couples Yx and $-Yz$ parallel to the planes xy and yz.

Hence P at A is replaced by the forces X, Y, Z at O and the couples $Zy-Yz$, $Xz-Zx$, and $Yx-Xy$, parallel to the planes yz, zx, and xy, respectively.

203.] **Parallel Forces.** Suppose a rigid body to be acted on by any number of parallel forces applied at given points in the body. Take any origin, O, of co-ordinates, and through it draw three rectangular axes, that of z being parallel to the common direction of the forces. Then the force P_1, acting at (x_1, y_1, z_1) may be replaced, as in last Art., by

$$P_1 \text{ at } O \text{ along } Oz,$$

and the couples $\quad P_1 y_1$ and $-P_1 x_1$

parallel to the planes yz and zx.

Replacing each force in this manner, the whole system will be equivalent to a force

$$P_1+P_2+\ldots+P_n, \text{ or } \Sigma P \text{ at } O,$$

together with the couple

$$P_1 y_1+P_2 y_2+\ldots+P_n y_n, \text{ or } \Sigma Py,$$

parallel to the plane yz, and the couple

$$-P_1 x_1-P_2 x_2-\ldots-P_n x_n, \text{ or } -\Sigma Px,$$

parallel to the plane zx.

These two couples compound a single couple whose axis is found by drawing $OL = \Sigma Py$, on any scale, and OM (in the negative sense of the axis of y) $= \Sigma Px$, on the same scale, and completing the parallelogram $OLGM$ (Fig. 233). If OG, the diagonal, is denoted by G,

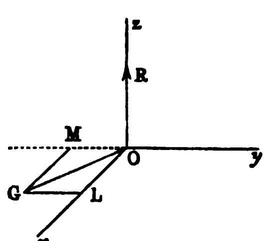

Fig. 233.

$$G = \sqrt{(\Sigma Px)^2+(\Sigma Py)^2}$$

and $\quad R = \Sigma P,$

R being the resultant force.

204.] **Centre of Parallel Forces.** Since the resultant of two parallel forces, P_1 and P_2, acting at the points A_1 and A_2 divides

the line A_1A_2 in a point g such that $\frac{A_1 g}{A_2 g} = \frac{P_2}{P_1}$, and since, by elementary geometry (see Art. 84), the distance of g from any plane $= \frac{P_1 x_1 + P_2 x_2}{P_1 + P_2}$, where x_1 and x_2 are the distances of A_1 and A_2 from this plane, it follows, by repeating this construction, that the distances, $\bar{x}$, $\bar{y}$, $\bar{z}$, of the *centre of parallel forces* from the planes yz, zx, and xy are given by the equations

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P}.$$

205.] **Conditions of Equilibrium of a System of Parallel Forces.** A system of parallel forces has been reduced (Art. 203) to a single force, R, and a single couple, G. Now since these cannot in combination produce equilibrium (ϵ, Art. 200), we must have $R = 0$, and $G = 0$, separately.

Since G cannot be $= 0$ unless $\Sigma Px = 0$ and $\Sigma Py = 0$, the conditions of equilibrium are

$$R = 0, \tag{1}$$

$$\Sigma Px = 0, \ \Sigma Py = 0. \tag{2}$$

DEF. The moment of a force with respect to a plane to which it is parallel is the product of the force and its perpendicular distance from the plane.

Hence for the equilibrium of parallel forces—*The sum of the forces must vanish, and the sum of their moments with respect to every plane parallel to them must also vanish.*

EXAMPLES.

1. A heavy triangular table, ABC, is supported horizontally on three vertical props at the vertices; prove that the pressures on the props are equal.

Let P, Q, R be the pressures at A, B, C, and let a vertical plane through A and the centre of gravity of the table cut the side BC in a, its middle point. For equilibrium the sum of the moments of the forces P, Q, R, and W (the weight of the table) with respect to this plane must $= 0$. But the moments of P and W are each $= 0$, since these forces lie in the plane. Hence the moments of Q and R are equal and opposite. Now the distance of Q from the plane $= Ba.\sin AaB$, and the distance of $B = Ca.\sin AaC$; and since $Ba = Ca$, these distances are equal. Therefore $Q = R$; and similarly it can be shown that $R = P$; therefore, &c.

2. A heavy triangular plate hangs in a horizontal plane by means of three vertical strings attached to its vertices; at what point in its area must a given weight be placed so that the system of strings may be least likely to break?

Ans. At the centre of gravity of the board. For if $W =$ the weight of the board and P the sustained weight, we have

$$P+Q+R = W+P,$$

or the sum of the tensions is constant, wherever P is placed. Hence if any one is less than $\frac{1}{3}(W+P)$, some other must be greater than this value. It is evident, therefore, that the best arrangement makes each tension $= \frac{1}{3}(W+P)$; but this happens (as proved in last example) when P is placed at the centre of gravity.

3. A heavy elliptic cylinder is sustained in a vertical position by three props applied at three points on the circumference of its base; how should the props be placed in order that the cylinder may be least likely to be upset?

Let the base of the cylinder have any form, ABC (Fig. 234), and let G be the projection of its centre of gravity on the plane of the base.

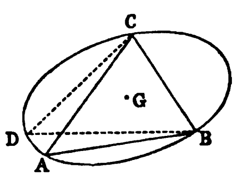

Fig. 234.

Then, if the props are applied at A, B, and C, the perpendiculars from G on the sides of the triangle ABC must be all equal when the equilibrium is most stable. For, suppose that the cylinder is about to be upset round the line AB; then the moment of the force required to upset it is $W.p$, where W is the weight of the cylinder and p the perpendicular from G on AB. Again, suppose that the cylinder is about to be upset about AC; then the moment of the force required to upset it is $W.q$, where q is the perpendicular from G on AC. Hence if p and q are unequal, advantage will be gained by increasing the smaller of them, even though the greater must be consequently diminished; and it follows that the maximum advantage is gained when p and q are equal. In the same way it can be shown that the perpendicular from G on BC must, in the most advantageous case, be equal to that from G on AB; and therefore the perpendiculars from G on the sides ABC must be all equal.

Hence the problem amounts to inscribing in a given curve a triangle on the sides of which the perpendiculars from a given point shall be equal. In the particular case in which the base is an ellipse, we have to find a circle concentric with the ellipse, such that a triangle circumscribed to the circle shall be inscribed in the ellipse. Now (Salmon's *Conic Sections*, p. 330, 5th edition), let the ellipse have for equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0$, and the circle $x^2+y^2-r^2 = 0$; then the discriminant of $k(x^2+y^2-r^2)+\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0$ is

$$k^3 . r^2 + \left(1 + r^2 \frac{a^2+b^2}{a^2b^2}\right)k^2 + \frac{r^2+a^2+b^2}{a^2b^2} . k + \frac{1}{a^2b^2};$$

and the required condition being $\Theta^2 = 4\Delta . \Theta'$, we have two values for r, namely, $r_1 = \frac{ab}{a+b}$; and $r_2 = \frac{ab}{a-b}$. The first value alone is admissible, because $\frac{ab}{a-b} > b$, and the circle in this case either cuts the ellipse or entirely encloses it.

Since an infinite number of triangles can be inscribed in the ellipse and circumscribed to the circle of radius $\frac{ab}{a+b}$ (Salmon, *ibid.*), the problem is capable of an infinite number of solutions. It is easy to see that in the cases in which it is possible to have a real system of in- and circum- scribed triangles for the ellipse and the circle of radius $\frac{ab}{a-b}$, the centre of the ellipse is outside the area of the triangle. This circle is, therefore, irrelevant to our question.

4. A heavy square board, $ABCD$, of uniform thickness, is hung by three vertical strings, one of which is attached to a corner, A, of the board. The plane of the board being horizontal, find the points, E and F, in the area to which the other two strings should be attached in order that it may be most difficult to overturn the board by placing a weight anywhere on it.

It is evident that advantage is gained by taking the points E and F on the edges of the board.

Assume AE to be the direction of the line joining the points of application of two of the strings, and suppose that a weight, P, placed somewhere in the area ABE is on the point of overturning the board about the line AE. Then the tension of the string at $F = 0$; and if W is the weight of the board, acting at G, the weight P required to upset it is

$$W \times \frac{\text{distance of } G \text{ from } AE}{\text{distance of } P \text{ from } AE}.$$

Hence the greater the distance of P from AE, the less the requisite value of P, or, in other words, the more easily will the board be upset. It is evident, therefore, that the applied weight should be placed at B; and in the same way, if the board is to be upset round the lines AF and FE, the applied weights should be placed at the corners D and C, respectively.

Again, in the arrangement of greatest advantage, the board is upset with equal ease round each of the lines AE, AF, and FE. For, if it be more easily upset round one of these lines than round another, advantage will be gained by making it a little more stable with regard to the first. Hence, since the weights placed at B, D, and C are all equal, we have

$$\frac{\text{distance of } G \text{ from } AE}{\text{distance of } B \text{ from } AE}=\frac{\text{distance of } G \text{ from } AF}{\text{distance of } D \text{ from } AF}=\frac{\text{distance of } G \text{ from } EF}{\text{distance of } C \text{ from } EF}.$$

The angles EAB and FAD are, therefore, equal, and each

$$= \tan^{-1}(\sqrt{2}-1).$$

5. A heavy elliptic table is supported on three vertical props; how must they be placed so that it may be most difficult to upset the table by placing a weight on it?

Ans. The props must be placed at three points, A, B, C, on the circumference of the ellipse; and if γ is the eccentric angle of C, that of B is $\frac{2}{3}\pi+\gamma$, and that of A is $\frac{4}{3}\pi+\gamma$. The weight which, most advantageously applied, will then just upset the table is half its own weight.

This may be seen as follows. Assuming any line in the area as the line joining two props, the least weight that will be required to upset the table must be placed at the point of contact of a tangent parallel to the assumed line, since the weight will have maximum leverage at this point. Also, it must be equally easy to upset the table round the three lines AB, BC, CA; that is, the requisite weights placed at C', A', B', the points of contact of the tangents, must be all equal. If, then, x, y, z be the perpendiculars from the centre on the lines BC, CA, AB, and P, Q, R the perpendiculars on the parallel tangents, we must have

$$\frac{x}{P-x}=\frac{y}{Q-y}=\frac{z}{R-z};$$

or if α, β, γ be the eccentric angles of A, B, C,

$$\frac{\cos\dfrac{\alpha-\beta}{2}}{1-\cos\dfrac{\alpha-\beta}{2}}=\frac{\cos\dfrac{\beta-\gamma}{2}}{1-\cos\dfrac{\beta-\gamma}{2}}=\frac{-\cos\dfrac{\alpha-\gamma}{2}}{1+\cos\dfrac{\alpha-\gamma}{2}},$$

a negative sign being used in the last, since (γ, β, α being in ascending order of magnitude) $\dfrac{\alpha-\gamma}{2}$ is evidently $>\dfrac{\pi}{2}$. Hence $\beta=\dfrac{2}{3}\pi+\gamma$, $\alpha=\dfrac{4}{3}\pi+\gamma$; and the weight required to upset the table $= W\dfrac{x}{P-x}$, or $\frac{1}{2}.W$. Any one position of C is, therefore, as good as any other; and if C is made the extremity of either axis, the line AB is parallel to the other at a distance equal to $\frac{3}{4}$ of the first axis from it.

6. A rectangular board is held with its plane horizontal by three vertical strings attached to three of its corners; find the point in its area at which a weight must be placed so that the tensions of the strings shall be given multiples of the weight of the board.

Ans. Let W be the weight of the board; let the strings be applied at the corners A, B, C; let $AC = 2a$, $AB = 2b$; and let the tensions of the strings at A, B, C be lW, mW, nW, respectively.

Then the weight must be placed at a point whose distances from AB and AC are
$$\frac{2n-1}{l+m+n-1}.a \quad \text{and} \quad \frac{2m-1}{l+m+n-1}.b.$$
The magnitude of the weight is, of course, $(l+m+n-1)\,W$.

7. A uniform circular lamina is placed with its centre upon a prop; find at what points on its circumference three weights, w_1, w_2, w_3, must be placed that it may remain at rest in a horizontal position (Walton's *Mechanical Problems*, p. 73).

Ans. The angles which the weights subtend in pairs at the centre of the lamina are the supplements of the angles of a triangle whose sides are proportional to the weights.

206.] **Reduction of a System of Forces acting in any manner on a Rigid Body.** Let any origin, O (Fig. 232), be assumed arbitrarily, and let any system of rectangular axes, Ox, Oy, and Oz, be drawn through it. If, then, forces P_1, P_2, P_3,... act on the body at points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),... each force can be replaced by three components acting at O along the axes, together with three couples whose axes coincide with the co-ordinate axes. The force P_1, for example, is equivalent to X_1, Y_1, Z_1 at O and three couples, $Z_1y_1-Y_1z_1$, $X_1z_1-Z_1x_1$, and $Y_1x_1-X_1y_1$. Adding the components of the forces, and also the axes of the couples, in the directions Ox, Oy, and Oz, the whole system of forces is equivalent to

the force ΣX along Ox,
„ ΣY „ Oy,
and „ ΣZ „ Oz;

and the system of couples is equivalent to

the couple $\Sigma\,(Zy-Yz)$, or L, in the plane yz,
„ $\Sigma\,(Xz-Zx)$, or M, „ zx,
and „ $\Sigma\,(Yx-Xy)$, or N, „ xy.

(Of course the axes of L, M, N are drawn along the axes of x, y, and z, respectively.)

Hence if R be the magnitude of the resultant of translation,
$$R = \sqrt{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2};$$
and if G be the magnitude of the resultant couple,
$$G = \sqrt{L^2+M^2+N^2}.$$

The direction-cosines of R are $\frac{\Sigma Y}{R}$, $\frac{\Sigma Y}{R}$, and $\frac{\Sigma Z}{R}$; and those of G are $\frac{L}{G}$, $\frac{M}{G}$, and $\frac{N}{G}$.

Thus, *any system of forces acting on a rigid body can be replaced by a single resultant force acting at an arbitrary origin, the magnitude and direction of this force being the same for all origins, and a single resultant couple the magnitude and direction of whose axis are both dependent on the origin chosen.*

It has been already remarked (Art. 200) that G is not only the axis of the resultant couple (corresponding to a resultant force acting at O), but also the sum of the moments of the forces about a line at O drawn in the direction of G; and since the axes of couples have been proved to follow the parallelopiped and parallelogram laws, it follows that the sum of the moments of the forces about this line is greater than the sum of their moments about any other line at O; and also that the sum of the moments of the forces about any other line through O is the resolved part of G in the direction of this line.

Remark. The magnitude and direction of G are constant at all points along the same right line parallel to R. For R may be supposed to act at any point in this line, and the vector G may be moved parallel to itself to the point at which R is supposed to act. The axis G is called the *axis of principal moment* at O.

207.] **Moment of a Force round any Line.** Let a force of given magnitude act in a given direction at a given point A, and let its moment be required about a given right line passing through a given point P. With reference to any three rectangular axes, let (x, y, z) be the co-ordinates of A; let (ξ, η, ζ) be those of P; and let (X, Y, Z) be the components of the force.

Then the moment of the force round a line through A parallel to the axis of x is
$$Z(y-\eta)-Y(z-\zeta),$$
while its moments round the lines through A parallel to the axes of y and z are, respectively,
$$X(z-\zeta)-Z(x-\xi) \text{ and } Y(x-\xi)-X(y-\eta).$$

Denote these component moments by L, M, N, respectively. Then, if the line through A about which the total moment is

required makes angles whose direction-cosines with the axes of reference are l, m, n, the required moment is

$$lL + mM + nN.$$

208.] Poinsot's Central Axis. Any system of forces acting on a rigid body has been proved to be equivalent to a single resultant force, R, acting at an arbitrary origin, O, and a single resultant couple, G. Let ϕ be the angle between R and G, and resolve G into two components, OK and On (Fig. 235) along and perpendicular to R, respectively. On is the axis of a couple in the plane ROx, perpendicular to On. Now let each force of this couple be made equal to R, and the arm, OP *, is consequently equal to $\frac{On}{R}$; that is,

$$OP = \frac{G \sin \phi}{R}. \qquad (1)$$

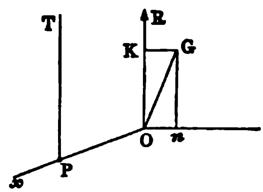

Fig. 235.

One of these forces may be applied at O to destroy the resultant, R, at this point, and there finally remains a resultant force, R, at P along PT (parallel to OR), together with a couple whose axis is OK, or $G \cos \phi$. Denoting OK by K, we have then

$$K = G \cos \phi. \qquad (2)$$

The axis OK may, of course, be drawn at P along PT [(a), Art. 200].

Hence *the whole system of forces is equivalent to a resultant force equal to R acting along the line PT and a couple in a plane perpendicular to this line.*

The line PT thus determined is called *Poinsot's Central Axis.*

To construct Poinsot's Central Axis for any system of forces— *Reduce the forces to a resultant force, OR, acting at any origin, O, and a couple whose axis is OG; then on a line perpendicular to the plane of OR and OG measure off a length, OP†, equal to* $\frac{G \sin \phi}{R}$, *where* ϕ *is the angle between OR and OG. A line through the point P parallel to OR is the required Central Axis.*

---

* The point P should be represented on the production of the line xO through O, according to the convention of Art. 200. The inaccuracy in the figure was detected too late for correction.

† The *sense* of OP is determined by the convention of Art. 200.

209.] **Theorem.** A given system of forces has but one Central Axis.

This important proposition may be proved *ex absurdo* thus:—

Whatever origin be chosen, the resultant force acting at it is constant both in magnitude and in direction. Now, if it be possible, let the system of forces be equivalent to a resultant R acting at O and a couple whose axis is OK, and also to a resultant force R acting at O' and a couple whose axis is $O'K'$, the lines OK and $O'K'$ being, of course, in the direction of R. Then it is evident that the force R at O and the couple OK should equilibrate the *reversed* force R and *reversed* couple $O'K'$, at O'. But the couples give a single couple, $OK \sim O'K'$, and the forces give also a couple which, being in a plane perpendicular to the first couple, cannot with it produce equilibrium. Therefore, &c.

Since this axis is unique, equation (2) of the last Article shows that for all origins the quantity $G \cos \phi$, or the *projection of the axis of the resultant couple along the direction of the resultant force is constant, and equal to Poinsot's moment.*

210.] **Theorem.** The sum of the moments of the forces round Poinsot's Axis is less than the sum of their moments round any other axis of principal moment. This is, of course, evident from what has just been said. The deduction of the axis of principal moment at any point from Poinsot's may, however, for clearness, be specially exhibited.

Let Oz (Fig. 236) be Poinsot's Axis and OK $(= K)$ the moment of the forces round it. Let O' be any point in the body, and let it be proposed to find the principal moment at this point; $O'O$ is a line drawn through O' perpendicular to Poinsot's Axis. At O' introduce two equal and opposite forces, $O'R$ and $O'R'$, each $= R$. Then OR and $O'R'$ form a couple, whose axis, $O'n$, is perpendicular to the plane $ROO'R'$ and equal to $R\,.\,OO'$. Transfer the axis OK to $O'K'$ (Art. 200), and draw the diagonal, $O'G$, of the rectangle determined by $O'n$ and $O'K'$. Then $O'G\,(= G)$ is the axis of principal moment at O', and it is evidently $> O'K'$. Hence Poinsot's is the least principal moment.

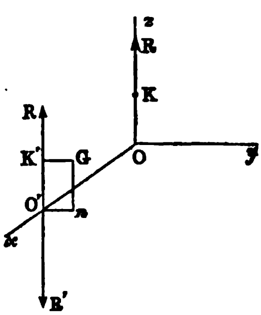

Fig. 236.

211.] **Definition of a Wrench** It has just been shown that any given system of forces acting on a rigid body can be reduced to a single force, R, and a single couple, K, such that the axis of the couple is coincident with the line of action of the force, and that this reduction, for the given force system, is unique.

A force acting along a line and a couple whose axis coincides with this line constitute together what is called a *wrench.*

The ratio of the moment of the couple, K, to the magnitude of the force, R, is evidently a *linear magnitude*, and is called *pitch.*

The right line about which the wrench takes place, when contemplated in conjunction with the pitch, is called a *screw.*

Thus, then, *a screw is a definite right line in space associated with a definite pitch.*

The right line itself about which the wrench takes place—the axis of the wrench—may be denoted by the symbol a, and the pitch associated with it may be denoted by the symbol p_a. It is evident that the complete determination of a screw (pitch included) requires *five* constants, since the axis may be determined by two equations of the forms

$$x = az + m, \quad y = bz + n,$$

which involve the four constants a, m, b, n; while the pitch is specified by another constant.

When the force and the axis of the couple—this latter drawn according to the convention of Art. 200—are in the same sense along the axis of the wrench, the pitch is positive; when they are in opposite senses, it is negative.

The force which acts in a wrench is called by Sir R. Ball the *intensity* of the wrench.

A force alone may be regarded as a wrench of zero pitch.

A couple alone may be regarded as a wrench of infinite pitch.

212.] **Wrench of Two Forces.** Let it be required to find the wrench of which two forces, P and Q, represented in magnitudes and lines of action by the two non-intersecting lines AP and BQ (Fig. 237), are equivalent.

Let AB be the shortest distance between the lines of action of the two given forces, and denote the length AB by h.

Then, following the rule of Art. 208, reduce the forces to a resultant acting at A together with a couple, by introducing two forces, Aq and Aq', equal, opposite, and parallel to Q. Com-

pounding AP and Aq, we get $Ar = R =$ resultant force; and drawing, in the sense determined by the convention of Art. 200, An to represent $Q \times h$, the moment of the couple (BQ, Aq'), the force and couple for the origin A are Ar and An. If O (on AB) is any point on AB, the couple corresponding to it is got by combining with An the axis Am which represents $R \times AO$, this line Am being perpendicular to Ar and in the plane of the lines An, AP, Aq; and if O is Poinsot's origin, the resultant, Ac, of An and Am coincides in direction with Ar.

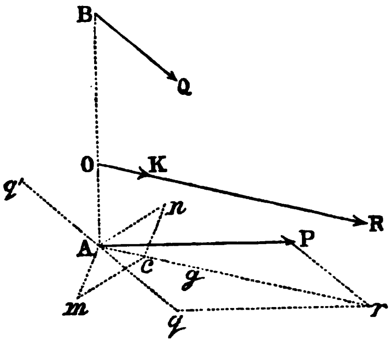

Fig. 237.

Now since Acn is a right angle,

$$\cos Anc = \frac{cn}{An} = \frac{R \times AO}{Q \times h}.$$

But $\qquad \angle Anc = \angle qAr; \qquad \therefore \quad Q \times \cos Anc = Ag,$

where g is the foot of the perpendicular from q on Ar. Hence

$$\frac{AO}{h} = \frac{Ag}{Ar},$$

or

$$\frac{AO}{OB} = \frac{Ag}{gr} = \frac{\text{projection of } Q \text{ along } R}{\text{projection of } P \text{ along } R};$$

so that *Poinsot's origin divides the shortest distance between P and Q inversely as the orthogonal projections of these forces along the direction of their resultant of translation.* Or, again, O may be determined by drawing from g the line gO parallel to the line rB.

The wrench to which P and Q are equivalent is represented in the figure by (OR, OK).

213.] **Two Intersecting Rectangular Screws.** Suppose OX (Fig. 238) to be the axis of a wrench the force in which is represented by the length OX $(=X)$, and the moment of the couple by OM. If p_x is the pitch of this screw, the moment of the couple is

$$p_x \, . \, X.$$

Also, let OY and OL represent the force and couple in another wrench intersecting OX at right angles, and let p_y be the pitch of this second screw.

It is required to find the resultant wrench to which these two wrenches are equivalent.

Replace the forces X and Y by their resultant, OD; and also resolve the moments, OM and OL, into components along and perpendicular to OD.

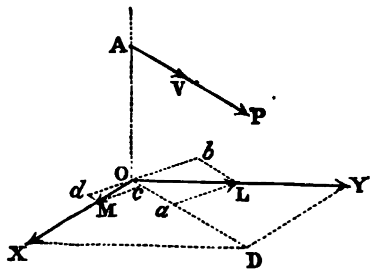

Fig. 238.

If θ denotes the angle DOX, we shall have along OD a moment, $Oa + Oc$, equal to $p_x.X\cos\theta + p_y.Y\sin\theta$; or if $OD = P$, we have along OD

$P(p_x.\cos^2\theta + p_y.\sin^2\theta)$.

Perpendicular to OD, the resultant moment is $Ob - Od$, or

$$P(p_y - p_x)\sin\theta\cos\theta.$$

Now a force OD ($= P$) and a couple whose axis, $Ob - Od$, is perpendicular to it are equivalent to a force equal and parallel to OD at a distance, OA, from OD, along the perpendicular to the plane of OD and the axis of the couple, such that

$$P \times OA = Ob - Od = P(p_y - p_x)\sin\theta\cos\theta;$$

$$\therefore \quad OA = (p_y - p_x)\sin\theta\cos\theta. \qquad (1)$$

Hence the two given wrenches are equivalent to the wrench consisting of the force P at A and the couple whose axis $AV = P(p_x.\cos^2\theta + p_y.\sin^2\theta)$; so that if p_θ denotes the pitch of the resultant screw,

$$p_\theta = p_x.\cos^2\theta + p_y.\sin^2\theta. \qquad (2)$$

The whole process may, of course, be reversed; i.e. starting with the single wrench about the screw AP, we may resolve it in an infinite number of ways into a pair of wrenches about two intersecting rectangular screws. The positions of these screws may be assigned by the distance OA and the angle θ; and when this is done, the component pitches, p_x and p_y, are given by (1) and (2).

214.] **The Cylindroid.** *Given two intersecting rectangular screws, it is required to find the locus of all screws which result from wrenches of any variable intensities about these two given screws.* That is, given two right lines, OX and OY, and two linear constants, p_x and p_y, associated with them, if a wrench in which the force is X and the couple $p_x.X$ act about OX, the magnitude X being anything whatever; and if a wrench in

which the force is Y and the couple $p_y . Y$ act about OY, the magnitude Y being also anything whatever; find what surface is traced out by the axis of the resultant wrench, as X and Y are separately or simultaneously varied in any manner.

Taking OX and OY (Fig. 238) as axes of x and y, and OA, their common perpendicular, as axis of z, the equations of AP are obviously

$$z = (p_y - p_x) \sin\theta \cos\theta, \tag{1}$$

$$y = x \tan\theta, \tag{2}$$

the angle θ depending on the magnitudes X and Y.

Hence, whatever θ may be, we have

$$z(x^2+y^2) - (p_y - p_x)\,xy = 0, \tag{a}$$

which is the equation of the surface traced out by the line AP as X and Y are varied. This surface is called the *Cylindroid*.

215.] **To construct the Cylindroid.** Easy methods of constructing the cylindroid at once present themselves. It is sufficient to give one. Taking two rectangular axes, Ox and Oy, and a perpendicular, Oz, to them, we are to imagine a right line which begins by lying along Ox to travel up along Oz, while it always remains parallel to the plane xy and rotates round Oz, the angle, θ, through which it has rotated, and the corresponding distance, z, through which it has risen being connected by equation (1) of last Article.

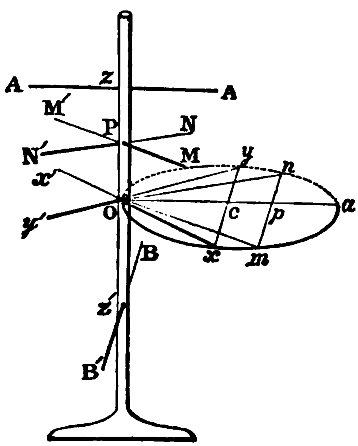

Fig. 239.

Let PM (Fig. 239) be any position of the moving line, its projection on the plane of xy being Om, and $\angle mOx = \theta$. Then, putting $p_y - p_x = 2h$, we have

$$OP = h \sin 2\theta.$$

Draw Oa bisecting the angle xOy, and equal to $2h$, and describe a circle on Oa as diameter, c being its centre, and Om meeting it in m. From m draw the chord mpn perpendicular to Oa. Then $\sin 2\theta = \cos 2\,mcp$; $\therefore\ OP = cp$.

When $\theta = \frac{\pi}{4} = xOa$, OP is greatest and equal to h, and the moving line, zA, is then parallel to Oa, the distance Oz being equal to h. Hence $Pz = ap$.

Thus we get a simple method of constructing the surface :—Divide up the whole diameter aO into any number of parts, ap, &c. (equal for simplicity). On the axis, Oz, take the length $Oz = oa =$ radius of circle; beginning with the point z, measure off parts, zP, &c., successively equal to the parts ap, &c.; then through any point, P, on Oz draw two parallels, PM and PN, to the lines Om and On, joining O to the extremities of the corresponding chord of the circle.

The ruled surface traced out thus by all the pairs of lines, such as PM and PN, is the cylindroid.

It is obvious, of course from the equation $z = h \sin 2\theta$, that through each point P on the axis Oz there are two generators, which coincide at the point z with a parallel to Oa. When P moves upwards from O along Oz, θ runs from 0 to $\frac{\pi}{4}$, until z is reached; when θ increases beyond $\frac{\pi}{4}$, the moving point P descends from z towards O, and in its descent gives the second generator PN at P, which is parallel to On. When $\theta = \frac{\pi}{2}$, P is at O and the generator is Oy. As θ increases beyond $\frac{\pi}{2}$, the moving point P travels downwards, along Oz', until $\theta = \frac{3}{4}\pi$, when z' is reached, Oz' being equal to h, and the generator being $z'B'$, which is parallel to the tangent at O to the circle. As θ increases beyond $\frac{3}{4}\pi$, the moving point moves up again towards O, which it reaches when $\theta = \pi$, the generator then coinciding with Ox, its original position. Thus all through the motion the generator has continuously revolved in the same sense—counter clockwise.

Another way of looking at the matter is this—imagine a pair of scissors placed with the rivet at z and the blades closed and coinciding with $A'zA$; then let the rivet be gradually brought down along zO while the blades gradually open in such a way that when they are parallel to a pair of chords Om and On, the rivet has descended through a distance equal to ap.

(A vivid figure of the cylindroid will be found in Ball's *Theory of Screws.*)

216.] **Angle between two Screws.** In order to make our equations in the sequel universally applicable without ambiguity, it becomes necessary to give a definite meaning to the angle between two screws, since *à priori* the expression is not definite.

The following definition of the angle between two screws will be found to be of universal application whether the pitches are both positive, or both negative, or one positive and the other negative :—

Let the axis of each screw be marked with an arrow-head pointing in the sense in which the *force* acts along the screw. The two screws being denoted by a and β, place a watch with its back towards a and its face towards β, the shortest distance between them passing perpendicularly through its face. Then the angle through which the arrow on a must be rotated, in a sense opposite to that of the watch-hand rotation, so that this arrow shall be parallel to and in the sense of the arrow on β, is the angle between the screws.

217.] **Theorem.** *Any two given screws determine a cylindroid.* Let AP (Fig. 238) and BQ* be the axes of any two given screws whose pitches are, respectively, p_θ and p_ϕ, the line AB being the shortest distance between them. Let $AB = h$. Then what we have to show is that it is possible to find a single pair of rectangular lines, OX and OY, such that if the wrench of pitch p_θ about AP is resolved into two wrenches about these lines, and if the wrench of pitch p_ϕ, about BQ is also resolved into two wrenches about OX and OY, we shall get the same value in each case for the pitch about OX and also the same value for the pitch about OY.

Let ω be the angle between AP and BQ; let AP and BQ make angles θ and ϕ with the sought line OX, the point O being on AB at a distance z from B; and assume that each resolution gives a pitch p_x about OX, and a pitch p_y about OY. Then we have

$$p_\theta = p_x \,.\cos^2\theta + p_y \,.\sin^2\theta,\quad p_\phi = p_x \,.\cos^2\phi + p_y \,.\sin^2\phi\,; \qquad (1)$$

$$z+h = (p_y - p_x)\sin\theta\cos\theta,\qquad z = (p_y - p_x)\sin\phi\cos\phi, \qquad (2)$$

* BQ is not represented in the figure; but, for definiteness, B is supposed to lie on OA between O and A, while the projection of BQ on the plane OXY lies between OX and OD.

where $\theta = \omega + \phi$. Hence

$$p_\theta + p_\phi = p_y + p_x - (p_y - p_x)\cos\omega\cos(\omega + 2\phi); \tag{3}$$

$$p_\theta - p_\phi = (p_y - p_x)\sin\omega\sin(\omega + 2\phi); \tag{4}$$

$$h = (p_y - p_x)\sin\omega\cos(\omega + 2\phi); \tag{5}$$

so that we have

$$\tan(\omega + 2\phi) = \frac{p_\theta - p_\phi}{h}; \tag{6}$$

$$p_y + p_x = p_\theta + p_\phi + h\cot\omega; \tag{7}$$

$$p_y - p_x = \sqrt{h^2 + (p_\theta - p_\phi)^2}\operatorname{cosec}\omega; \tag{8}$$

which give definite values for p_x, p_y, and ϕ. Hence the lines OX and OY can be determined, and therefore also a single cylindroid containing the two given screws.

218.] **Composition of Wrenches.** *The resultant of any two wrenches is a wrench about a screw on the cylindroid determined by the two given wrenches.*

For, let p_1 and P_1 be the pitch and intensity of one, and p_2 and P_2 the pitch and intensity of the other. Also let p_x and p_y be the pitches of the two principal (or rectangular) screws, Ox and Oy, of the cylindroid. Then (Art. 213) replacing the first wrench by its components round Ox and Oy, we get a moment $p_x . X_1$ round Ox, and a moment $p_y . Y_1$ round Oy, the components of P_1 parallel to Ox and Oy being X_1 and Y_1. Similarly, replacing the second wrench by its components, we have finally the moments $p_x(X_1 + X_2)$ and $p_y(Y_1 + Y_2)$ round Ox and Oy, respectively. But if we take the resultant of the forces P_1 and P_2, as if they acted at a point, and if its components parallel to Ox and Oy are X and Y, we know that $X = X_1 + X_2$, and $Y = Y_1 + Y_2$. Therefore round Ox and Oy we have simply wrenches of intensities X and Y, which (Art. 213) give a single wrench about that screw on the cylindroid which is parallel to the direction of the resultant of translation of the given forces P_1 and P_2.

Hence the proposition of the *parallelogram of forces* for forces acting at a point becomes simply a proposition of the *parallelogram of screws* for the composition of wrenches.

Hence also three wrenches will be in equilibrium if they take place about three screws on the same cylindroid, whose directions are so related that the intensity of the wrench on any one screw

is proportional to the sine of the angle between the directions of the other two screws—the well-known *law of Sines.*

And, generally, the resultant of any number of wrenches about screws situated on the same cylindroid may be found by transferring all the forces in the wrenches to a single point, finding the resultant of these forces, and taking the screw on the cylindroid which is parallel to the direction of this resultant. A wrench about this screw with intensity equal to the resultant force is the resultant wrench sought.

COR. A wrench about any given screw on a cylindroid can be resolved into wrenches about any two assigned screws on the same cylindroid. For, a force acting along any given line can be resolved into two components along any two lines which meet it if they all lie in the same plane. In this way the intensities of the two component wrenches along the two assigned screws are determined.

219.] **Distribution of Pitch.** The pitches belonging to the various screws on a cylindroid may be graphically represented thus.

Taking the two principal screws of the cylindroid as axes, construct the conic whose equation is

$$x^2.p_x+y^2.p_y = k^3, \tag{1}$$

where k is any constant length. If r is the radius of this conic making an angle θ with the axis of x (i.e., the screw of pitch p_x), we have

$$p_x.\cos^2\theta+p_y.\sin^2\theta = \frac{k^3}{r^2}. \tag{2}$$

But by (2) of Art. 213 the left-hand side is the pitch of the screw whose axis is parallel to r. Hence

$$p_\theta = \frac{k^3}{r^2}, \tag{3}$$

which graphically represents p_θ in precisely the same way as the moment of inertia of the lamina is represented.

The conic (1) is called the *pitch conic* of the cylindroid. It is an ellipse if the principal pitches have the same sign, and a hyperbola if they have opposite signs.

In the latter case there will be two screws of zero pitch, viz., those parallel to the asymptotes of the pitch hyberbola. In every case there will be *two* screws having a given pitch, and they are parallel to two equal diameters of the pitch conic.

This conic possesses the following noteworthy property. *If the wrench on any screw of the cylindroid is replaced by a force and a couple at the centre of the pitch conic (centre of the cylindroid), the axis of this couple will lie along the perpendicular to the diameter of the pitch conic which is conjugate to the direction of the force*—or, in other words, the plane of the couple will be that of the axis of the cylindroid and this conjugate diameter.

For, let the axis of the given wrench make an angle θ with the axis of x at the centre, O, of the cylindroid, and let P be the intensity of the wrench. Then the component wrenches at O to which the given one is equivalent are ($P\cos\theta$, $Pp_x\cos\theta$) and ($P\sin\theta$, $Pp_y\sin\theta$). The two couples, $Pp_x\cos\theta$ and $Pp_y\sin\theta$, at O compound into a couple, G, making with Ox an angle ψ such that $\cot\psi = \frac{p_x}{p_y}\cot\theta$. Hence

$$\tan\theta\tan\left(\frac{\pi}{2}+\psi\right) = -\frac{p_x}{p_y},$$

which is the well-known equation connecting the directions of two conjugate diameters of the conic, the square of whose axes are $\frac{k^3}{p_x}$ and $\frac{k^3}{p_y}$, so that the line perpendicular to G is the diameter conjugate to the direction of the given screw (θ).

220.] **Screw Motion of a Rigid Body.** It will be shown in a subsequent chapter that if a rigid body occupying a position which we may denote by (A), be displaced in any manner so as to occupy another position (B), the change from (A) to (B) could have been effected by rotating the body round a certain axis, and then giving it a motion of translation along this axis; in other words, Poinsot's result for a system of forces holds for the displacements of the individual points of a rigid body—viz., *the displacement can be produced by giving the body a twist* about a screw.*

The ratio of the motion of translation along the axis of the

* Sir R. Ball uses this term *twist* to denote a rigid-body motion which consists of a translation along, accompanied by a rotation about, a line. The term *twist* is, however, so generally used to signify a *strain* of a natural solid—which is wholly distinct from a rigid-body motion—that it is advisable to call the attention of the student to its technical signification in Sir R. Ball's theory. Except in connection with the theory of Screws, we shall use this word subsequently (in discussing bent and twisted wires, &c.) to denote *strain*.

screw to the circular measure of the angle of rotation about it is called the *pitch* of the screw; so that, as in the case of forces and couples, *the pitch is still a linear magnitude.*

A motion of translation alone may be regarded as a twist of infinite pitch.

A motion of rotation alone may be regarded as a twist of zero pitch.

221.] **Reciprocal Screws.** If a rigid body is acted upon by a wrench about a screw α, what is the work done by giving the body a small twist about another screw β?

Let AP and Ox (Fig. 238) represent the screws α and β, respectively, their pitches being p_α and p_β. Let the force in the wrench be P, and the angle of rotation about β be ω, while θ is the angle (Art. 216) between the screws.

Replace the force P and the couple $P.p_\alpha$ by their components at the point A parallel and perpendicular to Ox. The components of the couple are $P.p_\alpha \cos\theta$ and $P.p_\alpha \sin\theta$; and these we may suppose transferred to the point O. The components of the force are $P\cos\theta$ and $P\sin\theta$. Transfer these to O, introducing (Art. 202) the couples whose axes along Oy and Ox are $Ph\cos\theta$ and $-Ph\sin\theta$, where $h = OA =$ shortest distance between the screws. Hence the given wrench is replaced by a force $P\cos\theta$ acting along Ox, a force $P\sin\theta$ acting along Oy, a couple $P.p_\alpha\cos\theta - Ph\sin\theta$ whose axis is along Ox, and a couple $P.p_\alpha\sin\theta + Ph\cos\theta$ whose axis is along Oy. For the displacement of translation $\omega.p_\beta$ along Ox the only work done is $P\cos\theta \times \omega.p_\beta$, which is due to the first component force; and for the rotation ω round Ox the only work done (Art. 201) is $(P.p_\alpha\cos\theta - Ph\sin\theta)\,\omega$, which is due to the first component couple. Hence the whole work done is

$$P.\omega\left[(p_\alpha + p_\beta)\cos\theta - h\sin\theta\right]. \qquad (a)$$

The expression in brackets is called the *virtual coefficient* of the two given screws

This virtual coefficient will in the sequel be denoted by $\varpi_{\alpha\beta}$.

It is obvious from symmetry that if the body were acted upon by a wrench with force P about β, and it were displaced by a screw motion about α, with spin ω, the same amount of work would be done as before. If the virtual coefficient vanishes, the two screws are said to be *reciprocal*—i.e., two screws are reciprocal when, if a body receive a twist of any amplitude

(with, of course, a given constant pitch) about one of them, no work will be done against a wrench acting on the body along the other screw; or, again, if it were acted on by a wrench on one, it would not move at all if only free to twist about the other.

The following results follow at once from the relation of reciprocity of two screws.

Two intersecting screws ($h = 0$) will be reciprocal either if they are at right angles, or if the sum of their pitches $= 0$.

Two screws at right angles will be reciprocal if they intersect. If they do not intersect, the condition of reciprocity will be fulfilled if either pitch $= \infty$. Put into ordinary language this is as follows—if two rectangular lines intersect, a body free only to twist round one of them would not move if acted upon by any wrench on the other. If they do not intersect, a body free only to twist about one of them will not move if acted upon solely by a couple about the other—which is evident from first principles. A body free only to twist about a line will not move if acted upon solely by a force along the line—the equivalent of which, in the language of screws, is that a screw is its own reciprocal if its pitch is zero, or infinite.

Again, *if a screw is reciprocal to two given screws, it is reciprocal to every screw on the cylindroid determined by these two screws.* Let θ and ϕ be any two screws of pitches p_θ, p_ϕ, and let η be a screw of pitch p_η, which is reciprocal to the first two. Then if a body is acted on by a wrench (P, Pp_θ) of *any* intensity, P, on the screw θ, no work is done by giving the body a twist $(\omega, \omega p_\eta)$ of *any* amplitude, ω, about η. The same holds for the screws ϕ and η. But a wrench of any intensity on any other screw, ψ, on the cylindroid (θ, ϕ) can be replaced by component wrenches of certain intensities (Art. 218) on θ and ϕ; and since no work is done against these component wrenches by a twist on η, no work will be done against the wrench on ψ; therefore ψ and η are reciprocal. We may therefore speak of the screw η as being *reciprocal to the cylindroid* (θ, ϕ).

222.] **Reciprocal Screws on a Cylindroid.** *Two screws on a cylindroid are reciprocal if they are parallel to a pair of conjugate diameters of the Pitch Conic.*

Let α, β be any two screws on a cylindroid making angles

θ and θ' with the axis of x. Suppose $\theta' > \theta$. Then the shortest distance, h, between them is (Art. 217)

$$\tfrac{1}{2}(p_y - p_x)(\sin 2\theta' - \sin 2\theta), \text{ or } (p_y - p_x)\cos(\theta' + \theta)\sin(\theta' - \theta).$$

Hence their virtual coefficient is

$$[p_x + p_y + (p_x - p_y)\cos(\theta' + \theta)\cos(\theta' - \theta)]\cos(\theta' - \theta)$$
$$+ (p_x - p_y)\cos(\theta' + \theta)\sin^2(\theta' - \theta),$$

or $$(p_x + p_y)\cos(\theta' - \theta) + (p_x - p_y)\cos(\theta' + \theta),$$

or $$2\,(p_x \cos\theta\cos\theta' + p_y \sin\theta\sin\theta'),$$

and the vanishing of this is the condition that the directions θ and θ' should be conjugate in the conic $x^2 p_x + y^2 p_y = k^3$. For another proof see Ball's *Theory of Screws*, p. 37.

223.] **Theorem.** *A cylindroid can be constructed so as to be reciprocal to any four given screws.* This theorem is thus proved by Sir Robert Ball (*Theory of Screws*, Art. 26).

The determination of a screw requires five conditions (Art. 211); therefore if a screw is reciprocal to five given screws, it will be completely determined, since we shall have five equations of the form

$$(p_a + p_\eta)\cos\theta - h\sin\theta = 0, \tag{a}$$

η being the required screw. But if a screw is reciprocal to *four* given screws, it will not be completely determinate: it must describe a certain surface-locus. This locus is a cylindroid. For, let a, β, γ, δ be the four given screws, and, if possible, let there be *three* screws, λ, μ, ν, reciprocal to these four and not lying on one cylindroid. Then, by last Art., *every* screw, θ, on the cylindroid (λ, μ) is reciprocal to the four given screws; so is every screw, ϕ, on the cylindroid (μ, ν); and so is every screw on the cylindroid (θ, ϕ). Thus the sought screw does not describe a surface-locus, but a family of surfaces, which is impossible. Hence λ, μ, ν must be co-cylindroidal, and their cylindroid is reciprocal to the four given screws.

To show how this cylindroid may be constructed, we proceed thus.

Arrange the four given screws in the descending order of their pitches; let this order be $p_a, p_\beta, p_\gamma, p_\delta$. Let k be a magnitude intermediate to p_β and p_γ. Then on the cylindroid (a, γ) find the two screws (λ, λ') whose pitches are each k; also on the cylindroid (β, δ) find the two (μ, μ') whose pitches are each k. Draw the *two* lines (Art. 241) which intersect the four

screws λ, λ', μ, μ'; let these lines (σ, σ') be made screws each with pitch $-k$.

Then the screw σ is reciprocal to the screw λ, because $p_\sigma + p_\lambda = 0$ and the two screws intersect.

Similarly σ is reciprocal to λ'; therefore (Art. 221) σ is reciprocal to the cylindroid (λ, λ'), i.e., to the cylindroid (α, γ). In the same way σ is reciprocal to the cylindroid (β, δ); and similarly σ' is reciprocal to both cylindroids. Hence the cylindroid (σ, σ') is the required cylindroid, since every screw on it is thus proved to be reciprocal to all the screws α, β, γ, δ.

224.] **Problem.** *To construct the screw which is reciprocal to any five given screws.* Let the five given screws be α, β, γ, δ, ϵ; construct the cylindroid reciprocal to the first four and also that reciprocal to the last four; these two cylindroids must intersect in the screw reciprocal to all five. Moreover there can be only *one* solution; for, since there are five equations given to determine the (five) unknown quantities which define the required screw, there must be a *definite* number of solutions (proceeding, possibly, from some algebraical equation obtained by eliminating four of the unknowns from the five equations of reciprocity); and if *two* screws could be found reciprocal to the given set of five, an infinite number could be found—viz., all those on the cylindroid of these two. There can be, therefore, *only one screw reciprocal to five given screws.*

225.] **Theorem.** *On any cylindroid can be found one, and only one, screw reciprocal to any given screw.*

Let ϵ be the given screw. Find any four screws reciprocal to the cylindroid. Find (last Art.) the single screw reciprocal to ϵ and these four. This last screw must lie on the cylindroid, since every screw reciprocal to the four lies on the cylindroid.

226.] **Theorem.** *Given seven screws placed in any manner in space; then there is one determinate system of equilibrating wrenches on this system of screws.*

Since the pitches are all supposed given, we have to show that the several forces (or intensities) of the wrenches are fully determinate—at least their mutual ratios are so, just as it is the mutual ratios of three equilibrating forces acting along three given coplanar concurrent lines which are determinate.

Let the given screws be α, β, γ, δ, ϵ, ζ, η. Construct the

single screw, ψ, which is reciprocal to the last five. The five wrenches on these give a resultant wrench reciprocal to ψ; and since the resultant of the wrenches on α and β is equal and opposite to the resultant of the five, the resultant of the wrenches on α and β is reciprocal to ψ. But the resultant of these two is a wrench on some screw on the cylindroid (α, β); this screw (ξ) must therefore be reciprocal to ψ, and is therefore known (last Art.). The law of sines, or parallelogram of wrenches (Art. 218) for the three given screws α, β, ξ, determines the ratio of the forces (or intensities) of the wrenches on α and β. Similarly for every other pair of the given seven.

The problem of this Article may, of course, be put in this way—*Given completely a wrench on a screw* η, *to resolve it into six wrenches on six given screws*, i. e., to find the intensities of these six component wrenches.

Sir R. Ball points out that this is the generalisation of the ordinary statical process of reducing a given force to three component forces and three couples.

The actual determination of the six component intensities of the wrenches on six given screws, $\rho_1, \rho_2, \rho_3, \ldots, \rho_6$, equivalent to a given wrench on a given screw, ρ, can be effected as follows.

Let (R, Rp_ρ) be the wrench on ρ; let the component wrench on ρ_1 be $(R_1, R_1 p_1)$ where R_1 is the intensity (to be determined) and p_1 the (given) pitch of ρ_1. Similarly let $(R_2, R_2 p_2)$, &c., be the other component wrenches.

Now the work done by the wrench (R, Rp_ρ) in twisting a body about any screw through any angle is equal to the sum of the works done by the component wrenches in the same twist.

Let the body be twisted about the screw ρ itself through an angle ω. Then the work done by the wrench (R, Rp_ρ) is $2\,Rp_\rho \,.\, \omega$ (Art. 221). The work done by the component wrench $(R_1, R_1 p_1)$ is $R_1 \varpi_{\rho\rho_1} \,.\, \omega$; and so on for the others. Hence

$$2\,Rp_\rho = R_1 \varpi_{\rho\rho_1} + R_2 \varpi_{\rho\rho_2} + \ldots + R_6 \varpi_{\rho\rho_6}. \qquad (1)$$

Now consider a twist of any amplitude, ω, about the screw ρ_1, and equate the work done in it by the wrench (R, Rp_ρ) to the sum of the works of the component wrenches. Thus we get

$$R \varpi_{\rho\rho_1} = 2\,R_1 p_1 + R_2 \varpi_{\rho_1\rho_2} + \ldots + R_6 \varpi_{\rho_1\rho_6}. \qquad (2)$$

Similarly, considering twists round the other screws, $\rho_2, \ldots, \rho_6$,

in succession, we obtain five more equations like (2); so that we have a system of linear equations with known coefficients, $p_1, p_2 \ldots, \varpi_{\rho_1\rho_2}, \ldots$, for the required forces $R_1, R_2, \ldots$ in terms of the given force R.

Substituting the values of $\varpi_{\rho\rho_1}, \varpi_{\rho\rho_2}, \ldots, \varpi_{\rho\rho_6}$ given by (2) and the similar equations in (1), we have

$$R^2 p_\rho = R_1^2 p_1 + R_2^2 p_2 + \ldots + R_1 R_2 \varpi_{\rho_1\rho_2} + \ldots$$
$$= \Sigma R_1^2 p_1 + \Sigma R_1 R_2 \varpi_{\rho_1\rho_2}, \tag{3}$$

which is analogous to the expression for the resultant of any number of forces.

The six screws of reference can be chosen with such relations among themselves as will greatly simplify the values of the component wrenches—just as a choice of rectangular axes simplifies the components of a force.

Let ρ_1 be any screw; ρ_2 any screw reciprocal to ρ_1; ρ_3 any screw reciprocal to ρ_1 and ρ_2; ρ_4 any screw reciprocal to ρ_1, ρ_2, ρ_3; ρ_5 any screw reciprocal to $\rho_1, \rho_2, \rho_3, \rho_4$; and ρ_6 the single screw (Art. 224) reciprocal to the remaining five. Thus (2) becomes

$$R\,\varpi_{\rho\rho_1} = 2\,R_1 p_1,$$

since $\varpi_{\rho_1\rho_2} = 0, \ldots, \varpi_{\rho_1\rho_6} = 0$. This determines R_1; and similarly $R\varpi_{\rho\rho_2} = 2\,R_2 p_2$, &c.; moreover (3) becomes

$$R^2 p_\rho = \Sigma\, R_1^2 p_1.$$

Such a system of screws, viz., one in which every pair of screws is reciprocal, is called a system of co-reciprocal screws.

227.] **Degrees of Freedom of a Rigid Body.** The position of a rigid body in space is completely defined by *six* independent variables, viz., the three co-ordinates of some point in it with reference to assumed rectangular axes, and the three angles (see Routh's *Rigid Dynamics*, Chap. IX) which in the well-known theory of the motion of a rigid body about a fixed point determine the positions of all points in the body relatively to this fixed point. The body may, however, be so hampered in any case that these six variables are not all independent. If each of them may be anything whatever independently of any of the others, the body is perfectly free or has *freedom of the sixth order*, or *six degrees of freedom*. If the variables are connected by one equation, so that virtually only five are independent (the sixth being known as soon as any five are assumed), the body has *freedom of the fifth order*, or *five degrees of freedom*.

If they are connected by two equations, or, in other words, if the position of the body depends on only *three* independent variables, the body has *freedom of the third order*, or *three degrees of freedom*; and so on.

A rigid body occupying any position can be brought into an indefinitely near position by giving it a small motion of translation whose components parallel to fixed rectangular axes are $(\delta a, \delta b, \delta c)$, which are the components of the translation of any point, A, in the body, and rotating it round axes of reference at A parallel to the fixed axes through angles $(\delta\theta_1, \delta\theta_2, \delta\theta_3)$. See Chap. XV, or Routh, *ibid.*

The component absolute motions of any point in the body are expressed by the equations

$$\delta x = \delta a + (z-c)\,\delta\theta_2 - (y-b)\,\delta\theta_3, \text{ \&c.},$$

(a, b, c) being the co-ordinates of A with reference to the axes through the fixed origin.

If the only motion possible for the body is a twist (of any amplitude) about a given screw, the body has one degree of freedom. For, the only variable on which its motion depends is the amplitude, ω, of the twist about the given line, a, the translation being $\omega . p_a$, which (since p_a is given) is known when ω is assumed. The value of δx is $[lp_a + m(z-c) - n(y-b)] . \omega$, where (l, m, n) are the direction-cosines of the axis of the screw. Since, then, the adjacent position of the body depends on only *one* variable, the body has one degree of freedom.

If the constraints of a rigid body are such that every position adjacent to the one which it occupies can be obtained by some combination of twists (of variable amplitudes) about two given screws, it has two degrees of freedom, and it can likewise twist about every screw on the cylindroid determined by these two.

For, an adjacent position now depends on two independent variables, viz., two twists of amplitudes ω and ω' about the two given screws a, β; and the value of δx will be the sum of the previous expression and one exactly similar referring to β. Moreover, since two twists on a and β always compound a twist on some screw of their cylindroid, the last part of the proposition is evident.

In the same way, three degrees of freedom are equivalent to the possibility of attaining all consecutive positions by twisting about three *given* screws, and, *mutatis mutandis*, the above

enunciation holds for this case, except that the other screws about which twisting may take place do not lie on any cylindroid.

And so on for the remaining degrees of freedom.

That besides three given (non-cocylindroidal) screws there will be an infinite number of screws about which the body can twist is evident, because the resultant of any three twists is a twist about some screw whose position depends on the amplitudes of the three; and since these may be varied in any manner, their resultants will give an infinite number of twists about an infinite number of screws.

Just as the screws of possible twist coming from *two* given screws are infinite in number, but yet very specially related—forming a cylindroid—so those of possible twist coming from *three* given screws, although infinite, are related, and their assemblage in space is called a *screw complex of the third order*. A screw complex of the second order is a cylindroid.

Similarly for screw complexes of the fourth and fifth orders. What is a screw complex of the sixth order? It is the assemblage of screws about which a body could twist if every consecutive position of the body can be attained by twisting about six given screws. But such a body is perfectly free; therefore it could twist about every line in space. Hence the assemblage of all lines in space is the screw complex of the sixth order.

We may formally define a screw complex of any order, m, thus—*A screw complex of the m^{th} order is an assemblage of screws in space such that any one screw of the assemblage can be determined from any m of them and cannot be determined by any smaller number;* that is, a twist of any assumed amplitude, or a wrench of any assumed intensity on the one screw in question, can always be exhibited as compounded of twists of proper amplitudes, or wrenches of proper intensities, on the m screws selected.

Thus, in the complex of the second order (cylindroid) a wrench of any intensity on any one screw of the complex can, by the law of sines (Art. 218), be resolved into two wrenches, of appropriate intensities, about *any two* selected screws of the complex.

228.] **Examples of Degrees of Freedom.** The following are some very simple instances of the various degrees of freedom of a rigid body.

One degree of freedom. (Every position, consecutive to a given one, that can be attained by the body is producible by a twist of variable amplitude about one fixed screw.) A nut moving on a fixed axis with a thread cut on it. A body sliding along a fixed axis, but guided so as to prevent rotation (screw of infinite pitch). A body capable of rotating on a fixed axis, but prevented from moving along it (screw of zero pitch; e.g. a compound pendulum).

Two degrees of freedom. (Every position, consecutive to a given one, that can be attained by the body is producible by some combination of twists of variable amplitudes about two fixed screws.) A nut moving on an axis having a thread cut on it, and this axis itself rigidly attached to another nut which can move on a fixed axis with a thread cut on it. A body with a fixed axis (spindle) stuck through it, so that the body can move along and rotate about the axis, the rotation and translation being quite independent. A rigid body with one fixed point, O (ball and socket joint), a string of given length being attached to another point, P, in the body and to a point fixed in space; or, instead of being thus held by the string, the point P may be constrained to any fixed curve on a sphere.

To find the cylindroid corresponding to any given case of freedom of the second order, all we have to do is to find any two screws about which twists may be given, and these two determine the whole cylindroid.

Three degrees of freedom. (Every position consecutive to a given one attainable by some combination of twists of variable amplitudes about three fixed screws.) A body moveable on a spindle which has a nut rigidly attached to it, this nut being capable of moving on an axis fixed in space with a thread cut on it. A body moveable round a fixed point (ball and socket joint) —the well-known example in the theory of Precession and Nutation.

To find the complex corresponding to any given case of freedom of the third order, find any three screws about which twisting may take place, and these determine the whole complex.

Four degrees of freedom. Body on spindle which is capable itself of spindle motion on an axis fixed in space.

Five degrees of freedom. Body on spindle which has a ball and socket motion round a fixed point.

229.] **Determination of Screws on a Complex.** Consider two screws, α, β, which determine a cylindroid. All the other screws of the complex are simply axes of wrenches which are the resultants of all possible wrenches on α and β. Now (Art. 218), if wrenches of any intensities, P, Q, act on α and β, the position of the resultant screw (and therefore its pitch) depends only on the *ratio* of P to Q.

Hence the position of any screw on the cylindroid which is determined by the given screws, α, β, depends on only *one* variable.

Similarly the position of any screw on the complex of the third order determined by three given screws, α, β, γ, depends on only *two* variables, viz. the *ratios* of the three (variable) intensities of wrenches on these screws. And, generally, the position of a screw on a complex of the m^{th} order determined by m given screws depends on $m-1$ variables—ratios of intensities of wrenches on the given screws.

230.] **Screws Reciprocal to a given Complex.** *All screws which are reciprocal to a given complex of the order m form themselves a complex of the order $6-m$.*

Firstly, they form a complex, i.e. an assemblage specially related. For, consider all the screws reciprocal to a given cylindroid. A rigid body which is free to twist about any of these screws would not do so if it were acted upon by wrenches on any of the screws on the cylindroid; hence twisting solely about the screws in this reciprocal assemblage must be equivalent to a certain limitation in the freedom of the body, so that the assemblage is a complex.

Secondly, the order of the complex is $6-m$. For, any screw of this reciprocal complex, being reciprocal to m screws (viz. any m determining the given complex), satisfies by its determining constants m equations with known coefficients, such as (a), Art. 223. This screw would therefore require $5-m$ further conditions for its complete determination. But (last Art.) the number of these conditions is one less than the degree of the complex to which it belongs; therefore the degree of this reciprocal complex is $6-m$.

It is, of course, obvious that if a rigid body has m degrees of freedom, so that it is capable of twisting about any screw on a certain complex of order m, any system of forces acting on it and

reducing to a wrench on any screw of the reciprocal complex will be in equilibrium with the constraints of the body; and conversely, *when a rigid body acted on by any system of forces has a certain specified degree of freedom, the condition of its equilibrium is that the given force system must reduce to a wrench on some screw of the complex which is reciprocal to the complex of screws about which the body can twist.* This, as Sir R. Ball observes (*Theory of Screws*, p. 41), is the most general equilibrium theorem for a rigid body.

231.] **Number of Conditions determining a Complex of given Order.** In order to determine completely a quadric surface we know that nine conditions are necessary; and to determine, in the same sense, a cylindroid we shall show that eight conditions are necessary; and, generally, to determine a complex of any order, m, we shall show that $m(6-m)$ conditions are necessary.

For simplicity begin with a cylindroid. If its centre and axes are unknown, taking any origin and rectangular axes, we must in equation (a) of Art. 214 assume an expression of the form

$$x\cos\alpha + y\cos\beta + z\cos\gamma + p$$

for x, with similar expressions for y and z, and also an unknown quantity for $p_y - p_x$. This gives us *ten* independent constants; but the conditions of rectangularity of the new planes of reference reduces this to eight—which is the proper number of unknown coefficients in the general equation of a cylindroid.

Generally, a complex of order m is determined if m screws are given; and since each screw requires 5 constants for its complete determination, we have thus $5m$ constants given. Now these will give us more information than we want—we want merely the complex, and, in addition, we know completely m special screws in it. We must therefore diminish the number $5m$ by the number of conditions required to specialize the m screws. Now (Art. 229) each screw is specialized by $m-1$ data, therefore the data which specialize m of them are $m(m-1)$ in number. Hence the conditions required to determine the complex, without informing us of particular screws in it, are $5m - m(m-1)$, or $m(6-m)$ in number.

Cor. The number of conditions which determine any complex and the number which determine its reciprocal complex are the same. Consequently the most general complex of the fifth order

is the reciprocal of a single screw, and the most general complex of the fourth order is the reciprocal of a cylindroid.

232.] **Reciprocal of a Single Screw.** Let any line be the axis of a screw of pitch p_a. Then every line in space can be the axis of a screw reciprocal to the given one; for if ω is the angle between the given line and any other line OL, and if h is the shortest distance between the lines, we have merely to give the pitch, p_λ, to the screw on OL, such that

$$p_\lambda + p_a = h \tan \omega.$$

Consider the screws reciprocal to a given screw that can be drawn through a given point, O.

Through O (Fig. 240) draw a line OA parallel to the axis of the given screw, a; let OP be the perpendicular from O on a. Then the shortest distance, h, between OL and a is the perpendicular from P on the plane of OL and OA. Let $OP = p$, and with O as centre and OP (or any other length) as radius describe a sphere. (The axis of the screw a is the line through P parallel to OA.)

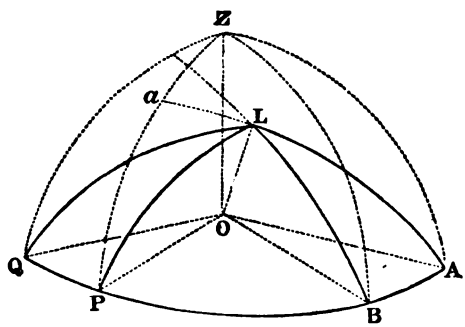

Fig. 240.

At O draw OZ at right angles to OA and OP; produce the great circle AL to meet the great circle PZ in a. Then the shortest distance between a and OL is $p \sin Pa$; and $\omega = LA$.

Hence $$p_\lambda + p_a = p \,.\, \sin Pa \cot aL.$$

But by Napier's Analogies applied to the triangle PaL,

$$\sin Pa = \cot a PL \,.\, \tan aL,$$

therefore $$p_\lambda + p_a = p \,.\, \tan LPA, \qquad (a)$$

which determines the pitch of the screw on OL.

All screws of the same pitch at O lie in a plane. For if p_λ is constant, the angle LPA is constant, i.e. the line OL moves in the plane POL which makes a constant angle with the plane POA. (For a different proof see Ball's *Theory of Screws*, p. 85.)

233.] **Reciprocal of a Cylindroid.** Every right line, in general, meets a cylindroid in three points, through each of which passes a screw on the surface. If the right line is the axis of a screw reciprocal to the cylindroid, this screw (λ) is reciprocal to the three screws which it thus meets. But when two reciprocal screws intersect, we must have (Art. 221) either the sum of their pitches equal to zero, or the angle between them a right angle. Now it is impossible that the pitches of all three screws on the cylindroid should be equal, because from Art. 219 we see that there are only *two* screws on the surface which have the same pitch. Hence the screw λ must be perpendicular to one of the screws which it meets.

Now consider all the screws that can be drawn from a given point, O, reciprocal to the cylindroid. It will (Art. 221) be sufficient to make these screws from O reciprocal to any *two* screws on the surface. Through O draw a plane perpendicular to the axis of the cylindroid, and let the two screws at the point where it cuts the axis be α and β. Through O (Fig. 240) draw OA and OB parallel to α and β, and OZ parallel to the axis of the cylindroid, OP and OQ being the directions of perpendiculars from O on α and β, respectively.

Then if OL is the axis of a screw reciprocal to α and to β, we have by last Art.

$$p_\lambda + p_\alpha = p \tan LPA,$$
$$p_\lambda + p_\beta = q \tan LQA,$$

where p and q are the lengths of the perpendiculars from O on α and β.

Hence we have

$$p \tan LPA - q \tan LQA = p_\alpha - p_\beta, \tag{α}$$

so that the line OL moves subject to this condition.

The problem of the locus of OL is therefore this—given two intersecting lines, OP and OQ, if two planes are drawn through these lines making angles θ and ϕ with the plane of OP and OQ, such that

$$p \tan \theta - q \tan \phi = k, \tag{β}$$

where p, q, k are constants, what is the surface-locus of the line of intersection of the planes?

It is at once obvious that it is *a cone of the second degree.* Hence all the screws reciprocal to a given cylindroid that can be drawn through a given point lie on a cone of the second degree.

(A simple geometrical proof of this theorem is given by Sir R. Ball, *Theory of Screws*, p. 23. We have thought it advisable to present the matter in a different light, as there is an advantage in looking at the subject from different points of view.)

We may remark that if the two planes are drawn through the two lines so as to satisfy the equation

$$p \cot \theta - q \cot \phi = k,$$

the surface-locus of their line of intersection will be *a plane*, and we have thus the extension of the useful 'cotangent formula' of Art. 35, vol. i.

By what precedes, it is evident that this cone is the locus of the feet of perpendiculars from O on the generators of the cylindroid, because every screw reciprocal to the cylindroid intersects the surface in two screws of pitch equal and opposite to its own, and one screw at right angles.

Hence the parallel to the nodal line (axis of z) of the cylindroid drawn through any point O, since it is perpendicular to a generator, must belong to the reciprocal cone drawn from O. Moreover the pitch of the screw whose axis is this line is ∞; for it is at right angles to every screw of the cylindroid, and when $\omega = \frac{\pi}{2}$ while h is not zero (Art. 221) the condition of reciprocity requires the sum of the pitches $= \infty$; hence the pitch of the screw through O parallel to the nodal line $= \infty$.

It follows that on the reciprocal cone can be found screws of all pitches from $-\infty$ to $+\infty$ (while, of course, the pitches on the cylindroid itself range from p_y to p_x, the principal pitches, Art. 219). For, every two screws of the same pitch on the cylindroid are intersected by some generator of the cone, the pitch of which is equal and opposite to the common pitch of these two screws. Hence the two screws of zero pitch (when they exist, Art. 219) are intersected by a generator of zero pitch on the cone; and we have just seen that this cone has also a generator of infinite pitch.

234.] **Reciprocal to Complex of Third Order.** We shall notice, finally, the screws reciprocal to a complex of the third order which can be drawn through any point, O. This number is *three*, as is thus shown. It is sufficient to take any three screws, α, β, γ, of the given complex, and to find the reciprocals to these. Draw the cylindroid (α, β); then all the screws

through O reciprocal to α and β form a cone of the second degree; and observe that one generator of this cone is the perpendicular from O on β. Again, draw the cylindroid (β, γ); then all the screws through O reciprocal to β and γ form another cone of the second degree, and one generator of it also is the perpendicular from O on β. These cones, having the same vertex, intersect in *four* right lines, one of which we know—viz. the perpendicular from O on β. The screws reciprocal to α, β, and γ must lie on these common lines of intersection. But it is easy to see that the common generator, which is the perpendicular from O on β, is not relevant. For, whatever screw γ may be, the pitch of the screw on this perpendicular is fixed, viz. (see Art. 233) a pitch equal and opposite to the pitches of the two screws on the cylindroid (α, β), other than β, which it intersects. Since, then, γ may be any screw whatever, it cannot be restricted to being reciprocal to the screw on the perpendicular from O on β.

For a different proof that only three screws reciprocal to a complex of the third order can be drawn from a given point see Ball's *Theory of Screws*, p. 122.

EXAMPLES.

1. The sum of the pitches of the two screws which pass through any point on the axis of a cylindroid is constant.

2. A cubical block (represented by Fig. 228) is free to twist about its diagonal OO'; determine a wrench—

(a) about AB,

(b) about AD,

so that the block may be in equilibrium.

Ans. In (a) the wrench is one of infinite pitch, i.e. a couple about AB. In (b) the pitch of the screw on OO' being p, that of the screw on AD is $a-p$, where a is the length of an edge of the block, so that the wrench is $[P, (a-p)P]$, where P is a force of any magnitude.

3. A right cone is capable of twisting about a screw coincident with one of its generating lines; find the wrench about a given diameter of its base which will keep it in equilibrium.

Ans. If the axis of the given screw of twist (pitch p) is BA, where B is the vertex and A a point on the circumference of the base, O the centre of the base, OP the radius of the base about which the wrench is to take place, P being on the circumference of the base, $\angle POA = \theta$, $c =$ height of cone, the required wrench is

$$[P, -(p + c \tan\theta) P].$$

4. A body which has freedom of the second order is acted upon—

(a) by a single force,

(b) by a single couple;

what is the condition of equilibrium?

Ans. In case (a) the line of action of the force must intersect both the screws of zero pitch on the cylindroid which defines the possible motions of the body; and in (b) the axis of the couple must be parallel to the nodal line of the cylindroid.

5. At a given point, O, are compounded three wrenches of fixed pitches, a, b, c, along three fixed rectangular lines, Ox, Oy, Oz; the intensities of these wrenches being all varied in any manner, find the surface-locus traced out by the Poinsot centre.

Ans. Its equation is

$$\frac{\overline{b-c}^2}{x^2}+\frac{\overline{c-a}^2}{y^2}+\frac{\overline{a-b}^2}{z^2}+\frac{\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}{xyz}=0.$$

The section of this surface by any plane through any axis of coordinates is an ellipse (and the axis itself). The force and the principal couple at O are always related thus—the force being a central radius vector of a fixed ellipsoid, the axis of the principal couple coincides in direction with the central perpendicular on the tangent plane to this ellipsoid at the extremity of the radius vector, and varies inversely as this perpendicular.

6. The axes of three coplanar screws of pitches p_α, p_β, p_γ form a triangle whose sides are a, b, c, respectively; prove that the pitches of the two screws (other than that perpendicular to their plane) which can be drawn through any point, O, in their plane and reciprocal to them are the roots of the equation

$$\frac{ap}{x+p_\alpha}+\frac{bq}{x+p_\beta}+\frac{cr}{x+p_\gamma}=0,$$

where p, q, r are the perpendiculars from O on the sides a, b, c (all reckoned positive when O is inside the triangle).

(It is, of course, evident that any screw of infinite pitch perpendicular to their plane is reciprocal to all three.)

7. Find the directions of the two reciprocal screws at O in the last problem.

8. In a screw complex of the fourth order show that all screws of given pitch must intersect two fixed right lines.

(Consider the reciprocal cylindroid; take the two screws on it whose pitches have the given value with contrary sign.)

9. Show that in a screw complex of the fourth order the locus of those screws which are parallel to a given line is a plane. (Ball, *Theory of Screws*, p. 146.)

(Take the one screw on the reciprocal cylindroid which is per-

pendicular to the given direction; then the plane through this screw parallel to the given direction is the locus.)

10. Construct the cylindroid reciprocal to four screws of zero pitch.

[Draw the two lines which, as we shall see a little farther on, can be drawn to intersect all the four given screws; attribute zero pitch to these two, and construct their cylindroid.]

11. A perfectly free body is acted upon by five given forces, show how it can be moved in a particular manner in such a way that no work is done by or against the forces. (Ball, *Theory of Screws*, p. 152.)

[Draw the cylindroid, as in last example, for any four; on this cylindroid find the screw reciprocal to the fifth; this is the single screw (Art. 224) which is reciprocal to all five; and any twist on this will be unaccompanied by work.]

12. A perfectly free rigid body is acted upon by three screws; what are the conditions of equilibrium?

Ans. As to the situations of the screws, they must be co-cylindroidal; and as to the intensities of the wrenches on them, they must satisfy the law of sines (Art. 218).

13. For the same case what are the conditions of equilibrium of six screws and of seven screws, respectively?

Ans. For six, the screw reciprocal to five must be reciprocal to the sixth, and the intensities must be related. For seven; there is no condition as to the positions of the screws—the only condition is a relation between the intensities of their wrenches (see Art. 226).

235.] **Theorem.** A system of forces can be reduced to two forces in an infinite number of ways. For they can be reduced to a resultant force, R, acting at any point, together with a couple. Now the forces of the couple can be made of any magnitude by varying its arm; and one of them can be combined with R. There will then remain the resultant of R and this force together with the remaining force of the couple. Therefore, &c.

Of course the wrench to which all pairs of forces equivalent to a given force system reduce is unique; and since we have shown (Art. 221) that the wrench of two forces takes place about a screw which intersects the shortest distance between the lines of action of the two forces, we see that—*Poinsot's axis intersects the shortest distance between the lines of action of every pair of forces to which the given force system can be reduced.*

Suppose that AP and BQ (Fig. 237, p. 18) are a pair of forces

to which a given force system can be reduced, and let $p = \frac{K}{R}$ = the pitch of the Poinsot screw to which they are equivalent. Then the distance (OA) of either line (AP) from the Poinsot axis is the product of the pitch and the cotangent of the inclination of the other line (BQ) to the Poinsot axis.

For if $\theta = PAr$, $\phi = qAr$, since Ac represents K and An represents Qh, we have $K = p \,.\, R = Qh \sin\phi$. But

$$\frac{AO}{OB} = \frac{Q\cos\phi}{P\cos\theta}; \quad \therefore \quad \frac{AO}{h} = \frac{Q\cos\phi}{R};$$

$$\therefore \quad AO \times R = Qh\cos\phi;$$

$$\therefore \quad AO = p\cot\phi.$$

Similarly $$BO = p\cot\theta.$$

The two lines of action, AP, BQ, of any pair of forces equivalent to a given wrench are sometimes called *reciprocal lines*.

They possess the following property—if any point, S, be taken on either line (AP), the axis of principal moment at this point is the perpendicular to the plane containing S and the other line (BQ).

This property is at once obvious, since to get G, the axis of principal moment at S (supposed on AP), we introduce at S two forces equal and opposite to Q; then the couple Q at B and $-Q$ at S is in the plane of S and BQ, and its axis, G, is, of course, perpendicular to this plane.

The relation between the two lines is thus reciprocal, so that either line is the envelope of the planes of principal couples at all points on the other line.

The two forces P and Q along AP and AQ may, of course, be regarded as two wrenches each of zero pitch, and therefore as determining a cylindroid. If in Article 217 we put $p_\theta = p_\phi = 0$, we find $p_y = h\cot\frac{\omega}{2}$, $p_x = -h\tan\frac{\omega}{2}$; also the origin of the cylindroid bisects the distance h, and its axes are parallel to the internal and external bisectors of the angle between AP BQ. The equation of the cylindroid is

$$z(x^2+y^2) - hxy\,\mathrm{cosec}\,\omega = 0.$$

The two principal pitches have opposite signs, and the given forces act along the two screws of zero pitch of this cylindroid.

236.] **Theorem.** When a system of forces is reduced to a pair of forces represented in magnitudes and lines of action by two right lines, the volume of the tetrahedron formed by these lines is constant, however the reduction is made.

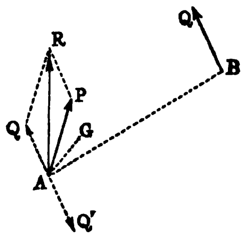

Fig. 241.

Let the system of forces be reduced to P and Q, and let these be supposed to act at the extremities, A and B, of the shortest distance between them. Now to get the force and couple corresponding to the origin A, introduce at this point two opposite forces, AQ and AQ', each equal and parallel to Q.

Compounding P and Q we get the resultant force, R; and taking the forces Q at B and Q' at A we get a couple whose axis, AG, is at right angles to the plane $QBAQ'$ and equal to $Q \, . \, AB$. Since AB is perpendicular to both P and Q, it is clear that AG is in the plane QAP and at right angles to AQ.

Now since (Art. 208) $G \cos\phi = K$, we have

$$Q \, . \, AB \, . \sin QAR = K.$$

But $\sin QAR = \dfrac{P}{R} \, . \sin PAQ$. Hence

$$P \, . \, Q \, . \, AB \, . \sin PAQ = K \, . \, R.$$

Now the volume of the tetrahedron formed by the lines AP and BQ

$= \frac{1}{3}$ area $ABQ \times$ perpendicular from P on the plane ABQ;

$= \frac{1}{6} \, BQ \, . \, AB \times AP \, . \sin PAQ$;

$= \frac{1}{6} \, P \, . \, Q \, . \, AB \, . \sin PAQ$.

Hence if Δ denotes the volume of the tetrahedron,

$$\Delta = \tfrac{1}{6} K \, . \, R.$$

This theorem has been proved in various ways. For an elegant demonstration by Möbius, see *Crelle's Journal*, vol. iv, p. 179, or Jullien's *Problèmes de Mécanique Rationnelle*, vol. i, p. 71.

237.] **Symmetrical Reduction of a System of Forces.** A system of forces can be reduced to two forces equal in magnitude, equally inclined at opposite sides to Poinsot's Axis, and equally distant from this axis.

Suppose the forces replaced by R acting along Poinsot's Axis, Oz, and a couple, K. Take any point, O' (Fig. 236); draw $O'O$ perpendicular to Oz and produce it to O'' so that $O'O = OO''$. Let R acting at O be replaced by $\frac{1}{2}R$ acting at O' and $\frac{1}{2}R$ acting at O''. Also let the forces of the couple act at O' and O''; for this purpose these forces must each be made $= \frac{K}{2x}$, x being OO'. Now the resultant of $\frac{1}{2}R$ and $\frac{K}{2x}$ at O' is a force

$$= \frac{1}{2}\sqrt{R^2 + \frac{K^2}{x^2}},$$

acting towards the right, and the resultant of $\frac{1}{2}R$ and $\frac{K}{2x}$ at O'' is a force of the same magnitude acting towards the left of the figure.

If ω is the angle made with Poinsot's Axis by these new forces at O' and O'',

$$\tan\omega = \frac{K}{Rx}.$$

If we choose x so that $\frac{K}{x} = \sqrt{3}R$, each of the two symmetrical forces is equal to R, and they are inclined at an angle of 60° to Poinsot's Axis.

238.] **Analytical Condition for a Single Resultant.** We have just seen that a system of forces acting on a rigid body is, in general, equivalent to *two* forces. Let the forces be replaced by a single resultant force, R, acting at an arbitrary origin, O, and a couple G. Now the direction-cosines of R referred to axes Ox, Oy, and Oz, are (Art. 206),

$$\frac{\Sigma X}{R}, \quad \frac{\Sigma Y}{R}, \quad \text{and} \quad \frac{\Sigma Z}{R};$$

and those of G are

$$\frac{L}{G}, \quad \frac{M}{G}, \quad \text{and} \quad \frac{N}{G}.$$

Hence, if ϕ is the angle between G and R,

$$\cos\phi = \frac{L\Sigma X + M\Sigma Y + N\Sigma Z}{GR}. \tag{1}$$

Now if the resultant couple is in a plane containing R, one of its forces can be made to destroy R, and there will remain a single force; but if G and R are not at right angles to each

other, the system of forces cannot be equivalent to a single force. The required condition is, therefore, $\cos\phi = 0$, or

$$L\Sigma X + M\Sigma Y + N\Sigma Z = 0, \tag{2}$$

provided that ΣX, ΣY, *and* ΣZ *do not all vanish*; for if they do, R will also vanish, and ϕ will be illusory. In fact, in this case, since L, M, and N alone exist, the system of forces is equivalent to a couple.

239.] **Theorem.** The quantity $L\Sigma X + M\Sigma Y + N\Sigma Z$ has the same value for all systems of rectangular axes assumed anywhere in space.

From (1) of the last Art., it $= R \,.\, G\cos\phi$, or $R \,.\, K$, where K is Poinsot's moment (Art. 208).

Hence, if this quantity vanishes for any one set of axes, the force and the axis of the accompanying couple corresponding to any origin are at right angles.

The value of this quantity can be exhibited in another form, which also shows that it is independent of any particular set of axes.

Substituting for L, M, and N the values (Art. 206), $\Sigma(Zy - Yz)$, &c., the expression becomes

$$\begin{aligned}&(Z_1y_1 - Y_1z_1 + Z_2y_2 - Y_2z_2 + \dots)(X_1 + X_2 + \dots)\\ &+ (X_1z_1 - Z_1x_1 + X_2z_2 - Z_2x_2 + \dots)(Y_1 + Y_2 + \dots)\\ &+ (Y_1x_1 - X_1y_1 + Y_2x_2 - X_2y_2 + \dots)(Z_1 + Z_2 + \dots);\end{aligned}$$

or, substituting for X_1, Y_1, Z_1, ... in terms of the forces P_1, ... and their direction-cosines,

$$\begin{aligned}&[P_1(y_1\cos\gamma_1 - z_1\cos\beta_1) + P_1(y_2\cos\gamma_2 - z_2\cos\beta_2) + \dots]\\ &\qquad \times (P_1\cos\alpha_1 + P_2\cos\alpha_2 + \dots) + \&c. \dots.\end{aligned}$$

It is clear at once that the terms P_1^2, P_2^2, ... disappear, and the products P_1P_2, P_1P_3, ... alone remain.

Collecting the coefficient of P_1P_2 as a typical term, we have

$$\begin{aligned}P_1P_2[&(x_1 - x_2)(\cos\beta_1\cos\gamma_2 - \cos\gamma_1\cos\beta_2)\\ &+ (y_1 - y_2)(\cos\gamma_1\cos\alpha_2 - \cos\alpha_1\cos\gamma_2)\\ &+ (z_1 - z_2)(\cos\alpha_1\cos\beta_2 - \cos\beta_1\cos\alpha_2)].\end{aligned}$$

Now (see Salmon's *Geometry of Three Dimensions*, p. 31, third edition, or Frost's *Solid Geometry*, p. 39) if (P_1, P_2) denotes the angle between the directions of the forces P_1 and P_2, the

quantity in brackets $= d_{12} \cdot \sin(P_1, P_2)$, d_{12} being the shortest distance between the lines of action of the forces.

Hence

$$L\Sigma X + M\Sigma Y + N\Sigma Z = \Sigma P_1 P_2 \cdot d_{12} \cdot \sin(P_1, P_2). \qquad (1)$$

Again (Art. 236), the term involving $P_1 P_2$ on the right side of (1) denotes six times the tetrahedron formed by P_1 and P_2; therefore the quantity on the left side is equal to *six times the sum (with their proper signs) of the* $\frac{n(n-1)}{2}$ *tetrahedra which can be formed out of the pairs of lines representing the n forces*

$$P_1, P_2, \ldots, P_n.$$

This sum has, of course, no reference to any set of axes, and hence the necessarily *invariant* nature of $L\Sigma X + M\Sigma Y + N\Sigma Z$.

With regard to the sign to be given to any tetrahedron of the system, we define that—

The moment of a force with regard to a line is the component of the force perpendicular to the line multiplied by the shortest distance between the force and the line.

Hence $P_1 \cdot d_{12} \cdot \sin(P_1, P_2)$ is the moment of P_1 about the line of action of P_2. Now to determine the sign which must be given to any tetrahedron, let a watch be placed so that the direction in which either force acts passes perpendicularly from the back up through the face of the watch. If then the other force tends to produce rotation in the sense in which the hands rotate, the tetrahedron is to receive a negative sign, and if the rotation is the other way, a positive sign.

240.] **Conditions of Equilibrium of a Rigid Body acted on by any Forces.** The forces having been reduced to a resultant of translation, R, acting at any point, together with a corresponding couple, G, since a force and a couple cannot conjointly produce equilibrium ((ϵ), Art. 200) it is necessary that

$$R = 0 \text{ and } G = 0.$$

Substituting the values of R and G given in Art. 206, we see that these two are equivalent to the following *six* conditions:

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$
$$L = 0, \quad M = 0, \quad N = 0,$$

which are the analytical expressions of the fact that *the forces must have no component along any line and no moment about any axis.*

241.] **Particular Cases of Equilibrium.** (*a*) Equilibrium of three forces. *When three forces keep a body in equilibrium, their lines of action must be coplanar and concurrent* (*or parallel*).

For, let the forces be P, Q, R. Then the sum of their moments about every right line $= 0$. Take any point, p, on P, and from it draw a line meeting Q—in q, suppose.

Since the sum of moments about the line pq must be zero, and since the moments of P and Q about it are separately zero, this line must intersect R—in r, suppose.

Draw another line through p meeting Q in q'; then, as before, this line must meet R—in r', suppose. Now, since two points on each of the lines Q and R lie in the plane determined by the lines pqr and $pq'r'$, the lines Q and R must each lie wholly in this plane. Again, drawing any two lines whatever across Q and R, these must both be intersected by P; that is, P must lie in the plane of Q and R; hence all the forces are coplanar.

Finally, taking moments about the point (Art. 77) of intersection of Q and R, we see that P must pass through this point, and be equal and opposite to their resultant. If Q and R are parallel, P must be parallel to them, and equal and opposite to their resultant.

The case of Art. 19 is therefore the only case of equilibrium of three forces.

(*b*) Equilibrium of four forces. *If four forces keep a body in equilibrium, they must all lie on the same hyperboloid of one sheet.*

Any three non-intersecting right lines determine a hyperboloid of one sheet, because a surface of the second degree requires, in general, *nine* conditions for its determination, and the conditions that any one given right line ($x = az + m$, $y = bz + n$) should lie wholly on the surface are *three* in number; hence among the nine unknown coefficients in the equation of the surface there will be established nine (linear) equations if *three* given non-intersecting lines all lie on it. The surface is therefore determined.

Describe the hyperboloid determined by three of the forces, P, Q, R; then an infinite number of right lines can be drawn to intersect these three, and all such lines lie on the hyperboloid and constitute one system of its generators, while P, Q, R belong to the other system of generators (see Salmon's *Geometry of Three Dimensions*, Chap. VI). Every line intersecting P, Q, R

must, since the sum of the moments of the four forces about it $= 0$, also intersect the fourth force S; hence S passes through an infinite number of points lying on the hyperboloid, which is impossible unless S lies wholly on the surface.

The given forces, therefore, act along lines which are all generators of the same system of the same hyperboloid.

(*c*) Equilibrium of five forces. *If five forces keep a body in equilibrium, their lines of action must intersect two right lines.* If a right line could be drawn so as to intersect four of the forces, it would have to intersect the fifth, on account of the vanishing of the sum of the moments about it.

Now *two* right lines can, in general, be drawn to intersect any four non-intersecting right lines. For, let the four lines be denoted by P, Q, R, S. Construct the hyperboloid determined by (P, Q, R), and also the hyperboloid determined by (P, Q, S). These hyperboloids having two right lines for a part of their curve of intersection will have two other right lines for the remainder of the curve. For, let the equations of the line P be $(x = 0, y = 0)$, and those of Q be $(z = 0, w = 0)$; then the equation of any hyperboloid containing these lines is

$$x(mz+pw) + y(lz+qw) = 0;$$

another hyperboloid containing the same lines is

$$x(m'z+p'w)+y(l'z+q'w) = 0.$$

Now at all points of intersection of these two hyperboloids, for which x and y do not both vanish, and for which z and w do not both vanish—i. e. at all points of their curve of intersection, excluding the points on the two common generators—we have

$$\frac{mz+pw}{m'z+p'w} = \frac{lz+qw}{l'z+q'w}.$$

This equation, being homogeneous in z and w, denotes two planes passing through the line Q; but any plane through a generator must intersect the surface again in a right line; therefore these two planes cut the surface in two right lines, which are the remaining part of the curve of intersection of the two hyperboloids; and each of them intersects the generators (P, Q, R) and the generators (P, Q, S); i. e. each intersects the four lines P, Q, R, S. Each must, therefore, intersect the fifth force. Q.E.D.

Examples.

1. A rigid body is acted on by forces represented in magnitudes and lines of action by the sides of a gauche polygon taken in order; prove that the forces are equivalent to a couple, and that the sum of their moments about any line is represented by double the area of the projection of the polygon on a plane perpendicular to the line.

Let the forces be represented by the lines $AB, BC, CD, \ldots$ (Fig. 242), and let OQ be any axis.

On the axis take any point, O, and reduce the forces to a resultant, R, of translation at this point, together with a couple, G (Art. 206). This is done by introducing at O two forces parallel and equal to AB in opposed directions, two equal and opposite to BC, &c. Now (Art. 199) the resultant of translation vanishes, and the component couples are represented by double the areas of the triangles OAB, OBC, &c. If the axes of these couples are drawn at O, the sum of the moments of the forces about OQ will be represented by the sum of the components of the axes along OQ; but this is the same as double the sum of the projections of the areas of the triangles on a plane perpendicular to OQ; that is, the moment about OQ is represented by double the area of the projection of the polygon on a plane perpendicular to OQ.

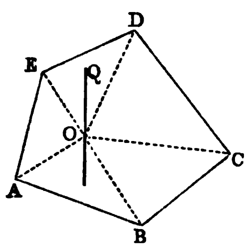

Fig. 242.

Again, since G is the greatest moment round any axis through O (Art. 206), it follows that the axis of the resultant couple is the line perpendicular to the plane on which the projected area of the polygon is a maximum.

2. When the resultant of translation vanishes, the forces will be in complete equilibrium if the sums of their moments round any three non-coplanar axes are separately equal to nothing.

For if L be the moment round the axis of x, the moment L', round a parallel axis through the point (a, β, γ) is $L + \gamma\Sigma Y - \beta\Sigma Z$. Hence $L' = L$, $M' = M$, $N' = N$; and since the moment round an axis through (a, β, γ) making angles λ, μ, ν with the axis of co-ordinates is $L' \cos\lambda + M' \cos\mu + N' \cos\nu$, it follows that the moments round all parallel axes are equal. For the three axes of moments we may take, therefore, three lines through the origin making angles $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and $(\lambda_3, \mu_3, \nu_3)$ with the axes of co-ordinates. Suppose then that

$$L\cos\lambda_1 + M\cos\mu_1 + N\cos\nu_1 = 0,$$

$$L\cos\lambda_2 + M\cos\mu_2 + N\cos\nu_2 = 0,$$

and

$$L\cos\lambda_3 + M\cos\mu_3 + N\cos\nu_3 = 0.$$

These require either that $L = M = N = 0$, or

$$\begin{vmatrix} \cos\lambda_1, & \cos\mu_1, & \cos\nu_1 \\ \cos\lambda_2, & \cos\mu_2, & \cos\nu_2 \\ \cos\lambda_3, & \cos\mu_3, & \cos\nu_3 \end{vmatrix} = 0.$$

The latter condition requires that the three axes of moments be in one plane. If they are not coplanar, we must have $L = M = N = 0$, i.e. the forces are in equilibrium.

3. A tetrahedron is acted on by forces applied perpendicularly to the faces at their respective centroids. If the force applied to each face is proportional to the area of that face, prove that the tetrahedron is in equilibrium, the forces being supposed to act all inwards or all outwards.

Let A, B, C, D be the vertices of the tetrahedron, and denote the areas of the faces opposite these vertices by A_1, B_1, C_1, D_1, respectively. Denote also the angle between the faces A_1 and B_1 by $\widehat{A_1B_1}$. Then evidently

$$A_1 = B_1\cos\widehat{A_1B_1} + C_1\cos\widehat{A_1C_1} + D_1\cos\widehat{A_1D_1};$$

or, if the forces perpendicular to the faces are denoted by P, Q, R, S,

$$P - Q\,.\cos\widehat{PQ} - R\,.\cos\widehat{PR} - S\,.\cos\widehat{PS} = 0,$$

which shows that there is no resultant force in a direction perpendicular to the face A_1; similarly there is no resultant force in directions perpendicular to the other faces; therefore the resultant of translation vanishes.

To show that there is no resultant couple, let each force be replaced by three equal forces acting at the angles of the corresponding face. Thus the force P is to be replaced by three forces each equal to $\frac{1}{3}P$ acting at the points B, C, D perpendicularly to the face BCD. Let us calculate the sum of the moments of the forces about the edge BC. For this purpose, let the forces $\frac{1}{3}Q$ and $\frac{1}{3}R$ at D be each resolved in the direction of the force $\frac{1}{3}P$ at this point, i.e. perpendicularly to the face BCD. Supposing the forces to act outwards, the components of $\frac{1}{3}Q$ and $\frac{1}{3}R$ are $-\frac{1}{3}Q\,.\cos\widehat{PQ}$ and $-\frac{1}{3}R\,.\cos\widehat{PR}$; therefore the sum of the moments of the forces at D about BC is proportional to

$$(A_1 - B_1\,.\cos\widehat{A_1B_1} - C_1\,.\cos\widehat{A_1C_1})p',$$

or $$D_1\,.\,p'\,.\cos\widehat{A_1D_1},$$

or, again, $$D_1\,.\,p\,.\cot\widehat{A_1D_1},$$

p' being the perpendicular from D on BC, and p the perpendicular from D on the base ABC. But this last expression is three times the volume of the tetrahedron multiplied by $\cot\widehat{A_1D_1}$. In the same way, the sum of the moments of the forces at A is represented by three times the volume of the tetrahedron multiplied by $\cot\widehat{A_1D_1}$; and as these moments are in opposite senses, the forces have no moment round

the edge BC, and similarly no moment round any of the edges. Hence by the last example they are in equilibrium.

For another simple method of proof see Collignon's *Statique*, p. 354.

4. Prove that a solid body of any shape is in equilibrium if it is acted on throughout its surface by normal forces, each force being proportional to the superficial element on which it acts.

One very simple method of proof consists in imagining a surface precisely equal and similar to that of the given body to be traced out in a weightless fluid which is subject to any pressure.

5. If a curved surface whose edge is a plane curve is acted on all over its surface by normal forces, each proportional to the element of surface on which it acts, prove that these forces have a single resultant if they all act towards the same side of the surface.

6. Forces perpendicular and proportional to the areas of the faces act at the centres of the circles circumscribing the faces of a tetrahedron; prove that they are in equilibrium, if they all act inwards or outwards.

They meet in the centre of the circumscribed sphere. The proposition is evidently true also for any polyhedron bounded by triangular faces.

Taking the results of this example and example 3 together, we see that forces proportional to the areas and perpendicular to them are in equilibrium if they act at the orthocentres of the triangular faces of any polyhedron.

7. Find the force necessary to keep a heavy door in a given position, the hinge line being inclined to the vertical and the hinges smooth.

Let i be the inclination of the hinge line to the vertical, and α the given inclination of the plane of the door to the vertical plane containing the hinge line. Then if W is the weight of the door, a the distance of its centre of gravity from the hinge line, and θ the angle between the normal to the plane of the door and the vertical, the moment of the weight about the hinge line is

$$Wa \cos \theta.$$

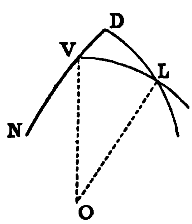

Fig. 243.

This is the moment of the required force. To find θ, let lines parallel to the hinge line and the vertical be drawn through any point, O, and through this point let a plane be drawn parallel to the plane of the door. Round O let any sphere be described; let V and L (Fig. 243) be the points where these lines meet the sphere; DL the circle in which the plane of the door intersects the sphere, and N the point in which the normal, ON, to the door intersects it. Then $VL = i$, $\angle DLV = \alpha$, and $NV = \theta$, and we have from the spherical triangle VDL,

$$\sin VD = \sin i \sin \alpha,$$

or

$$\cos \theta = \sin i \sin \alpha,$$

since N is the pole of DL. Hence the moment of the required force is

$$Wa \sin i \sin a,$$

and when its point of application and direction are known, its magnitude is therefore known.

8. A beam can turn in every direction about one end which is fixed; the other end rests on a rough inclined plane. Find the limiting position of equilibrium. (See Walton's *Mechanical Problems*, p. 191, third edition.)

Let AB (Fig. 244) be the beam, A the fixed end, DPH the rough inclined plane, PH the intersection of this plane with a horizontal plane through A, APD the vertical plane through A perpendicular to the inclined plane, BD a line parallel to PH, AO a perpendicular from A on the inclined plane, DQ a perpendicular on the horizontal plane, i the inclination of the plane, a the angle, ABO, between the beam and this plane, and μ the coefficient of friction.

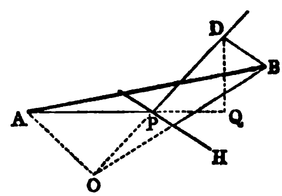

Fig. 244.

Now *suppose first that the beam is perfectly inelastic.* Then the end B describes on the inclined plane a circle whose centre is O, and if it is about to slip, the force of friction assumes a direction perpendicular to OB in the inclined plane. The extreme position of the beam will be denoted by the angle, θ or DOB, between the plane, AOB, through the beam normal to the inclined plane and the vertical plane, AOD.

The forces acting on the beam are its weight, the reaction of the smooth joint at A, and the total resistance of the inclined plane at B. This last force we shall consider as composed of a normal reaction, R, and a force of friction, μR, acting perpendicularly to BO. For the equilibrium of the beam take moments about a vertical axis through A. The moment of the normal reaction at B is $R \sin i \times BD$, or $R \sin i \,.\, BO \sin\theta$, or again, $R \sin i \,.\, AB \cos a \sin\theta$. To find the moment of μR, resolve it into $\mu R \cos\theta$ along BD and $\mu R \sin\theta$ parallel to OD; and resolve this latter again into a horizontal component, $\mu R \sin\theta \cos i$, and a vertical component, $\mu R \sin\theta \sin i$. The moment of μR is then equal to the sum of the moments of $\mu R \cos\theta$ and $\mu R \sin\theta \cos i$; that is, it is equal to

$$\mu R \cos\theta \times AQ + \mu R \sin\theta \cos i \times BD.$$

Hence the equation of moments is

$$R(\sin i - \mu \cos i \sin\theta)\, BD = \mu R \cos\theta \,.\, AQ.$$

But
$$AQ = AP + PQ = \frac{AO}{\sin i} + (OD - OP)\cos i$$

$$= \frac{AB \,.\, \sin a}{\sin i} + AB \cos i \cos a \cos\theta - AB \sin a \cot i \cos i$$

$$= AB\,(\sin i \sin a + \cos i \cos a \cos\theta);$$

therefore

$$(\sin i - \mu \cos i \sin\theta)\cos a \sin\theta = \mu \cos\theta\,(\sin i \sin a + \cos i \cos a \cos\theta),$$

or
$$\sin i \cos a \sin\theta = \mu \cos i \cos a + \mu \sin i \sin a \cos\theta,$$

or
$$\sin\theta - \mu \tan a \cos\theta = \mu \cot i.$$

Putting $\mu \tan a = \tan\beta$, we have θ from the equation

$$\sin(\theta - \beta) = \mu \cot i \cos\beta. \qquad (1)$$

If there is no horizontal plane through A obstructing the beam, it will be possible for the end B to describe a complete circle round O. Let us inquire the condition that the beam should rest in all possible positions. For this there must be no *limiting* position of equilibrium, or, in other words, the value of θ in (1) must be imaginary.

The required condition is, then, $\mu \cot i \cos\beta > 1$,

that is,
$$\mu > \frac{\tan i}{\sqrt{1 - \tan^2 i \tan^2 a}}.$$

Let us next *suppose that the beam is elastic*, or that, in virtue of a compression of the beam, B is not constrained to move in the circle whose centre is O. Supposing, then, that the beam has been jammed against the plane, if the coefficient of friction is gradually diminished, B will begin to move in some other direction than that perpendicular to OB, and this direction will be exactly opposite to that in which the force of friction acts. Now the reaction at A, the total resistance at B, and the weight of the beam lie in one plane which must, therefore, be *the vertical plane through the beam*. The total resistance at B must, moreover, lie inside or on the cone of friction described round B. Hence if the position of the beam is such that the vertical plane through it *touches* this cone, equilibrium will be at its limit, since the line of action of the total resistance is the line of contact of the vertical plane with the cone.

Let the lines and planes of the figure be projected on a sphere described about B as centre with arbitrary radius. Then the cone of friction will appear as a small circle of angular radius, NC (Fig. 245), equal to λ, the angle of friction. Let N be the point in which the normal to the inclined plane at B meets the sphere; A, the point representing the beam, and ACV the vertical plane through the beam touching the cone of friction. Now the vertical line at B lies in the vertical plane, ACV, through the beam, and it makes an angle equal to i with the normal to the inclined plane. Hence, take a point V in ACV so that $NV = i$, and we have NV, the circle answering to the vertical plane through B normal to the inclined plane (a plane

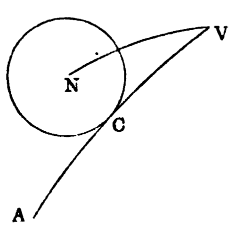

Fig. 245.

which is parallel to the plane APD, Fig. 244). In the spherical triangle NVC we have, then,

$$\sin NV . \sin NVC = \sin NC,$$

or

$$\sin i \sin \theta = \sin \lambda;$$

$$\therefore \quad \sin \theta = \frac{\sin \lambda}{\sin i}.$$

This second solution supposes that *the only condition to which the total resistance is subject is that of making with the normal an angle not greater than the angle of friction.* The supposition of perfect rigidity, on the contrary, restricts the direction of the force of friction in the inclined plane, making it perpendicular to the line OB.

9. A heavy elastic beam rests on two rough inclined planes whose intersection is a horizontal line. Show that every position of the beam may be one of equilibrium if the inclination of each plane is less than the angle of friction for that plane and the beam.

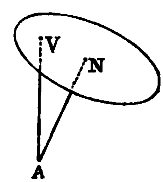

Fig. 246.

Let A (Fig. 246) be one end of the beam, AN the normal to the plane on which A rests, and AV the vertical at A. Then if the beam is sufficiently elastic, it may be jammed against the planes, and the only condition to which the total resistances at its ends are subject are the conditions of making with the normals angles not greater than the corresponding angles of friction. Hence in the extreme position in which the end A is about to slip, the vertical plane through the beam must touch the cone of friction described round the normal, AN. But this is manifestly impossible, since the angle λ is $> VAN$; for the vertical line is included within the cone, and through this line no plane can be drawn to touch the cone. There can, therefore, be no *limiting* equilibrium at either end in any position of the beam.

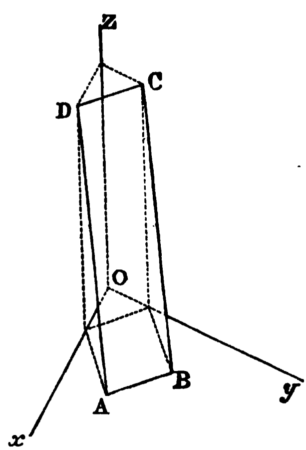

Fig. 247.

10. A ladder, $ABCD$ (Fig. 247), whose centre of gravity divides it into two given segments, rests with one end, AB, on the ground, the upper end, CD, resting symmetrically against two equal rough vertical planes which include a given angle; find its limiting inclination to the ground.

On account of the equal roughness of the vertical walls and the symmetrical position of the ladder, the total resistances at C and D are equal; moreover they have a single resultant passing through the middle point of CD, since the two normal

pressures and the two forces of friction have resultants passing through this point.

At each of the points A and B the total resistance makes the angle of friction with the normal, and the resultant of these forces acts at the middle point of AB, making the angle of friction with the vertical. The resultant resistance above and that below must meet in the vertical through the centre of gravity of the ladder.

Let λ be the angle of friction at the ground; $\lambda' =$ that for each wall; $a =$ lower and $b =$ upper segment of ladder made by its centre of gravity; $\theta =$ limiting inclination of ladder; $\phi =$ angle made with vertical by the resultant of the total resistances at C and D. Then, by the 'cotangent formula' of Art. 35, we have

$$(a+b)\tan\theta = \frac{a}{\mu} - b\cot\phi, \tag{1}$$

where $\mu = \tan\lambda$.

The angle ϕ may, of course, be found by the ordinary method of determining the magnitude and direction of the resultant of forces from their several components; but we prefer to employ for the purpose the *method of spherical projection*, which is more simple, and which will be frequently employed in the sequel. The method consists in constructing a sphere of any radius, and drawing through its centre lines and planes parallel to the lines and planes in our figure; these will intersect the surface of the sphere in points and circles, respectively,—as illustrated in examples 7, 8, 9 already.

Let O (Fig. 248) be the centre of the sphere; OZ a parallel to the vertical; ON and ON' parallels to the normals to the planes zy and zx, respectively; ZC and ZD planes parallel to these planes respectively; OR and OR' lines in the planes ZN and ZN', each inclined at the angle, λ', of friction to the corresponding normal; then OR and OR' represent the lines of action of the total resistances at C and D. If S is the middle point of the arc RR', the resultant of the resistances acts in OS, and the arc $ZS = \phi$.

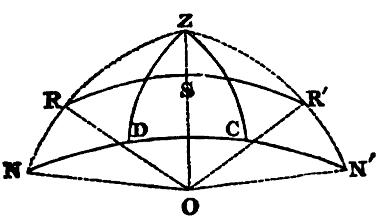

Fig. 248.

If α is the angle between the walls, $DC = \alpha$, and $NN' = \pi - \alpha$; therefore the angle $RZS = \frac{\pi}{2} - \frac{\alpha}{2}$; and applying Napier's Analogies to the triangle RZS, we have

$$\cot\phi = \mu' \operatorname{cosec}\frac{\alpha}{2}.$$

Hence (1) gives

$$(a+b)\tan\theta = \frac{a}{\mu} - b\mu' \operatorname{cosec}\frac{\alpha}{2}, \tag{2}$$

which determines the limiting inclination.

11. If the vertical walls are unequally rough, show that the initial motion of the ladder cannot be one in which the line CD moves down parallel to its original position.

12. If the walls are unequally rough, show that the initial motion cannot be one in which one corner (D) is for the moment at rest, while slipping takes place at two other corners (C and B).

13. A solid rectangular block is placed with one of its faces on an inclined plane so rough as to prevent slipping, while tumbling is possible; to investigate the positions of equilibrium.

Let ABC (Fig. 249) be the face on the inclined plane. All the different positions may be obtained by turning the block round the edge, AI, through any corner of the base, which is perpendicular to the inclined plane. Draw the horizontal line Ax in the inclined plane. Let G be the centre of gravity of the block; O that of the face ABC; GP the vertical line through G meeting the face ABC in P. Since GO is perpendicular to the inclined plane, $\angle PGO = i =$ inclination of plane, so that the sides of the triangle OGP are all constant whatever be the position of the block; therefore if the successive positions of P are marked on the face ABC, they trace out in it a circle with centre O.

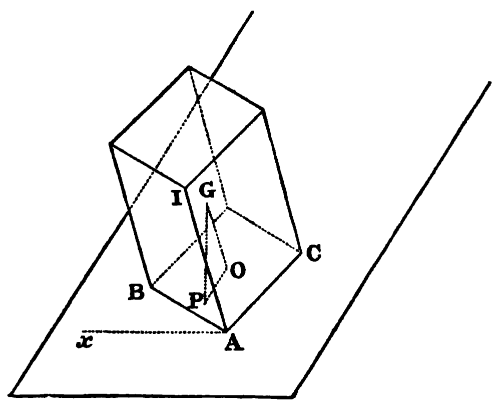

Fig. 249.

Again, since Ax is the line of intersection of the inclined plane and a horizontal plane, it is at right angles to the plane of two intersecting normals to these planes; it is therefore at right angles to the plane of GO and GP, and hence to OP in all positions of the block.

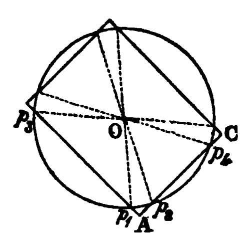

Fig. 250.

Therefore if Fig. 250 represents the base of the block and the circle traced out in it by the motion of P, the points in which the circle intersects the sides of the face being p_1, p_2, p_3, p_4, we see that if we turn the block round AI so that any one of the lines Op_1, Op_2, Op_3, Op_4 is at right angles to Ax, we obtain a position in which the block is about to tumble; in other words, make a perpendicular to any of the lines $Op_1, Op_2, \dots$

(drawn in the plane of the base) horizontal, and we obtain a limiting position.

If $2a$ and $2b$ are the lengths of the edges AB and AC, and c is the distance of G from the base, the conditions that the circle should intersect all the edges of the base are $c \tan i > a$ and $c \tan i > b$, where i is the inclination of the plane.

Obviously if the block be any solid body having a base of any form, the solution is the same. If O is the projection of G on the base, and $OG = c$, describe a circle round O with radius $c \tan i$; let p be any point of intersection of this circle with the contour of the base; then make a perpendicular to Op horizontal, and we obtain a limiting position.

14. A heavy uniform bar rests with its extremities on two rough inclined planes whose line of intersection is horizontal; supposing that the bar is slightly elastic and can be jammed between the planes, investigate its positions of limiting equilibrium.

We may evidently consider the centre of the bar to be restricted to a fixed vertical plane which is perpendicular to both of the inclined planes. Take this plane as that of yz, the axis of x being the line of intersection of the inclined planes, and the axis of z a vertical line. Let (y, z) be the co-ordinates of the centre of gravity of the bar; $2a$ = length of the bar; θ = angle between the bar and a vertical line; ϕ = angle between vertical plane through the bar and the plane yz; i and i' the inclinations of the given planes; λ and λ' the angles of friction between them, respectively, and the bar.

Then the co-ordinates of the extremities of the bar are

$a \sin\theta \sin\phi$; $y + a \sin\theta \cos\phi$; $z + a\cos\theta$ for one extremity, A,

$-a \sin\theta \sin\phi$; $y - a \sin\theta \cos\phi$; $z - a\cos\theta$ for the other, B.

Since these lie on the inclined planes, we have

$$z + a\cos\theta = (y + a\sin\theta\cos\phi)\tan i, \qquad (1)$$

$$z - a\cos\theta = (y - a\sin\theta\cos\phi)\tan i'. \qquad (2)$$

Now, as in Example 9, if the first end is going to slip,

$$\sin i \sin\phi = \sin\lambda, \qquad (3)$$

since the vertical plane through the beam touches the cone of friction at this end. If the other end were about to slip, we should have

$$\sin i' \sin\phi = \sin\lambda'; \qquad (4)$$

so that both ends cannot slip at once unless

$$\frac{\sin\lambda}{\sin i} = \frac{\sin\lambda'}{\sin i'}.$$

Let t and t' stand for $\tan i$ and $\tan i'$; then, eliminating θ from (1) and (2), we have

$$[2z - (t - t')y]^2 \sec^2\phi + [(t - t')z + 2tt'y]^2 = a^2(t + t')^2.$$

Hence the positions in which either end is about to slip are such

that the centre of gravity lies on a certain ellipse—any position of this point on the ellipse being admissible—the corresponding value of ϕ being given by (3) or (4), and that of θ by (1) or (2).

We have now to determine, however, whether *both* ellipses are admissible or not—i.e. whether there are positions in which the end A is about to slip, while B remains at rest, and also positions in which B is about to slip while A remains at rest.

Assuming that A is about to slip, the vertical plane through the bar touches the cone of friction described around the normal at A to the inclined plane (i); but at the same time this vertical plane must not lie wholly outside the cone of friction at B, i.e. it must intersect this latter in two real right lines. Now if, for simplicity, we transfer the origin to B, the axes remaining unchanged in direction, the equation of the vertical plane through the bar is

$$x - y\tan\phi = 0,$$

and the cone of friction at B is

$$(y\sin i' + z\cos i')^2 - \cos^2\lambda'(x^2+y^2+z^2) = 0;$$

and these will intersect in a pair of real lines if

$$\sin i' \sin\phi < \sin\lambda',$$

or by (3),

$$\frac{\sin\lambda}{\sin i} < \frac{\sin\lambda'}{\sin i'}.$$

If this inequality is satisfied, it is only the end A that can slip; if the reverse holds, it is the end B that can slip. Thus both values of ϕ are not admissible.

15. If at any point, P, a plane, ϖ, be drawn perpendicular to the axis of principal moment at the point, find the envelope of ϖ as P moves along a given curve.

Simplicity will be gained by taking Poinsot's Axis, Oz (Fig. 236, p. 16), as axis of z. Let (a, β, γ) be the co-ordinates of P with reference to Oz and any two axes of x and y. Then, introducing two forces equal and opposite to R at P, we shall have the whole force system equivalent to R at P, Poinsot's couple K, and a couple $R\rho$, where ρ is the perpendicular from P on Oz. We may replace the couple $R\rho$ by two components parallel to Ox and Oy, and these will be $-R\beta$ and Ra; so that the component axes of the principal couple G at P are $(-R\beta, Ra, K)$. Hence the equation of the plane ϖ is

$$-R\beta(x-a) + Ra(y-\beta) + K(z-\gamma) = 0,$$

or

$$\beta x - ay - \frac{K}{R}(z-\gamma) = 0. \quad (1)$$

If the equations of the curve along which P moves are

$$a = \phi(\gamma), \quad \beta = \psi(\gamma),$$

substitute these values of a and β in (1), and eliminate γ from the resulting equation and its derived with respect to γ.

Verify, in particular, the result of Art. 235, that if P moves a right line, ϖ, will turn round another right line; that Poinsot's Axis

intersects the shortest distance between these two lines, dividing it in the ratio $\frac{\cot\theta}{\cot\phi}$ (Art. 212).

16. Find the surface traced out by the axes of principal moment at points taken along a right line intersecting Poinsot's Axis perpendicularly.

Let Ox (Fig. 236) be the assumed line, and let it be taken as axis of x, Poinsot's Axis, OK, being that of z. Let $OO' = x$, and let (y, z) be the co-ordinates of any point on $O'G$. Then, if $\phi = \angle GO'K'$, we have

$$\frac{z}{y} = \cot\phi = \frac{Gn}{O'n} = \frac{K}{R.x},$$

or

$$xz = \frac{K}{R}.y,$$

an equation which denotes a hyperbolic paraboloid. As the point O' moves out from O along Ox, the axes (such as $O'G$) of principal moment revolve towards the right; as O' moves in towards O, they revolve towards the left, and, after coincidence with Poinsot's Axis at O, they still revolve towards the left. At an infinite distance from O they are at right angles to Poinsot's Axis.

17. Find the surface traced out by the axes of principal moment at points taken all along any arbitrary curve.

From Example 15, the equations of the principal axis at the point (α, β, γ) with reference to Poinsot's Axis as axis of z, and any two rectangular axes of x and y are

$$\frac{x-\alpha}{-\beta} = \frac{y-\beta}{\alpha} = \frac{z-\gamma}{p},$$

where p is the pitch of the wrench to which the given forces are equivalent. From these we have

$$\alpha = p.\frac{px+y(z-\gamma)}{(z-\gamma)^2+p^2}; \quad \beta = p.\frac{py-x(z-\gamma)}{(z-\gamma)^2+p^2};$$

and if the point (α, β, γ) moves along the curve whose equations are

$$\phi(\alpha, \beta, \gamma) = 0, \quad \psi(\alpha, \beta, \gamma) = 0,$$

substitute the above values of α and β in these equations and then eliminate γ. The resulting equation in x, y, z is that of the surface traced out.

18. A plank, AB, laid on a rough inclined plane, has attached to its upper extremity, A, a cord which lies along the plane in the direction of the plank and is pulled with a constant force, P; find the limiting position of equilibrium of the plank.

Ans. Let W = weight of plank, i = inclination of the plane, λ = angle of friction, and θ = inclination of the plank to a horizontal line drawn in the inclined plane; then

$$\sin\theta = \frac{P^2 + W^2(1-\cos^2 i\sec^2\lambda)}{2PW\sin i}.$$

19. Show that the initial motion of the plank will be one of translation simply, in a direction making with a horizontal line in the inclined plane an angle ϕ determined by the equation

$$\tan\phi = \frac{P\sin\theta - W\sin i}{P\cos\theta},$$

where θ has the value found in last example.

20. If $P = 0$, explain the values of θ in the cases

$$i > \lambda, \quad i < \lambda, \quad i = \lambda.$$

21. Find the value of P so that the direction of slipping shall be at right angles to the direction of the plank, and find θ.

Ans. $P = W\sqrt{1-\cos^2 i \sec^2\lambda}$, and $\cos\theta = \dfrac{\tan\lambda}{\tan i}$.

[This case is the same as that in which the cord is replaced by a smooth pivot at the extremity A.]

22. A triangular prism is placed with its triangular face on a rough inclined plane, which is rough enough to prevent slipping; find the greatest height of the prism so that there may be at least one position of equilibrium.

Ans. If i = inclination of plane, and if the sides of the triangular face are a, b, c, in descending order of magnitude, the greatest height is

$$\tfrac{2}{3}\sqrt{2a^2+2b^2-c^2}.\cot i.$$

23. A heavy plate of any form rests on two rough fixed pegs A and B, the line joining which is not horizontal; the plate can turn round a pivot, without friction, at a point C; if C is raised so that the plate turns gradually about the fixed line AB, find the inclination of the plane ABC to the horizon when the plate begins to slip on the pegs.

24. A particle is acted on by any number of given forces, P_1, P_2,... ; prove that if R is their resultant,

$$R^2 = \Sigma(P^2) + 2\Sigma(P_1 . P_2 \cos \widehat{P_1P_2}),$$

where $\widehat{P_1P_2}$ denotes the angle between the directions of P_1 and P_2.

25. Prove that a system of forces acting on a rigid body may be replaced by two equal forces whose lines of action are perpendicular to each other, and each inclined at an angle of 45° to Poinsot's Axis: the forces act at the ends of a line bisected by this axis; the length of this line is $\dfrac{2K}{R}$, and each force is $\dfrac{R}{\sqrt{2}}$, R being the resultant of translation, and K Poinsot's moment.

26. Prove that the distance between the lines of action of the two rectangular forces which equivalently replace a given system of forces is a minimum when the forces are equal.

27. $ABCD$ is a tetrahedron; forces P, Q, R act along the edges BC, CA, AB in order, and forces P', Q', R' act along AD, BD, CD; prove that the condition for a single resultant is

$$\frac{PP'}{BC\,.\,AD}+\frac{QQ'}{CA\,.\,BD}+\frac{RR'}{AB\,.\,CD}=0.$$

28. A rough heavy body, bounded by a curved surface, rests upon two others which themselves rest on a rough horizontal plane; show that the three centres of gravity and the four points of contact lie in one plane.

29. A heavy beam rests on two smooth inclined planes; show that their line of intersection must be perpendicular to the beam and parallel to the horizon.

30. Prove that the moment of a force represented by the right line PQ, about a right line AB is six times the volume of the tetrahedron $ABPQ$ divided by AB.

31. Three equal heavy spheres hang in contact from a fixed point by strings of equal length; find the weight of a sphere of given radius which when placed upon the other three will just cause them to separate.

Ans. If W and a be the weight and radius of each of the three spheres, W' and r the weight and radius of the superincumbent sphere, and l the length of each string,

$$\frac{W'}{W'+3W}=\sqrt{\frac{3r^2+6ar-a^2}{3l^2+6al-a^2}}.$$

32. Three spheres are placed in contact on a rough horizontal plane, and a fourth sphere is placed upon them, there being no friction between the spheres themselves. Show that equilibrium is impossible.

33. Three equal spheres are placed in contact on a rough horizontal plane, and a fourth sphere is placed upon them, there being friction between the spheres themselves. Find the least coefficient of friction between the spheres which will allow of equilibrium.

Ans. If a is the radius of each of the equal spheres and r that of the superincumbent sphere, the least value of λ, the angle of friction, is given by the equation

$$\sin 2\lambda=\frac{2}{\sqrt{3}}\cdot\frac{a}{a+r}.$$

(The total resistance between the upper sphere and any one of the lower spheres must be capable of acting through the point of contact of the latter and the ground.)

34. Three forces whose lines of action are given, but not their magnitudes, have a single resultant. Prove that the surface traced out by the line of action of the resultant is a hyperboloid of one sheet.

(Draw any three lines across the given lines of action. Then the line of action of the resultant must always intersect these three.)

35. A heavy triangular plate of uniform thickness is suspended

from a fixed point by means of three strings attached to the point and to the vertices of the plate; prove that the tension in each string is proportional to the length of the string.

(Let O be the fixed point, A, B, C the vertices of the plate, and G its centre of gravity.

Then G must lie vertically under O. Take $3\,OG$ to represent the weight of the plate. Then, by Leibnitz's graphic representation [Art. 199], the force $3\,OG$ may be resolved into the forces OA, OB, OC. But a given force can have only one set of components along three given concurrent lines. Therefore, &c.)

36. At points on any right line the axes of principal moment of a given system of forces are drawn; prove that their extremities trace out another right line. (Wolstenholme's *Problems*, p. 387, 2nd edition.)

(At any point O on the given line draw R and G. Take as axes of x, y, and z the given line, the line OG, and a line at O perpendicular to R and the given line. Then at any point P on the given line at a distance x from O if the axis of principal moment be drawn, the co-ordinates of its extremity will be x, G, and $Rx \sin a$, where a is the angle which R makes with the given line. Hence the extremities lie on the line $y = G$, $z = Rx \sin a$.)

37. Prove that the axes of principal moment at points along any right line whatever trace out a hyperbolic paraboloid.

(With the same axes as in last example, the surface has for equation $xy = \dfrac{G}{R \sin a} . z$.)

38. Find the condition that a given right line should intersect Poinsot's Axis.

Ans. If the equations of the line are $x = mz + p$, $y = nz + q$, the required condition is

$$R\,[mL + nM + N + q\,(X - mZ) - p\,(Y - nZ)] = K(mX + nY + Z),$$

where X is used for ΣX, &c.

(It will be found that the equations of Poinsot's Axis can be put into the forms

$$x = \frac{X}{Z}z + \frac{KY - MR}{RZ}, \quad y = \frac{Y}{Z}z - \frac{KX - LR}{RZ},$$

the origin being anywhere.)

39. A given system of forces is to be reduced to two inclined at the angle a; prove that the shortest distance between their lines of action cannot be less than $\dfrac{2G}{R} \cot \dfrac{a}{2}$. (Wolstenholme's *Book of Math. Prob.*, p. 387, second ed.)

40. Given any system of forces, find the point on a given right line at which the axis of principal moment is least inclined to the line.

Ans. The foot of the shortest distance between Poinsot's Axis and the given line.

[Most easily seen by spherical projection. Let O be any point on the given right line; round O as centre describe a sphere of any radius; let the given right line, OL, cut the sphere in L; let the resultant of translation at O and the axis, G, of principal moment at O cut the sphere in R and G, respectively. Draw the great circle arcs LR, LG. Then at any distance, x, along OL from O, the axis of principal moment is the resultant of an axis equal and parallel to G, and an axis Rx perpendicular to the plane LOR. Let a line, OQ, drawn through O parallel to this latter meet the sphere in Q. Draw the great circle arc, QG, meeting LR in H, suppose. Then the resultant of G and Rx is an axis somewhere in the plane QG; but, Q being the pole of LR, the arc LH is perpendicular to QG, and therefore is the least arc that can be drawn from L to QG. Hence when Rx and G give a resultant along OH, the axis of principal moment is least inclined to OL. Poinsot's centre being always sought on a line perpendicular to R and to the axis of principal moment at any point, the rest follows.]

41. The first case considered in example 8 is, equally with the second, a geometrico-statical problem. Solve it without any mention of force.

(Express the condition that the vertical through the extremity A of AB is intersected by a line inclined at the angle λ to the normal at B, this line lying in the plane of the normal and a perpendicular to OB.)

42. AQB is any unclosed curve in space, A and B its extremities, and Q any variable point on the curve; P is any fixed point in space, $PQ = r$, $ds =$ element of length at Q, $\theta =$ angle between PQ and ds. If each element ds is acted upon by a force $k\dfrac{\sin\theta \,.\, ds}{r^2}$ perpendicular to the plane of PQ and ds, k being a constant, find the resultant force and couple of this force system.

Ans. Let (a, β, γ), (x_1, y_1, z_1), (x_2, y_2, z_2) be the co-ordinates of P, A, B; let $PA = r_1$, $PB = r_2$, L, M, N the component moments round axes through P; and let

$$\int\frac{dx}{r} = F, \int\frac{dy}{r} = G, \int\frac{dz}{r} = H.$$

Then

$$X = k\left(\frac{dG}{d\gamma} - \frac{dH}{d\beta}\right), \text{ with similar values of } Y, Z;$$

$$L = k\left(\frac{a - x_2}{r_2} - \frac{a - x_1}{r_1}\right), \text{ with similar values of } M, N.$$

The axis of resultant moment is the external bisector of BPA, and $= k \sin \tfrac{1}{2} BPA$.

Hence if the curve is closed, the force system has a single resultant, which passes through P.

CHAPTER XIV.

ASTATIC EQUILIBRIUM.

242.] **Definition.** When a body is in equilibrium under the action of forces applied at given points in the body, with fixed magnitudes, and directions fixed in space, it will, under certain conditions to be satisfied by the forces, continue in equilibrium when it is displaced in any manner. Each force, then, while continuing to act at the same point in the body and retaining its *direction* in space, alters, of course, its actual line of action.

Equilibrium which thus subsists in all positions of the body is called *Astatic Equilibrium.* Some results connected with the astatic equilibrium of coplanar forces for displacements in their plane have been given in Chap. V, Vol. I.

The astatic conditions of a rigid body acted on by any forces have been investigated at great length in Moigno's *Statique* (Dixième Leçon), and also in a memoir by M. Darboux (Bordeaux, 1877). Treated by the ordinary Cartesian method, the theory of Astatic Equilibrium is somewhat cumbrous. For this reason the method here adopted is of a different nature, viz. one which involves the elementary processes of Quaternions, with which the student is assumed to be familiar. This method is one which possesses great advantages in every respect over the Cartesian method, and, in particular, has a power of suggesting results which the older method would fail to suggest, and could demonstrate only by long and painful analysis.

The equilibrium of a rigid body is preserved by two conditions—namely, the vanishing of the Resultant of Translation of the acting forces, and the vanishing of the Principal Moment calculated for any origin.

The general displacement of a rigid body can always be produced by a motion of translation, together with a motion of rotation about some axis, and such is the displacement which

must be contemplated in discussing the Astatic Conditions. It is obvious, however, that a motion of translation can alter nothing in the equilibrium of the forces, so that we may confine our attention to a displacement produced by the rotation of the body about an axis.

243.] **Alteration of a Vector by rotation.** Suppose that a body is rotated about an axis drawn through an origin O in the direction of a unit vector σ, what does any vector, a, drawn from O to a fixed point in the body, become?

Let a sphere, described with O as centre, meet the directions of σ, a, and $Va\sigma$ in points represented in the figure by these letters. Suppose that after rotation the vector a takes the direction denoted by a'; let the angle between Oa and $O\sigma$, be θ, and let the body rotate through an angle ψ. Assume

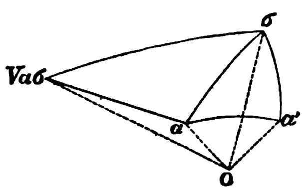

Fig. 251.

$$a' = xa + y\sigma + zVa\sigma;$$

$$\therefore\ Saa' = xa^2 + ySa\sigma,$$

$$S\sigma a' = xSa\sigma + y\sigma^2,$$

$$Sa'Va\sigma = z(Va\sigma)^2 = -za^2\sin^2\theta.$$

Now $Saa' = -T^2a(\cos^2\theta + \sin^2\theta\cos\psi)$, since $Ta' = Ta$;

also $S\sigma a' = -Ta.\cos\theta$, $Sa'Va\sigma = T^2a.\sin^2\theta\sin\psi$.

Hence we have

$$x = \cos\psi;\ y = (1-\cos\psi)Ta.\cos\theta = -(1-\cos\psi)Sa\sigma;\ z = -\sin\psi;$$

$$\therefore\ a' = a\cos\psi - (1-\cos\psi)\sigma Sa\sigma - \sin\psi\, Va\sigma, \qquad (A)$$

which determines the vector into which a is transformed by rotation.

244.] **General Astatic Conditions.** Let $a_1, a_2, a_3, \ldots$ be the vectors from a fixed origin to the points in the body at which forces represented in magnitudes and directions by the vectors $\varpi_1, \varpi_2, \varpi_3, \ldots$ are applied. Then the resultant of translation is $\varpi_1 + \varpi_2 + \varpi_3 + \ldots$, or $\Sigma\varpi$; and if G is the vector axis of the resultant couple for the assumed origin,

$$G = V(a_1\varpi_1 + a_2\varpi_2 + \ldots). \qquad (\xi)$$

The conditions of equilibrium are, then,

$$\Sigma\varpi = 0, \qquad (B)$$

$$\Sigma Va\varpi = 0. \qquad (\eta)$$

Now by the rotation of the body about any axis through O, each vector, $a_1, a_2, \ldots$ becomes altered according to (A); and substituting the altered value for each in (ξ), the couple, G', is given by the equation

$$G' = \cos\psi\,\Sigma\,Va\varpi + 2\sin^2\frac{\psi}{2}\,\Sigma\,(V\varpi\sigma Sa\sigma) + \sin\psi\,\Sigma\,(V.\varpi Va\sigma). \quad (\zeta)$$

$\Sigma\,\varpi$, of course, remains unchanged. This expression must vanish independently of ϕ and σ; hence we must have

$$\Sigma\,(V\varpi\sigma Sa\sigma) \equiv 0, \quad (1)$$

$$\Sigma(V.\varpi Va\sigma) \equiv 0. \quad (2)$$

Equation (2) at full length is

$$\sigma(Sa_1\varpi_1 + Sa_2\varpi_2 + \ldots) = a_1 S\varpi_1\sigma + a_2 S\varpi_2\sigma + \ldots,$$

or

$$h\sigma = \phi\sigma, \quad (3)$$

where $\phi\sigma$ is the linear vector function $\Sigma\,aS\varpi\sigma$ at the right side, and h is the scalar multiplier of σ at the left.

Now $\phi'\sigma$, the conjugate function (Tait's *Quaternions*, Chap. V.), is $\varpi_1 Sa_1\sigma + \varpi_2 Sa_2\sigma + \ldots$, or $\Sigma\,\varpi Sa\sigma$; and we can now show that ϕ and ϕ' are identical—i.e. ϕ is self-conjugate—in virtue of (η). For, operating on (η) with $V.\rho$, we have

$$\varpi_1 Sa_1\rho + \varpi_2 Sa_2\rho + \ldots = a_1 S\varpi_1\rho + a_2 S\varpi_2\rho + \ldots,$$

i.e.

$$\phi'\rho = \phi\rho,$$

where ρ is any vector. *Hence when* $\Sigma\,Va\varpi = 0$, *the function* $\phi\rho$, *or* $\Sigma\,aS\varpi\rho$, *is self-conjugate.*

Equation (1) is merely

$$V\phi\sigma.\sigma = 0. \quad (4)$$

Now, by supposition, (3) and (4) must be satisfied by all values of σ. This requires $h = 0$, and $\phi\sigma \equiv 0$. The quantity h is the Virial of the given forces.

We can show that both the conditions $h = 0$ and $\Sigma\,Va\varpi = 0$ are included in the identical vanishing of $\phi\sigma$. For, let $\sigma \equiv xi + yj + zk$, where (i, j, k) are a rectangular system of unit vectors; then $\phi\sigma = 0$ must be satisfied by all values of x, y, z. Hence we have

$$\Sigma\,aSi\varpi = 0,\ \Sigma\,aSj\varpi = 0,\ \ \Sigma\,aSk\varpi = 0. \quad (\theta)$$

Multiplying these by i, j, k, respectively, and adding, we have

$$\Sigma\,a\varpi = 0,$$

which, of course, gives $\Sigma\,Sa\varpi = 0$ and $\Sigma\,Va\varpi = 0$.

Hence the two equations

$$\Sigma\varpi = 0, \tag{λ}$$

$$\Sigma a S\varpi\sigma \equiv 0, \text{ or } \phi\sigma \equiv 0, \tag{μ}$$

include all the necessary astatic conditions.

The vanishing of the linear vector function $\phi\sigma$ for *all* values of σ is guaranteed by its vanishing when the unit vectors i, j, k are substituted for σ; so that we may take (λ) and the three equations (θ) as completely expressing the astatic conditions.

COR. If a body in equilibrium under the action of any forces is displaced round any axis, σ, the moment of the forces in the new position is given by the equation

$$G' = -2\sin^2\frac{\psi}{2}\, V\sigma\phi\sigma + \sin\psi\,(h\sigma - \phi\sigma). \tag{ξ}$$

245.] **Equation of Poinsot's Axis.** For any assumed origin, O' (Fig. 236, p. 16), let $G\,(=O'G)$ be the vector axis of the principal couple, and let Π (or $\Sigma\varpi$) be the Resultant of Translation.

Then the distance $O'O$ is $\dfrac{TG}{T\Pi}\sin\phi$, where ϕ is the angle $RO'G$;

therefore the vector $O'O$ is $\dfrac{TG}{T\Pi}\sin\phi\, UV\Pi G$, which is $\dfrac{V\Pi G}{T^2\Pi}$;

hence the equation of the line OK (Poinsot's Axis) is

$$\rho = x\Pi + \frac{V\Pi G}{T^2\Pi}.$$

246.] **Vector to Centre of Parallel Forces.** If at the extremities of two vectors, a_1, a_2, two parallel forces of magnitudes P_1 and P_2 act, the vector to their centre is

$$\frac{P_1 a_1 + P_2 a_2}{P_1 + P_2};$$

and for any number of parallel forces of magnitudes, $T\varpi_1$, $T\varpi_2$,... the vector to their centre is

$$\frac{\Sigma a T\varpi}{\Sigma T\varpi}.$$

247.] **One Force.** *To find the conditions that a system of forces should be astatically equivalent to a single force.*

Suppose that a force Π_1, acting at the extremity of a vector A_1 drawn to a point fixed in the body, astatically equilibrates the given system.

Denote $\Sigma\varpi$ by Π, and $\Sigma a S i\varpi$, $\Sigma a S k\varpi$, $\Sigma a S k\varpi$ by I, J, K respectively. Then we have, by (θ) and (λ),

$$A_1 S i\Pi = I, \quad A_1 S j\Pi = J, \quad A_1 S k\Pi = K,$$

which give the conditions

$$\frac{I}{S i\Pi} = \frac{J}{S j\Pi} = \frac{K}{S k\Pi}.$$

These conditions are always satisfied when the given forces form a parallel system, since $\frac{I}{S i\Pi}$ obviously becomes $\frac{\Sigma a T\varpi}{T\Pi}$, which is also the common value of $\frac{J}{S j\Pi}$ and $\frac{K}{S k\Pi}$, and the vector to the centre of parallel forces.

In general, the vectors $\frac{I}{S i\Pi}$, $\frac{J}{S j\Pi}$, $\frac{K}{S k\Pi}$ are those drawn from the origin to the points of application, P_i, P_j, P_k, of three systems of parallel forces which are obtained by resolving each force into three rectangular components in the directions i, j, k. Denote these vectors, respectively, by a_i, a_j, a_k.

In the present case, then, *the points* P_i, P_j, P_k, *must coincide.*

248.] **The Centres of three component parallel systems.** It is essential to have a clear idea of the points whose vectors from any origin we have just denoted by a_i, a_j, a_k. Resolve, in any position of the body, the force ϖ_1 at the point a_1 into three components parallel to any three rectangular fixed-space directions, i, j, k; and similarly resolve all the other forces. In all positions of the body these components of ϖ_1 are each absolutely constant in magnitude and in direction, since we assume each force of the system to retain the same direction in fixed space. And since these forces act each at one and the same point (or particle) of the body in all positions of the body, it follows that the set of components parallel to i, for example, constitute a system exactly the same as the weights of a number of particles forming a rigid body, or rigidly connected together even if they do not form a continuous solid. It is quite clear, then, that *their centre* (the point P_i) *must be an invariable point in the body* (or in rigid connection with it) however the body may be displaced, whether by translation, or by rotation, or both.

Similarly the other centres (of components parallel to j, and of those parallel to k), P_j, P_k, are both fixed points in the body.

So long, then, as the chosen fixed-space directions are the same, the points P_i, P_j, P_k are the same; but when all the forces, ϖ_1, ϖ_2, are resolved into a parallel set in any other direction, the centre of this parallel system will be a different point in the body.

We shall now prove that, *whatever this new direction may be, the centre of components parallel to it lies in the plane of the points P_i, P_j, P_k, corresponding to any three chosen directions.*

Let the new direction be that of the unit vector $xi+yj+zk$, where x, y, z are its direction-cosines. Let ρ be the vector to the new centre. Then

$$\rho = \frac{a_1 S(xi+yj+zk)\varpi_1 + \ldots}{S(xi+yj+zk)\varpi_1 + \ldots}$$
$$= \frac{x\Sigma aSi\varpi + y\Sigma aSj\varpi + z\Sigma aSk\varpi}{x\Sigma Si\varpi + y\Sigma Sj\varpi + z\Sigma Sk\varpi}$$

denote $\Sigma Si\varpi$, or $Si\Pi$, by $-a$; $\Sigma Sj\varpi$ and $\Sigma Sk\varpi$ by $-b$ and $-c$, respectively, so that a, b, c are the components of the resultant of translation in the directions of i, j, k. Then

$$a_i = -\frac{I}{a}, \text{ \&c.};$$

hence
$$\rho = \frac{ax a_i + by a_j + cz a_k}{ax + by + cz},$$

which proves that the extremity of ρ lies in the plane of the points P_i, P_j, P_k, since (Tait's *Quaternions*, § 30) the extremities of four vectors ρ_1, ρ_2, ρ_3, ρ_4 drawn from the origin will be coplanar if

$$p_1\rho_1 + p_2\rho_2 + p_3\rho_3 + p_4\rho_4 = 0,$$

and
$$p_1 + p_2 + p_3 + p_4 = 0.$$

This plane, which is therefore absolutely fixed in the body, we shall call the *plane of centres*. It depends simply on the magnitudes and directions of the forces and the points at which they are applied.

A special centre of parallel forces deserves to be noticed. It is that obtained by resolving all the forces parallel to their resultant of translation, Π. The vector to this point, which we shall call *the principal centre of the plane of centres*, is

$$\frac{a^2 a_i + b^2 a_j + c^2 a_k}{a^2 + b^2 + c^2}.$$

249.] **Two Forces.** *To find the conditions that a system of forces should be astatically equivalent to two forces.*

Let the two forces be Π_1 and Π_2 at points whose vectors are A_1 and A_2.

Hence
$$\Pi_1+\Pi_2+\Pi = 0, \qquad (1)$$
$$A_1 Si\Pi_1+A_2 Si\Pi_2 = -I, \qquad (2)$$
$$A_1 Sj\Pi_1+A_2 Sj\Pi_2 = -J, \qquad (3)$$
$$A_1 Sk\Pi_1+A_2 Sk\Pi_2 = -K. \qquad (4)$$

The last three equations show that I, J, K must be coplanar with A_1 and A_2. Hence $SIJK = 0$. (α)

But there is another condition to be satisfied; for if $lI+mJ+nK \equiv 0$, when l, m, n are given scalars, by multiplying (2), (3), and (4) by l, m, n, adding, and equating to zero the coefficients of A_1 and A_2 (for no such equation as $pA_1+qA_2 = 0$ is possible, p and q being scalars) we have
$$lSi\Pi_1+mSj\Pi_1+nSk\Pi_1 = 0, \qquad (5)$$
$$lSi\Pi_2+mSj\Pi_2+nSk\Pi_2 = 0, \qquad (6)$$
which, from (1), give by addition this second condition,
$$lSi\Pi+mSj\Pi+nSk\Pi = 0. \qquad (\beta)$$

Now it is easy to see that (α) and (β) signify that *the three centres, P_i, P_j, P_k of parallel forces are in one right line.* For we have assumed $lSi\Pi\,.\,a_i+mSj\Pi\,.\,a_j+nSk\Pi\,.\,a_k \equiv 0$; and we know that the extremities of three vectors will be collinear (Tait's *Quaternions*, § 30) if the sum of their multipliers in this equation is zero—which is asserted in (β). We shall call this the *Line of Centres.* Moreover, *the points of application of Π_1 and Π_2 must also lie on this line.* For (2) can be written
$$A_1 Si\Pi_1+A_2 Si\Pi_2+Si\Pi\,.\,a_i = 0;$$
and since, by (1), $Si\Pi_1+Si\Pi_2+Si\Pi = 0$, this equation shows that the points at which Π_1 and Π_2 act lie on a right line through P_i; similarly, they lie on a right line through P_j; therefore, &c.

Again, the forces Π_1 and Π_2 are obviously known in magnitudes and directions from equations (1) ... (4) as soon as their points of application are assumed; *and in all cases they are both parallel to a given plane*; for (5) gives
$$S(li+mj+nk)\Pi_1 = 0,$$

and a similar equation in Π_2, which show that Π_1 and Π_2 are both perpendicular to the vector $li+mj+nk$, as is also the Resultant of Translation of the given system, by (β).

Now $I=-a\alpha_i$, $J=-b\alpha_j$, $K=-c\alpha_k$.

Assume the vectors A_1 and A_2. Let

$$A_1 = \alpha_i + x(\alpha_i - \alpha_j); \quad A_2 = \alpha_i + y(\alpha_i - \alpha_j).$$

Then equations (2), (3), (4) give

$$Si\Pi_1 = -\frac{ay}{x-y}, \quad Sj\Pi_1 = -\frac{b(y-1)}{x-y}, \quad Sk\Pi_1 = -\frac{cny+bm}{n(x-y)},$$

$$Si\Pi_2 = \frac{ax}{x-y}, \quad Sj\Pi_2 = \frac{b(x-1)}{x-y}, \quad Sk\Pi_2 = \frac{cnx+bm}{n(x-y)},$$

which, of course, determine Π_1 and Π_2.

If we take Π_1 and Π_2 at right angles to each other, we have

$$Si\Pi_1 Si\Pi_2 + Sj\Pi_1 Sj\Pi_2 + Sk\Pi_1 Sk\Pi_2 = 0,$$

which gives

$$n^2(a^2+b^2+c^2)xy + bn(cm-bn)(x+y) + b^2(m^2+n^2) = 0.$$

Now it is obvious that the distance of the extremity of A_1 from P_i is $x.P_iP_j$, and the distance of the extremity of A_2 from P_i is $y.P_iP_j$; and this last equation shows that these distances (ξ_1 and ξ_2) are connected by an equation of the form

$$\xi_1\xi_2 + p(\xi_1+\xi_2) + q = 0,$$

and that, therefore, the points at which Π_1 and Π_2 are applied are conjugate points of an involution system on the line of centres $P_iP_jP_k$.

The distance of the centre of this involution system from P_i is

$$\frac{b(bn-cm)}{n(a^2+b^2+c^2)}P_iP_j,$$

so that the vector, Ω, to this centre is

$$\alpha_i - \frac{b(bn-cm)}{n(a^2+b^2+c^2)}(\alpha_i - \alpha_j),$$

or

$$\Omega = \frac{a^2\alpha_i + b^2\alpha_j + c^2\alpha_k}{a^2+b^2+c^2}, \qquad (7)$$

which is (Art. 248) the vector to the centre of a system of parallel forces whose common direction is that of the Resultant of Translation of the given system—*i.e.* the principal centre.

Hence, *when a system of forces can be astatically equilibrated by two rectangular forces, the points of application of these latter must*

lie on the line of centres, and be conjugate points of an involution system whose centre is the centre of a system parallel to the Resultant of Translation.

It remains to be proved that the line of centres is unique. For this purpose we shall show that, if the given forces are each resolved in the direction of any vector, the centre of this system will lie on the line $P_i P_j P_k$.

If i' is the assumed vector, let $i' = xi + yj + zk$; then the vector to the centre of forces parallel to i' is

$$\frac{a_1 Si'\varpi_1 + a_2 Si'\varpi_2 + \ldots}{Si'\varpi_1 + Si'\varpi_2 + \ldots}, \quad \text{or} \quad -\frac{xI + yJ + zK}{ax + by + cz}.$$

If I' is this vector, we have

$$(ax + by + cz) I' - ax \,.\, a_i - by \,.\, a_j - cz \,.\, a_k = 0\,;$$

and since the sum of the multipliers of I', a_i, a_j, a_k is zero, and the extremities of the latter three are collinear, the extremity of I' must lie on the line of centres.

The relations between the vectors A_1 and A_2 and the forces Π_1 and Π_2 will, perhaps, be better seen if we use θ for the unit vector in the direction of the line of centres.

Clearly, then, we may put, when Π_1 and Π_2 are rectangular,

$$\left.\begin{aligned} A_1 &= \Omega + x\theta, \\ A_2 &= \Omega - \frac{h^2}{x}\theta, \end{aligned}\right\} \tag{8}$$

$$I = -a\,\Omega - s\theta, \quad J = -b\,\Omega - s'\theta, \quad K = -c\,\Omega - s''\theta,$$

where h^2, s, s', s'' are all given constants, and x any variable scalar.

Equations (2), (3), (4) then determine Π_1 and Π_2, and give

$$\Pi_1 = -\frac{h^2\Pi + x\Theta}{h^2 + x^2}, \quad \Pi_2 = -\frac{x^2\Pi - x\Theta}{h^2 + x^2}, \tag{9}$$

where $\Theta = si + s'j + s''k$, so that Θ is a given vector parallel to the plane to which Π, Π_1, and Π_2 are all parallel.

It is easy to prove in different ways that the vectors Π and Θ are perpendicular to each other. One simple method consists in the fact that $S\Pi_1\Pi_2$ must be zero independently of the value of x. It may be seen otherwise thus:

$$aI + bJ + cK = -(a^2 + b^2 + c^2)\,\Omega - (as + bs' + cs'')\,\theta\,;$$

and this gives, by (7), $as + bs' + cs'' = 0$, which is the condition that Θ and Π should be perpendicular.

The condition $S\Pi_1\Pi_2 = 0$ further gives

$$h^2\Pi^2 - \Theta^2 = 0,$$

or, if R be $T\Pi$, $$T\Theta = hR.$$

Corresponding to different values of x we can represent the magnitudes and directions of Π_1 and Π_2. Since they equilibrate Π, they must, of course, be represented by the two sides of a right-angled triangle described, in a plane fixed in space, on Π as hypotenuse.

Draw from any origin, O, two right lines, OA and OB, in the directions of Π and Θ respectively. Then, if the components of Π_1 along OA and OB are X_1 and Y_1, we have

$$X_1 = \frac{h^2R}{h^2+x^2}, \quad Y_1 = \frac{hRx}{h^2+x^2};$$

therefore, if $\Pi_1 = \overline{OP_1}$, we have

$$\tan P_1OA = \frac{x}{h},$$

which determines the direction of Π_1.

Again, we can prove the following theorem:—

As their points of application along the line of centres vary, the two rectangular forces which astatically equilibrate the system trace out a hyperbolic paraboloid.

For if ρ is the vector to any point on the surface traced out by Π_1,

$$\rho = \Omega + x\theta + y\frac{h^2\Pi + x\Theta}{h^2+x^2},$$

where x and y are any variable scalars; or, if the centre of the line of centres is taken as origin of vectors,

$$\rho = x\theta + \frac{h^2y}{h^2+x^2}\Pi + \frac{xy}{h^2+x^2}\Theta.$$

The Cartesian equation of this surface, referred to the line of centres as axis of x, and those of Π and Θ as axes of y and z, respectively, is $$xy = hz.$$

Since the two forces Π_1 and Π_2 are equivalent to the given system, the discussion of these forces may replace the discussion of the system. For example, to find Poinsot's Axis, we have, from (8) and (9),

$$VA_1\Pi_1 + VA_2\Pi_2 = -V(\Omega\Pi + \theta\Theta).$$

Hence, from Art. 245, Poinsot's Axis is

$$\rho = x\Pi + \Omega - \frac{1}{R^2} V\theta\Theta\Pi;$$

or, if the centre of the line of centres is taken as origin of vectors,

$$\rho = x\Pi - \frac{1}{R^2} V\theta\Theta\Pi,$$

so that Poinsot's Axis will coincide with the Resultant of Translation at the centre of the line of centres if $V\theta\Theta\Pi = 0$, or, in other words, if the line of centres is perpendicular to the plane (of Θ and Π) to which the forces Π_1, Π_2, and Π are parallel. The body may, of course, be turned round so that this perpendicularity occurs.

A system of forces can be astatically equilibrated by two forces when all the forces of the system are parallel to one plane.

For, let the unit vector perpendicular to this plane be taken as k. Then $Sk\varpi_1 = 0$, $Sk\varpi_2 = 0$, ...; therefore $K = 0$, and the equations reduce to

$$\Pi_1 + \Pi_2 + \Pi = 0, \tag{10}$$

$$A_1 Si\Pi_1 + A_2 Si\Pi_2 + Si\Pi . a_i = 0, \tag{11}$$

$$A_1 Sj\Pi_1 + A_2 Sj\Pi_2 + Sj\Pi . a_j = 0. \tag{12}$$

The second shows that the extremities of the vectors A_1, A_2, and a_i are collinear, since by (10) the sum of their multipliers is zero. Similarly (12) shows that the extremities of A_1, A_2, and a_j are collinear; therefore the points of application of Π_1 and Π_2 lie on the line of centres.

A system of forces may be astatic for displacements about a particular axis without being astatic for displacements about other axes.

If σ is the unit vector in the direction of the axis of displacement, the conditions of continuous equilibrium are

$$\Sigma\varpi = 0,$$

$$\cos\psi\Sigma Va\varpi + 2\sin^2\frac{\psi}{2}\Sigma(V\varpi\sigma Sa\sigma) + \sin\psi\Sigma V.\varpi Va\sigma = 0;$$

the latter, holding for all values of ψ, gives

$$\Sigma Va\varpi = 0, \quad \Sigma(V\varpi\sigma Sa\sigma) = 0, \quad \Sigma V.\varpi Va\sigma = 0.$$

Consider the case in which all the forces lie in one plane, and let the axis of displacement be any axis perpendicular to this plane. Also take the origin of vectors in the plane.

Then these conditions, since

$$Sa_1\sigma = Sa_2\sigma = \ldots\ldots = 0, \quad S\varpi_1\sigma = S\varpi_2\sigma = \ldots\ldots = 0,$$

become $\quad \Sigma\varpi = 0, \quad \Sigma Va\varpi = 0, \quad \Sigma Sa\varpi = 0.$

It has been shown (Vol. I, pp. 130, 131) how these conditions are otherwise deduced, it being observed that $\Sigma Sa\varpi$ is obviously the Virial of the forces.

Such a system of forces as this can always be equilibrated (for the displacements considered) by a single force. For, let the force be Π_1 at the extremity of a vector A_1. Then the conditions are

$$\Pi_1 + \Pi = 0, \quad A_1\Pi_1 + \Sigma a\varpi = 0;$$

$$\therefore \quad \Pi_1 = -\Pi, \quad A_1 = \frac{\Sigma a\varpi}{\Pi},$$

the expression for A_1 being evidently a vector, since a_1, $\varpi_1, \ldots$ and Π are all coplanar.

We thus arrive at the 'Centre' of the system, and it is very easy to prove that (see Vol. I, p. 130) this point is characterised by the vanishing of the sum of the moments and of the Virial of the forces about it.

250.] **Three Forces.** *To investigate the astatical equivalence of a system of forces to three forces.*

Let the forces be Π_1, Π_2, and Π_3 at the extremities of vectors A_1, A_2, A_3.

Then

$$\Pi_1 + \Pi_2 + \Pi_3 = -\Pi, \qquad (1)$$

$$A_1 Si\Pi_1 + A_2 Si\Pi_2 + A_3 Si\Pi_3 = -I, \qquad (2)$$

$$A_1 Sj\Pi_1 + A_2 Sj\Pi_2 + A_3 Sj\Pi_3 = -J, \qquad (3)$$

$$A_1 Sk\Pi_1 + A_2 Sk\Pi_2 + A_3 Sk\Pi_3 = -K. \qquad (4)$$

Now, $Si\Pi$ being still denoted by $-a$, &c., (2) can be written

$$A_1 Si\Pi_1 + A_2 Si\Pi_2 + A_3 Si\Pi_3 - a\,a_i = 0,$$

while (1) gives $\quad Si\Pi_1 + Si\Pi_2 + Si\Pi_3 - a = 0.$

Hence (Art. 248) the extremities of A_1, A_2, and A_3 lie in a plane containing the point P_i, which has been shown to be the centre of a system of parallel forces obtained by resolving each force parallel to i.

The remaining equations show in the same way that these extremities lie in a plane containing P_j and P_k.

Hence *the points of application of the three forces which astatically equilibrate the system lie in the plane of the three centres*

I_i, P_j, P_k; that is, the plane containing the extremities of the vectors a_i, a_j, a_k. This plane we have called the *Plane of Centres.*

The astatic reduction to three forces is therefore always possible.

The equation of the plane of centres is, of course,

$$S(Va_j a_k + Va_k a_i + Va_i a_j)\,\rho = -Sa_i a_j a_k. \tag{5}$$

Operating on (2), (3), (4) with $S\,.\,VA_2A_3$, we have

$$Si\,\Pi_1\,.\,SA_1A_2A_3 = -SIVA_2A_3,$$
$$Sj\,\Pi_1\,.\,SA_1A_2A_3 = -SJVA_2A_3,$$
$$Sk\,\Pi_1\,.\,SA_1A_2A_3 = -SKVA_2A_3;$$

which give

$$\Pi_1 = \frac{1}{SA_1A_2A_3}(iSIVA_2A_3 + jSJVA_2A_3 + kSKVA_2A_3);$$

and in the same way we obtain Π_2 and Π_3, so that, when their points of application are assumed, the forces Π_1, Π_2, Π_3, are thus completely known.

It may be observed that the origin of vectors can always be so chosen that the vectors a_i, a_j, a_k shall be a rectangular system, the vectors i, j, k, remaining the same. For the points P_i, P_j, P_k depend only on the vectors i, j, k, and not on the origin of vectors; and, given three points P_i, P_j, P_k, two other points, Q and Q', can be found such that the lines QP_i, QP_j, and QP_k are a rectangular system, as are also the lines $Q'P_i$, $Q'P_j$, and $Q'P_k$.

The points Q and Q' are the points common to three spheres described on the sides of the triangle $P_i P_j P_k$ as diameters; they are equidistant from this plane on opposite sides, and lie on the perpendicular to it drawn through the orthocentre of the triangle.

We may suppose either of these points taken as origin of vectors, and treat I, J, K as a rectangular system.

For simplicity, denoting the vectors $VA_2A_3, VA_3A_1, VA_1A_2$ by $\epsilon_1, \epsilon_2, \epsilon_3$, we have

$$\Pi_1 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}\,(iSI\epsilon_1 + jSJ\epsilon_1 + kSK\epsilon_1), \tag{6}$$

$$\Pi_2 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}\,(iSI\epsilon_2 + jSJ\epsilon_2 + kSK\epsilon_2), \tag{7}$$

$$\Pi_3 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}\,(iSI\epsilon_3 + jSJ\epsilon_3 + kSK\epsilon_3). \tag{8}$$

It may be noticed that, for different systems of rectangular vectors i', j', k', the vectors to the centres of the corresponding systems of parallel forces are in the directions of conjugate diameters of a certain ellipsoid. For if

$$i' = xi + yj + zk, \quad j' = x'i + y'j + z'k, \quad k' = x''i + y''j + z''k,$$

it is obvious that I', J', K' are

$$xI + yJ + zK, \quad x'I + y'J + z'K, \quad x''I + y''J + z''K$$

respectively. And since $xx' + yy' + zz' = 0$, &c., it follows that

$$\frac{SII'SIJ'}{I^4} + \frac{SJI'SJJ'}{J^4} + \frac{SKI'SKJ'}{K^4} = 0, \text{\&c.},$$

showing that the vectors to the new centres are in the directions of conjugate diameters of the ellipsoid

$$\frac{S^2 I\rho}{I^4} + \frac{S^2 J\rho}{J^4} + \frac{S^2 K\rho}{K^4} = 1.$$

The origin of vectors is now a fixed point in the body, and the points P_i, P_j, P_k, are, of course, fixed points in the body; and, by the nature of astatic equilibrium, we may consider the body as being placed in any position whatever. Suppose, then, that it is so turned round the origin of vectors that I, J, and K coincide in directions with i, j, k respectively. This may be regarded as a sort of *initial position* of the body. Let the tensors of I, J, and K be t_1, t_2, and t_3 respectively. Then

$$iSI\epsilon_1 + jSJ\epsilon_1 + kSK\epsilon_1$$

becomes $t_1 iSi\epsilon_1 + t_2 jSj\epsilon_1 + t_3 kSk\epsilon_1$, that is, a self-conjugate linear vector function of ϵ_1. Denote it by $\phi\epsilon_1$. Then (6), (7), and (8) give

$$\Pi_1 = \frac{\phi\epsilon_1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}, \quad \Pi_2 = \frac{\phi\epsilon_2}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}, \quad \Pi_3 = \frac{\phi\epsilon_3}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}}. \quad (9)$$

But in the ellipsoid $S\rho\phi\rho = 1$, the normal at the extremity of a vector a is parallel to ϕa (Tait's *Quaternions*, Chap. VIII).

Hence we have the following theorem:—

The body being placed in the initial position, the forces applied at the extremities of any three vectors, A_1, A_2, A_3, drawn from the origin to points in the plane of centres, are in the directions of normals to the ellipsoid $S\rho\phi\rho = 1$ at the points where its surface is intersected by the vectors $VA_1A_2, VA_2A_3, VA_3A_1$.

If we wish the forces Π_1, Π_2, Π_3 to be a mutually rectangular set, we must take $S\phi\epsilon_1\phi\epsilon_2 = 0$, &c., from which it is evident

that the directions of ϵ_1, ϵ_2, and ϵ_3 must be conjugate diameters of the ellipsoid $$S\rho\phi^2\rho = 1.$$

Of course the centres P_i, P_j, P_k may be taken as those at which the forces are applied. The forces will then be in the directions i, j, k respectively, and in the initial position they will meet in a point. Their magnitudes are obviously a, b, c—by the very nature of a resolution of the given system of forces into three component parallel sets (Art. 248).

What we have just called the *initial position* of the body requires a few words to render its nature more clear. Any three directions i, j, k, fixed in space, being chosen, and the centres P_i, P_j, P_k corresponding to them being taken, join these centres to one of the two points, Q, at which they subtend right angles in pairs. When the body has a position given to it at random the lines QP_i, QP_j, QP_k are not parallel to the corresponding directions i, j, k; but (Q being obviously a point in rigid connection with the body) if the body is moved so that these lines OP_i, &c., are parallel to i, j, k, *the system of forces will have in this position a single resultant.*

For it is obvious from elementary principles (see Art. 248) that the given force system could always be replaced by a force at P_i parallel to i and equal to the sum of the components of all forces parallel to i, together with forces at P_j and P_k equal to the sums of components parallel to j and k. Hence these forces would act in the lines QP_i, QP_j, QP_k if these lines were placed parallel to i, j, k, and there would, therefore, be a single resultant.

Since any directions may be chosen for i, j, k, there will be an infinite number of triads of centres P_i, P_j, P_k—all lying in a plane fixed in the body—and of lines QP_i, &c., and therefore of initial positions. Hence an initial position is simply any one in which the forces have a single resultant.

If the vectors a_1, a_2, ... are measured from a Q-point (i.e. a point at which P_i, P_j, P_k subtend right angles in pairs), the function $\Sigma aS\varpi\rho$ will be self-conjugate in the initial position in which the single resultant passes through this point (Art. 244). Generally, in all questions relating to the equilibrium of a given force system, we may substitute for the system *three* forces, viz. those which are astatically equivalent to the system at any triad of centres, P_i, P_j, P_k.

251.] **Grouping of the Centres round the Principal Centre.** Simplicity will be gained by taking the principal centre as origin of vectors, and supposing that all positions of the body are obtained by rotation round this point as a fixed point in space. Further, of the three fixed-space vectors i, j, k, we shall take i in the (invariable) direction of the Resultant of Translation of the forces. This latter assumption will make $b = 0$, $c = 0$, since the sums of the components of the forces perpendicular to the direction of their Resultant of Translation are necessarily zero. The constant a becomes then the magnitude of this resultant. Moreover, since Ω, the vector from the origin to the principal centre, is zero, we must have (Art. 249) $I = 0$.

The vectors J and K, of course, remain. Let C be the principal centre; then J and K are vectors at C in the plane of centres, and we can easily see that the fixed-space directions j and k can be so chosen as to make J and K at right angles to each other. For, let j' and k' be any two rectangular unit vectors in the plane of jk, let θ be the angle between j' and j. Then

$$j' = j\cos\theta + k\sin\theta\,;\quad k' = -j\sin\theta + k\cos\theta\,;$$

and if J' and K' are the vectors corresponding to j' and k',

$$J' = \Sigma a S j'\varpi,\quad K' = \Sigma a S k'\varpi\,;\ \text{therefore}$$

$$J' = J\cos\theta + K\sin\theta\,;\quad K' = -J\sin\theta + K\cos\theta\,;$$

so that J' and K' will be rectangular if

$$\tan 2\theta = \frac{2\,SJK}{K^2 - J^2}.$$

We shall suppose, then, that j and k have been chosen so that J and K are rectangular; and these latter we shall call the *principal vectors* in the plane of centres; the axes of i, j, k thus chosen may be called the principal fixed space axes.

Now let $li + mj + nk$ be a unit vector, ρ, in any direction at C, its direction-cosines being l, m, n. The vector to the centre, P, which corresponds to this vector is

$$\frac{mJ + nK}{la}. \tag{1}$$

If the vectors J and K are taken as axes of y and z, respectively, and if their tensors are t and t', the co-ordinates of P being (y, z), we have

$$y = \frac{mt}{la},\qquad z = \frac{nt'}{la}, \tag{2}$$

a being, as before said, the magnitude of the Resultant of Translation of the given forces.

If ρ is any vector in a given plane, $px+qy+rz = 0$, we have $pl+qm+rn = 0$, so that the locus of P is the right line

$$\frac{q}{t}y+\frac{r}{t'}z+\frac{p}{a} = 0. \tag{3}$$

If ρ is any vector making a constant angle, θ, with the direction of the Resultant of Translation, we have

$$m^2+n^2 = \sin^2\theta;$$

therefore the locus of P is the ellipse

$$\frac{y^2}{t^2}+\frac{z^2}{t'^2} = \frac{\tan^2\theta}{a^2}. \tag{4}$$

And if we wish to astatically equilibrate the given system by three forces at the three centres, P, P', P'', corresponding to any three rectangular directions,

$$li+mj+nk,\ l'i+m'j+n'k,\ l''i+m''j+n''k,$$

the magnitudes of these forces will be, respectively, la, $l'a$, $l''a$, and they will be parallel to the assumed directions $li+mj+nk$, &c. (p. 79).

It is to be noted that, with the origin of vectors as now chosen, the general values (6) &c., of Art. 250 for Π_1, Π_2, Π_3 all assume the indeterminate form $\frac{0}{0}$, since the vectors A_1, A_2, A_3 are coplanar, the operations $S.VA_2A_3$, &c., by which they were obtained from (2), (3), (4) being illusory. But the student will have no difficulty in obtaining the result just given (which is obvious from first principles) from the general equations referred to.

A given system of forces can always be replaced, with complete astatical equivalence, by three equal and mutually rectangular forces. For we have only to choose the directions $li+mj+nk$, &c., parallel to which all the forces are resolved, so that

$$l = l' = l'' = \frac{1}{\sqrt{3}},$$

that is, the three equivalent forces are all inclined to the direction of the Resultant of Translation at $\cos^{-1}\frac{1}{\sqrt{3}}$, and each force $= \frac{a}{\sqrt{3}}$.

The corresponding centres (all lying, of course, on the ellipse (4)) form a triangle of which the centre of the ellipse is the centroid ('centre of gravity').

252.] **Principal Moment at the Principal Centre.** Replace the system by three mutually rectangular forces at any three centres, P, P', P'', whose vectors are

$$\frac{mJ+nK}{la}, \quad \frac{m'J+n'K}{l'a}, \quad \frac{m''J+n''K}{l''a},$$

the principal fixed space axes being those of reference. The corresponding forces are $la\,(li+mj+nk)$, &c. Then G, the axis of principal moment at C (the principal centre), being $\Sigma V a\varpi$, we have

$$G = V(Jj+Kk). \tag{1}$$

Also, i being the direction of Π, the Resultant of Translation, the condition for a single resultant is $SGi = 0$, or

$$SJk = SKj, \tag{2}$$

that is, the body must be turned round C until the body-vectors J and K satisfy this condition.

The position of the body being any whatever, Poinsot's moment is

$$S(Jk-Kj), \tag{3}$$

since it is $S\Pi G$.

253.] **Wrench of Maximum Pitch.** Poinsot's moment is evidently a numerical maximum in some position or positions of the body, since when there is a single resultant it is zero. When it is a maximum its differential must vanish for any small change of position of the body—i.e. for a rotation through any small angle $\Delta\psi$ round any axis. Equating to zero the differential of (3),

$$S(k\,dJ-j\,dK) = 0. \tag{1}$$

Now, if σ is a unit vector in the direction of any axis by small rotation round which the new position of the body is attained, it is obvious either from first principles, or from (A) of Art. 243, that

$$dJ = -\Delta\psi\,.\,VJ\sigma, \text{ and } dK = -\Delta\psi\,.\,VK\sigma. \tag{2}$$

Hence substituting in (1) we have

$$S(kJ-jK)\sigma = 0, \tag{3}$$

which must hold for every value of σ. But this is impossible unless

$$V(kJ-jK) = 0, \tag{4}$$

which obviously requires j, k, J and K to be coplanar.

Hence the principal body-vectors must be placed in the plane of the principal fixed-space vectors j, k.

If, then, θ is the angle between J and j, Poinsot's moment $= SJk - SKj = (t+t')\sin\theta$, which is a maximum when $\theta = \frac{\pi}{2}$; i.e. the body must be turned round the principal centre so as to make J coincide with k and K with $-j$, and the maximum pitch is

$$t+t'.$$

When J coincides with $-k$ and K with j, the pitch is $-(t+t')$, an algebraic minimum.

254.] **Minding's Theorem.** *Whenever the body is so placed that the forces have a single resultant, its line of action intersects two conics fixed in the body.* The equation of Poinsot's axis is $\rho = x\Pi + \frac{V\Pi G}{T^2\Pi}$ (Art. 245). Substituting ai for Π, and the value of G from Art. 252, this becomes

$$\rho = axi + \frac{1}{a}(jSiJ + kSiK). \tag{1}$$

We shall now by means of (2) of Art. 252 express j and k in terms of i, J, and K. Putting Vij for k, this condition gives

$$SKj - S.JVij = 0,$$

$$\text{or} \quad SKj - SJij = 0, \text{ or } SKj - S.VJi.j = 0,$$

$$\text{or, finally,} \quad S(K+ViJ)j = 0,$$

i.e. j is perpendicular to $K+ViJ$; and it is also perpendicular to i, so that it must be proportional to $Vi(K+ViJ)$. Assume, then,

$$yj = V(iK + iViJ),$$

where y is a scalar. This is the same as

$$yj = ViK - J - iSiJ, \tag{2}$$

since (Tait's *Quaternions*, Art. 90) $V.aV\beta\gamma = \gamma Sa\beta - \beta Sa\gamma$.

Now by squaring (2), denoting the tensors of J and K, as before, by t and t', and the unit vector perpendicular to J and K (i.e. to the plane of centres), by ν, we have

$$y^2 = t^2\sin^2\theta + t'^2\sin^2\theta' - 2tt'Si\nu, \tag{3}$$

where θ and θ' are the angles between i and J, and between i and K, respectively.

In exactly the same way, if in (2) of Art. 252 we put Vki for j, we find

$$y'k = V(-iJ + iViK);$$

and treating this as above, we find the same value for y' as for y, so that

$$yk = -ViJ - K - iSiK. \tag{4}$$

Substituting the values of j and k from (2) and (4) in (1), we have $$ay\rho = pi - JSiJ - KSiK + SiJViK - SiKViJ, \tag{5}$$
where p is a scalar. The last two terms of this are easily proved to be of the form $bi + cv$; for they are $SJiViK + SiKVJi$; and if q and r are any quaternions, $SqVr + SrVq = Vqr - V.VqVr$; hence these two terms are equal to $-VJK - V.VJiViK$. But the vectors VJi and ViK are each perpendicular to i, therefore the vector of their product is parallel to i, while for VJK we can use $tt'\nu$. Hence (5) becomes
$$ay\rho = zi - JSiJ - KSiK - tt'.\nu,$$
where z is simply a scalar. We may denote the linear vector function $-JSJ\sigma - KSK\sigma$ by $\chi\sigma$, so that
$$ay\rho = zi + \chi i - tt'.\nu,$$
$$= (\chi + z)i - tt'.\nu.$$

To find the point in which this line intersects the body-plane $K\nu$, put $SJ\rho = 0$. Now $SJ\chi i = -J^2 SJi = t^2 SJi$;
therefore $$(t^2 + z)SJi = 0,$$
which gives $z = -t^2$. Hence for the point of intersection
$$ay\rho = (\chi - t^2)i - tt'\nu; \tag{6}$$
therefore $$ay(\chi - t^2)^{-1}\rho = i - tt'(\chi - t^2)^{-1}\nu. \tag{7}$$

Now if for simplicity we use λ and μ for unit vectors in the directions J and K, so that $J = t\lambda$, $K = t'\mu$,
$$\chi\sigma = -t^2\lambda S\lambda\sigma - t'^2\mu S\mu\sigma;$$
also $$t^2\sigma = -t^2(\lambda S\lambda\sigma + \mu S\mu\sigma + \nu S\nu\sigma),$$
$$\therefore\ (\chi - t^2)\sigma = (t^2 - t'^2)\mu S\mu\sigma + t^2\nu S\nu\sigma, \tag{8}$$
and from this we find at once
$$(\chi - t^2)^{-1}\sigma = \frac{\mu S\mu\sigma}{t^2 - t'^2} + \frac{\nu S\nu\sigma}{t^2}, \tag{9}$$
if $S\lambda\sigma = 0$, i.e. if σ is coplanar with μ and ν, as ρ in (6) is supposed to be. [To invert the function $\chi - t^2$, assume
$$(\chi - t^2)^{-1}\sigma \equiv A\mu + B\nu,$$
where A and B are unknown scalars; then operate on both sides with $\chi - t^2$, and we get A and B at once.]

From (9) we have $(\chi - t^2)^{-1}\nu = -\dfrac{\nu}{t^2}$, so that (7) becomes
$$ay(\chi - t^2)^{-1}\rho = i + \frac{t'}{t}\nu. \tag{10}$$

From (6) and (10), by taking the scalar of the product,

$$a^2 y^2 S\rho\,(\chi - t^2)^{-1}\rho = S.(i+\frac{t'}{t}\nu)[(\chi-t^2)i - tt'.\nu]$$
$$= S.(i+\frac{t'}{t}\nu)[(t^2-t'^2)\mu Si\mu + (t^2 Si\nu - tt')\nu]$$
$$= t^2\sin^2\theta + t'^2\sin^2\theta' - 2tt'\,Si\nu,$$

(remembering that, since λ, μ, ν are rectangular unit vectors,
$(Si\lambda)^2+(Si\mu)^2+(Si\nu)^2 = 1$, or $(Si\nu)^2 = 1-\cos^2\theta-\cos^2\theta'$).
Hence we have

$$S\rho\,(\chi-t^2)^{-1}\rho = \frac{1}{a^2} \tag{11}$$

for the point in which the line of the single resultant intersects the body-plane $K\nu$. But if the lines λ, μ, ν are taken, respectively, as axes of x, y, and z, and the quadric

$$\frac{x^2}{t^2}+\frac{y^2}{t'^2}+\frac{z^2}{t^2+t'^2} = 1$$

is constructed, equation (11) denotes the focal conic

$$\frac{y^2}{t'^2-t^2}+\frac{z^2}{t'^2} = 1$$

in the plane yz, so that this conic is the locus of points in which the single resultant intersects the plane. Similarly the points in which it intersects the plane xz lie on the focal conic

$$\frac{x^2}{t^2-t'^2}+\frac{z^2}{t^2} = 1.$$

Thus the Theorem of Minding is proved.

[The proof here given proceeds on the basis of a proof given by Professor Tait in a more condensed form.]

255.] **Theorem.** *At every point there can be found two axes round either of which if the body is displaced by rotation through any angle, the sum of the moments of the forces about the axis is zero.*

Let O be any point (taken as origin of vectors), and let the body be rotated round an axis through O in the direction of the unit vector σ. Then taking as before $\phi\sigma \equiv \Sigma a S\varpi\sigma$,

$$\phi'\sigma \equiv \Sigma\varpi S a\sigma,\quad h = \Sigma S a\varpi,\quad \text{and}\quad G = \Sigma V a\varpi,$$

the new principal moment at O is, from (ζ), Art. 244, given by the equation

$$G' = G\cos\psi - 2\sin^2\frac{\psi}{2}V\sigma\phi'\sigma + \sin\psi\,(h\sigma-\phi\sigma); \tag{a}$$

and if the sum of the moments about the axis of rotation is zero, we must have $$SG'\sigma = 0,$$
independently of the value of ψ.

Now $$SG'\sigma = \cos\psi\, SG\sigma - \sin\psi\,(h + S\sigma\phi\sigma),$$
so that we must have
$$SG\sigma = 0,\quad S\sigma\phi\sigma = -h, \tag{b}$$
while, of course, $T\sigma = 1$.

The first of these shows that σ must be perpendicular to G, the axis of principal moment at O before rotation, while the second and third show that it must be a vector drawn to some point on the curve of intersection of a unit sphere with the quadric $S\sigma\phi\sigma = -h$. Now this curve is intersected by the plane $SG\sigma = 0$ in four points which lie in pairs (by symmetry) on two right lines drawn through O.

If, then, translation along the axis of rotation is prevented by suitable means, the body will be in equilibrium in every position produced by rotation round either of these lines.

256.] **Reduction to a Force and Two Couples.** A particular case of the reduction to three forces deserves to be noticed. Suppose the direction of the vector i to be chosen so as to coincide with that of the Resultant of Translation. Then j and k are perpendicular to this direction, and therefore b and c, the sums of the resolved parts of the forces in directions perpendicular to the Resultant of Translation, are each zero. Hence P_j and P_k are at infinity, while P_i is, of course, the centre of the plane of centres. The vectors J and K of course remain, and the directions of j and k may be so chosen that J and K are perpendicular to each other, as has been already shown (Art. 251). Let C be the principal centre of the plane of centres, and suppose that O, the origin of vectors, is chosen on the perpendicular through C to this plane, so that the points P_i (that is, C), P_j, and P_k subtend right angles in pairs at O, the two latter points being at infinity on the lines OJ and OK.

Let the force Π_1 be applied at C, i.e. $A_1 = \overline{OC} = -\dfrac{I}{a}$. Then, the body being placed in an initial position, the axes of the quadrics $S\rho\phi\rho = 1$ and $S\rho\phi^2\rho = 1$ are in the directions OC, OJ, OK. Hence the vectors ϵ_2 and ϵ_3 are in the plane JOK, and (the applied forces being taken as a rectangular system) ϵ_1 coincides

in direction with A_1; therefore the vectors A_2 and A_3 are in the plane JOK, and their extremities are at infinity. The forces Π_2 and Π_3 are then applied at infinity, and we can see that the magnitude of each is zero. For, denoting (as in the previous cases in which the origin of vectors is not taken at the principal centre) the tensors of I, J, K by t_1, t_2, t_3, since

$$-S\epsilon_1\epsilon_2\epsilon_3 = S^2 A_1 A_2 A_3,$$

it is very easy to show in equations (6), &c., of Art. 250 that

$$T\Pi_1 = \frac{1}{TA_1 . \sin p_1}\sqrt{t^2{}_1\cos^2\theta_1 + t^2{}_2\cos^2\phi_1 + t^2{}_3\cos^2\psi_1},$$

where p_1 is the angle between the direction of A_1 and the plane of A_2A_3, and θ_1, ϕ_1, ψ_1 are the angles between ϵ_1 and the directions of I, J, K. Similar values are obtained for

$$T\Pi_2 \text{ and } T\Pi_3.$$

Now, in the present case, $TA_2 = TA_3 = \infty$; therefore Π_2 and Π_3 are zero forces applied at infinity.

This result, of course, indicates a new mode of reduction—namely, to a force and two couples; and this is the mode of reduction adopted in all cases by Moigno.

Let the principal centre be taken as origin of vectors, and suppose a force Π_2 applied at the extremity of a vector γ, while a force $-\Pi_2$ is applied at the extremity of γ'. Then in equation (2) of Art. 250 we shall have the term $\gamma S i \Pi_2 - \gamma' S i \Pi_2$, or $(\gamma-\gamma') S i \Pi_2$. Denote $\gamma-\gamma'$ by μ; then μ is the vector joining the points at which the forces $(\Pi_2, -\Pi_2)$ constituting the couple act. Similarly, let μ' be the vector joining the points at which the forces $(\Pi_3, -\Pi_3)$ act. We may, for shortness, call μ and μ' the *vector arms* of the couples. Then our equations are (I and A_1 being zero)

$$\Pi_1 + \Pi = 0,$$

$$\mu S i \Pi_2 + \mu' S i \Pi_3 = 0,$$

$$\mu S j \Pi_2 + \mu' S j \Pi_3 = -J,$$

$$\mu S k \Pi_2 + \mu' S k \Pi_3 = -K.$$

The second requires $S i \Pi_2 = 0$, $S i \Pi_3 = 0$; i.e. the forces of the couples are in a plane perpendicular to the Resultant of Translation. Let the tensors of J and K be now t and t', as in Art. 251.

Suppose the body placed in the initial position; then $J = tj$,

$K = t'k$, and i is the unit vector perpendicular to the plane of centres. Hence we have

$$\mu Sj\Pi_2 + \mu' Sj\Pi_3 = -tj,$$
$$\mu Sk\Pi_2 + \mu' Sk\Pi_3 = -t'k.$$

Operate on these with $S.Vi\mu'$, and observe that

$$S\mu Vi\mu' = -Si\mu\mu' = -SiV\mu\mu' = TV\mu\mu';$$

also
$$SjVi\mu' = -Sk\mu';$$

then we have
$$Sj\Pi_2 = \frac{tSk\mu'}{TV\mu\mu'}, \quad Sk\Pi_2 = -\frac{t'Sj\mu'}{TV\mu\mu'}.$$

Hence
$$\Pi_2 = \frac{1}{TV\mu\mu'}(-tjSk\mu' + t'kSj\mu') \tag{1}$$

$$\Pi_3 = \frac{1}{TV\mu\mu'}(-tjSk\mu + t'kSj\mu). \tag{2}$$

Let Π_2 and Π_3 be perpendicular to each other. Then

$$\frac{Sj\mu Sj\mu'}{t^2} + \frac{Sk\mu Sk\mu'}{t'^2} = 0,$$

which shows that the directions of μ and μ' are those of a pair of conjugate diameters of the ellipse

$$S\rho\phi\rho = 1 \tag{3}$$

where $\phi\rho \equiv f^2\left(\frac{j}{t^2}Sj\rho + \frac{k}{t'^2}Sk\rho\right)$, and f (denoting any constant force magnitude) is introduced for homogeneity.

Assume the arms to be represented, not only in directions, but also in magnitudes, by a pair of semi-conjugate diameters. Then $TV\mu\mu'$ is constant and equal to $\frac{tt'}{f^2}$, the product of the semiaxes.

Hence, from (1) and (2),

$$\Pi_2 = f^2\left(-j\frac{Sk\mu'}{t'} + k\frac{Sj\mu'}{t}\right) = f^2\left(j\frac{Sj\mu}{t} + k\frac{Sk\mu}{t'}\right),$$

$$\Pi_3 = f^2\left(-j\frac{Sk\mu}{t'} + k\frac{Sj\mu}{t}\right) = f^2\left(j\frac{Sj\mu'}{t} + k\frac{Sk\mu'}{t'}\right).$$

Now the ellipse (3) can be written $T\psi\rho = 1$, where

$$\psi\rho = f\left(j\frac{Sj\rho}{t} + k\frac{Sk\rho}{t'}\right),$$

and $\psi\rho$ obviously denotes the vector to the corresponding point on the circumscribed circle. Hence we have simply

$$\Pi_2 = f.\psi\mu, \quad \Pi_2 = f.\psi\mu' \tag{4}$$

and we arrive at the following result:—

The body having been placed so that the plane of centres is perpendicular to the Resultant of Translation, and the principal vectors J and K, fixed in the body, coincide with the corresponding vectors j and k fixed in space, the system may be astatically equilibrated by a single force acting at the centre of the plane of centres, equal and opposite to the Resultant of Translation, together with two couples in this plane, the forces of these couples acting in two rectangular directions at the extremities of any pair of semi-conjugate diameters of a certain ellipse, their forces being all equal and of constant magnitude whatever pair of diameters be chosen, and the forces at the extremities of each semi-diameter of the ellipse being parallel to the corresponding semi-diameter of its circumscribed circle.

257.] **Larmor's Proof of Minding's Theorem.** Professor Larmor has treated questions relating to astatic equilibrium in the following manner (see *The Messenger of Mathematics*, No. 160, August, 1884). The position of any line in space may be defined by six constants, or 'co-ordinates,' which are connected by two equations. These co-ordinates are the direction-cosines, l, m, n, of the line, and the moments round the axes of reference of a unit force acting along the line. If ξ, η, ζ are the co-ordinates of any point on the line, these moments are $m\zeta - n\eta$, $n\xi - l\zeta$, $l\eta - m\xi$. Denote these by λ, μ, ν, respectively. Then the two equations connecting the six co-ordinates $(l, m, n, \lambda, \mu, \nu)$ are

$$l^2 + m^2 + n^2 = 1,$$
$$l\lambda + m\mu + n\nu = 0.$$

A single homogeneous equation of the n^{th} degree between these co-ordinates represents a complex of lines of the n^{th} order. By substituting in such an equation the values of λ, μ, ν in terms of $l, \ldots, \xi, \ldots$ we obtain the relation between the direction-cosines of all the lines of the system that can be drawn through a given point (ξ, η, ζ). The lines of the complex which lie in any plane envelope a curve; the lines common to two complexes form a *congruency*; those common to three complexes form a ruled surface.

Now replace the given force system by a single force, R, equal

and parallel to the Resultant of Translation, at the principal centre, and two couples, each in the plane of centres, each having its forces equal to R, one force of each couple acting at C, and the others at points on the two principal body-vectors J and K at distances t and t' from C (f in the last Article being taken as unity).

Taking, as previously, J, K, ν, as axes of x, y, z, respectively, let the direction-cosines of the resultant force and those of the couples be (l, m, n), (l_2, m_2, n_2), and (l_3, m_3, n_3), respectively. Then the whole system is equivalent to the three component forces

$$Rl,\ Rm,\ Rn,$$

together with the three component couples

$$R\,(tn_2 - t'm_3),\quad R\,.\,t'l_3,\quad -R\,.\,tl_2,$$

along the axes. Now this system can be reduced to a wrench (R, pR), where p is the pitch of the wrench on an axis whose co-ordinates are, suppose, l, m, n, λ, μ, ν.

Then expressing that the wrench has the same moments about the axes as the given forces, we have

$$\lambda + pl = tn_2 - t'm_3, \tag{1}$$

$$\mu + pm = t'l_3, \tag{2}$$

$$\nu + pn = -tl_2. \tag{3}$$

Multiplying these by l, m, n and adding, we have

$$p = -tm_3 + t'n_2, \tag{4}$$

while by squaring and adding, we have, after substituting for p,

$$\lambda^2 + \mu^2 + \nu^2 = t^2m^2 + t'^2n^2. \tag{5}$$

Also from (2) and (3),

$$\frac{(\mu + pm)^2}{t'^2} + \frac{(\nu + pn)^2}{t^2} = l_2^2 + l_3^2$$

$$= m^2 + n^2. \tag{6}$$

The axes of all the wrenches form a complex which is given by (5), while (6) denotes the complex formed by all those which have a given pitch.

Now if $p = 0$, the wrench reduces to a force. Then (6) gives $\frac{\mu^2}{t'^2} + \frac{\nu^2}{t^2} = m^2 + n^2$. Dividing (5) by t^2, and subtracting from (6), we have

$$\frac{\lambda^2}{t'^2 - t^2} + \frac{\mu^2}{t'^2} = n^2. \tag{7}$$

Also

$$\frac{\lambda^2}{t^2 - t'^2} + \frac{\nu^2}{t^2} = m^2. \tag{8}$$

Equations (7) and (8) denote a congruency of lines, and if we wish to find the points in which it intersects the plane xy, we put $\zeta = 0$ in the values of λ, μ, ν. Then (7) gives

$$\frac{\eta^2}{t'^2 - t^2} + \frac{\xi^2}{t'^2} = 1,$$

which shows that the points lie on a focal conic of the quadric before discussed. Similarly for the points in which the lines of single force intersect the plane yz.

258.] **Stability and Instability of Equilibrium.** *A rigid body acted upon by any system of equilibrating forces, each of which is constant in magnitude, direction, and point of application, is in stable or unstable equilibrium, according as the Virial of the forces is a minimum or a maximum.*

Consider a small angular rotation, $\Delta\psi$, round an axis coinciding with any unit vector σ. Then according as the couple, G, called into existence by this displacement tends to diminish or to increase the angular displacement, the equilibrium is stable or unstable.

Observe that in Art. 243, in which our fundamental equation for the alteration of vectors is obtained, the angular rotation, ψ, is taken as positive when it is in the sense of the versor of σ, and negative when in the contrary sense. Now if ψ is positive, for stability G must project along σ in the sense of $-\sigma$; in other words, $SG\sigma$ is positive.

But, rejecting infinitesimals of the second order, we have

$$G = (h\sigma - \phi\sigma).\Delta\psi;$$

$$\therefore\ SG\sigma = -(h + S\sigma\phi\sigma).\Delta\psi.$$

Hence *for stability* $h + S\sigma\phi\sigma$ *must be negative.*

Now consider the alteration produced in the Virial, $\Sigma S a \varpi$, by rotation. We have

$$da = -\Delta\psi V a\sigma,$$

$$\therefore\ \frac{dh}{d\psi} = -\Sigma S V a\sigma.\varpi$$

$$= -\Sigma S\varpi a\sigma,$$

and $$\frac{d^2h}{d\psi^2} = \Sigma S.\varpi V a\sigma.\sigma$$

$$= \Sigma S\varpi(a\sigma - Sa\sigma)\sigma = \Sigma(-S\varpi a - S\varpi\sigma S a\sigma)$$

$$= -(h + S\sigma\phi\sigma);$$

so that by the previous result h, the Virial, is a minimum for stability.

It is, of course, obvious that for such a force system as we are discussing it is sufficient to calculate stability for rotation round axes through *any* origin, and that the Virial is the same with respect to all origins; for, changing the origin of vectors amounts only to writing $a+\epsilon$ for a, and

$$\Sigma S(a+\epsilon)\varpi = h + S\epsilon\Pi = h, \quad \because \Pi = 0.$$

EXAMPLES.

1. A system of forces, each of which is constant in magnitude, direction, and point of application in a rigid body, keeps the body in equilibrium in a certain position. If they keep it in equilibrium in another position differing infinitely little from the previous one, the same line of points in the body being common to both positions, prove that, any angular displacement whatever being given to the body round this line, equilibrium will continue to subsist.

Let σ be a unit vector along the line common to two positions. Then the second position is obtained by an infinitesimal rotation, $\Delta\psi$, round this line, so that in the value of G' given by (ξ) of Art. 244 we may neglect the term in $\sin^2\frac{\psi}{2}$ in comparison with the last; and since $G' = 0$, by hypothesis,

$$\phi\sigma = h\sigma.$$

But, this equation holding, we shall always have the coefficient of $\sin^2\frac{\psi}{2}$ equal to zero, whatever ψ may be. Hence $G' = 0$ for all displacements round the line.

This result is also very easily proved by the ordinary Cartesian method. For, taking the common line as axis of x, the co-ordinates of a point (x, y, z) become by rotation round this axis

$$(x;\ y\cos\psi - z\sin\psi;\ z\cos\psi + y\sin\psi).$$

If X, Y, Z are the components of the force at this point, the moments round the axes vanishing in the first position (that in which $\psi = 0$), we have

$$\Sigma Zy - \Sigma Yz = 0;\ \Sigma Xz - \Sigma Zx = 0;\ \Sigma Yx - \Sigma Xy = 0. \qquad (a)$$

Hence the new moments, L', M', N', are

$$\begin{aligned} L' &= -\Sigma Zx + \cos\psi\,\Sigma Xz + \sin\psi\,\Sigma Xy, \\ M' &= \Sigma Yx - \cos\psi\,\Sigma Xy + \sin\psi\,\Sigma Xz, \\ N' &= \sin\psi\,\Sigma(Yy + Zz). \end{aligned}$$

Now we are given that these vanish when $\Delta\psi$ is put for ψ; i.e. we are given

$$\Sigma Xy = 0; \ \Sigma Xz = 0; \ \Sigma(Yy + Zz) = 0.$$

These and (a) give $\Sigma Zx = 0$, $\Sigma Yx = 0$; so that L', M', and N' all vanish whatever ψ may be.

2. If a system of forces acting on a rigid body is astatic for all displacements of rotation round each of two intersecting lines, it is astatic for all displacements of rotation round all lines in the plane of these two.

Let σ and σ' denote the two given lines. Then we have by (ξ), Art. 244, $\phi\sigma = h\sigma$, and $\phi\sigma' = h\sigma'$; and it is at once obvious that, with these conditions satisfied, the coefficients of $\sin^2\frac{\psi}{2}$ and $\sin\psi$ will each vanish if $x\sigma + y\sigma'$ is written for σ, whatever x and y may be.

3. If a rigid body acted upon by any forces is placed so that the forces reduce to a single resultant at the principal centre, C, show that if it is turned through any angle round any axis at C lying on a certain cone, the sum of the moments of the forces round the axis is zero.

Taking in the initial position the vectors J and K (which coincide in directions with j and k) as axes of x and y, respectively, the axis of displacement may be any one joining C to the curve of intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ with the cylinder $tx^2 + t'y^2 = -h$.

For this case $\phi\sigma \equiv \Sigma\alpha S\varpi\sigma = tjSj\sigma + t'kSk\sigma$, as we see by replacing the given force system by three forces as in Art. 252, and observing that in the position of the body before displacement $J = tj$, $K = t'k$. Also the value of h is easily seen to be $-(t + t')$.

If the body is not in the initial position, $\phi\sigma$ (the vectors being measured from C) will be $JSj\sigma + KSk\sigma$.

The result holds, of course, with respect to any point on the line of action of the single resultant when this has any position.

4. If a rigid body acted on by any forces is turned so as to have a maximum Virial with respect to the principal centre, show that J and K coincide with j and k at the principal centre, and the maximum value is $t + t'$.

5. Prove that there are two positions of the body for which the forces reduce to identically the same wrench. (Prof. Larmor.)

[Consider in (6) of Art. 257 the co-ordinates $l, m, n, \lambda, \mu, \nu$ given, and make the values of p equal. The axis of this wrench is a line of the congruency determined by the complex of axes (5) and by the complex denoted by

$$\left(\frac{\mu^2}{t'^2} + \frac{\nu^2}{t^2} - m^2 - n^2\right)\left(\frac{m^2}{t'^2} + \frac{n^2}{t^2}\right) = \left(\frac{m\mu}{t'^2} + \frac{n\nu}{t^2}\right)^2.$$

Hence eight such axes pass through every point.]

6. Determine the locus of axes of wrench in which the two pitches are equal and opposite. (Prof. Larmor.)

Ans. The congruency determined by the complex (5) and the complex $\frac{m\mu}{t'^2}+\frac{n\nu}{t^2}=0$.

7. Show that a heavy magnetic solid can be astatically equilibrated by two forces, or by a force and a couple, and discuss its equilibrium.

Whatever may be the directions of magnetisation inside the body—i.e. the directions of the indefinitely small magnets of which we may imagine the solid to consist—the magnetic forces produced by the Earth are all parallel to a certain vertical plane, the plane of the magnetic meridian; hence all the applied forces are parallel to one plane, and therefore (Art. 249) they can be astatically equilibrated by two forces, or by a force and a couple.

Take the centre of gravity, G, of the body as origin of vectors, the axis of k vertically up, and that of j in the plane of the magnetic meridian. Let P and Q be the north and south poles of any one of the elementary magnets, let ϖ be the Earth's magnetic force exerted on P, and let μ be the vector QP. Let Π be the single force, applied at the extremity of the vector a, and let Π' be the force in the couple whose vector arm is β, which are to astatically equilibrate the weight and the magnetic forces. The weight being $-Wk$, the equations of astatic equilibrium are

$$\Pi - Wk = 0, \tag{1}$$

$$\beta Sj\Pi' = -\Sigma\mu Sj\varpi = J, \text{ suppose,} \tag{2}$$

$$-Wa+\beta Sk\Pi' = -\Sigma\mu Sk\varpi = K.... \tag{3}$$

The vectors J and K are vectors fixed in the body, and depend on the direction and intensity of magnetisation at each point.

Equation (2) shows that β, the vector arm of the equilibrating couple, is fixed in direction in the body, being parallel to J. Also (3) shows that the point of application of the force Π is any one whatever on a right line parallel to J, since

$$a = -\frac{K}{W}+xJ,$$

where x is a variable scalar. The direction of this line depends, therefore, only on the magnetism and not on the weight of the body, so that it would not be altered by adding a non-magnetic portion to the body.

The length of the arm β, and also its actual position, are arbitrary. We may assume a and β at will, and then the direction (in fixed space) and magnitude of Π' are known. If θ is the angle which the direction of the force Π' makes with j, equation (3) can be written

$$-Wa+J\tan\theta = K,$$

so that if a is assumed, the direction of Π' is known.

We may take two points, D and E, on the vectors J and K, respectively, such that $\overline{GD} = \frac{J}{W}$, $\overline{GE} = -\frac{K}{W}$; then, A being any point on the line drawn through E parallel to J, the last equation is

$$\overline{GD}\,.\tan\theta = \overline{GA} - \overline{GE}$$
$$= \overline{EA},$$

so that $\tan\theta = \frac{EA}{GD}$.

The principal couple at G in any position is $WVak + V\beta\Pi'$; and there will be a single resultant if

$$Sk\beta\Pi' = 0,$$

that is (since Π' is, like all the other forces, parallel to the plane jk), if the body is turned so that the two body-vectors J and K lie in the magnetic meridian. For all displacements in this plane there is an astatic centre (Art. 249, and p. 129, vol. I.), at which a single force will keep the body in astatic equilibrium. Such displacements may be produced by fixing an axis in the body perpendicular to the plane of the magnetic meridian.

If the direction of magnetisation is constant throughout the body, μ is of constant direction, so that $\Sigma\mu Sj\varpi = \mu\Sigma Sj\varpi$, and the vectors J and K are coincident in direction with μ, while the locus of the point A is a line parallel to μ through the centre of gravity of the body.

Let I denote the intensity of magnetisation at any point (both as regards magnitude and direction) and let γ denote similarly the intensity of the Earth's magnetic force (i.e. its force per unit pole). Then if dm denotes the strength of the element pole at the extremity of μ, we have by definition

$$\mu dm = I dx dy dz,$$

where $dx\,dy\,dz$ is the volume of the element of the body at the point considered. If f is the tensor of γ, or the Earth's resultant magnetic force per unit pole, the expression $\Sigma\mu Sj\varpi$ is obviously $\Sigma\mu Sj\gamma dm$, or $-f\cos\delta\Sigma\mu dm$, (where δ is the dip) or $-f\cos\delta\int I dx dy dz$, or $fIS\cos\delta$, where $S = \int dx\,dy\,dz =$ the volume of the magnetic portion of the body, I being assumed constant throughout the body. Hence

$$J = fIS\cos\delta;\ K = fIS\sin\delta,$$

and
$$a = \frac{fS}{W}(\cos\delta\tan\theta - \sin\delta).I.$$

Again, the position of the astatic centre is easily found.

For (Art. 249) if ϵ is the vector to it, we have

$$\epsilon = \frac{\Sigma a\varpi}{\Pi} = \frac{\Sigma\mu\gamma dm}{-Wk} = -\frac{S}{W}\frac{I\gamma}{k}.$$

But if i is a unit vector perpendicular to the magnetic meridian,

drawn towards the east, $\frac{\dot{\gamma}}{k} = f(\sin\delta + i\cos\delta)$, since the angle between j and k is $\frac{\pi}{2}+\delta$. Hence

$$\epsilon = -\frac{fS}{W}(I\sin\delta + I'\cos\delta),$$

if we denote by I' the vector VIi perpendicular to I in the plane of the magnetic meridian. The tensors of I and I' are equal. Hence a very simple construction for the astatic centre, or point at which a single force will keep the body in astatic equilibrium for displacements in the plane of the magnetic meridian. The product of S and the tensor of I is the magnetic moment of the whole body, which may be denoted by M; and fM is the maximum moment exerted on the body by the Earth's magnetic attraction. If $[m]$ denotes the strength of the unit pole, M may be represented by the product

$$[m] \times l,$$

where l may be regarded as the *length of the simple equivalent magnet*. Again, f is of the form $\frac{force}{[m]}$, so that our value of ϵ is a linear magnitude, as it ought to be. As f is known to be about $\cdot 47$ dynes, if W denotes the number of dynes in the weight of the body, the astatic centre is at a distance of $\frac{\cdot 47}{W} \times l$ from the centre of gravity, and on a line making an angle equal to the dip with the direction of magnetisation of the body.

8. Prove from first principles that if a body is astatic for displacements round any axis, it is astatic for displacements round all axes parallel to the given one.

9. Prove that the moment of a force ϖ acting at the end of a vector a about an axis through the origin in the direction of a unit vector σ is

$$-S\sigma a\varpi.$$

10. If a system of forces is astatic, prove that if each force is resolved into two components, one parallel to any given axis and the other perpendicular thereto, the set of components parallel to the axis and the set of components perpendicular to it are separately astatic.

The direction of the axis parallel to which all the forces are resolved may be taken as that of i, and the two components of any force, ϖ, are then $-iSi\varpi$ and $\varpi + iSi\varpi$. It will be found that each set satisfies the necessary conditions of making the vector sum of the forces vanish and the linear vector function $\Sigma a S\varpi'\rho$ vanish when i, j, and k are written for ρ, denoting by ϖ_1', ϖ_2',... the forces at the extremities of a_1, a_2,... in either resolved system.

11. A rigid body is in equilibrium under a system of forces; find the condition that there should exist some axis for all displacements round which the body is astatic.

If σ is a unit vector in the direction of the required axis, the vector couple produced by rotation being

$$\cos\psi\Sigma V\alpha\varpi - 2\sin^2\frac{\psi}{2}V\sigma\phi\sigma + \sin\psi(h\sigma - \phi\sigma),$$

it is necessary and sufficient that this should vanish identically. Hence we must have

$$\phi\sigma = h\sigma.$$

But there are three directions of ρ for which $\phi\rho = g\rho$, and three corresponding values of g. (Tait's *Quaternions*, Chap. V.) Hence the necessary condition is that h, or $\Sigma S\alpha\varpi$, must be one of the three principal parameters of the function $\phi\rho$ or $\Sigma\alpha S\varpi\rho$; in other words, $\frac{1}{\sqrt{h}}$ must be one of the semiaxes of the quadric $S\rho\phi\rho = 1$.

12. Supposing that an axis exists for all displacements round which the equilibrium is astatic, prove that if each force is resolved into two components, one parallel and the other perpendicular to the axis, each of these component sets is astatic for displacements round the axis.

13. In a non-equilibrating system of forces, if each force is resolved into two components, one parallel to an axis and the other perpendicular to it, find the conditions that the second set should be astatic for displacements round the axis.

Ans. In the first place the axis must be parallel to the resultant of translation of the given system; and if Π is this resultant and R its magnitude, we must have in addition

$$V\Pi\phi\Pi = R^2\Sigma V\alpha\varpi,$$

$$S\Pi\phi\Pi = -R^2\Sigma S\alpha\varpi,$$

the origin of vectors being anywhere.

The first of these equations shows that the axis of the principal couple at the origin must be at right angles to the resultant of translation of the given forces, i.e. there must be a single resultant. We may suppose that the origin is chosen on the line of action of this single resultant, so that we have $V\Pi\phi\Pi = 0$, i.e. $\phi\Pi = h\Pi$, where $h = \Sigma S\alpha\varpi$, by the second equation. Hence h must be one of the three principal parameters of the function ϕ, and the resultant must coincide in direction with one of the axes of the quadric $S\rho\phi\rho = 1$.

14. For a system of forces each of which retains its magnitude, direction, and point of application in a rigid body, prove that there are four positions of the body for which the forces reduce to a single resultant passing through a given point.

(See Schell, *Theorie der Bewegung und der Kräfte*, vol. II., p. 247.)

15. When the force system is equivalent to a couple, prove that there are four positions of equilibrium of the body. (Schell, *ibid.*)

CHAPTER XV.

THE PRINCIPLE OF VIRTUAL WORK APPLIED TO ANY SYSTEM OF BODIES.

259.] **Forces applied to a Particle.** It has been shown in Art. 199, p. 2, that the resultant of any number of forces applied to a particle may be represented by the side required to close the polygon of the forces. And whether the polygon $OP_1P_2\ldots P_n$ be plane or gauche, it is clear (as in Art. 55) that the sum of the projections of the sides, taken in order, along any line OA, is equal to zero.

Let the projections of the sides be denoted by $Q_1, Q_2, \ldots Q_n$. Then $Q_1+Q_2+\ldots+Q_n=0$. Multiplying this by OA, an arbitrary length along the line OA, we have

$$Q_1 \,.\, OA + Q_2 \,.\, OA + \ldots + Q_n \,.\, OA = 0.$$

But if p_1 is the projection of OA along OP_1, we have (see Art. 56)

$$Q_1 \,.\, OA = OP_1 \,.\, p_1.$$

If, then, the sides OP_1, $P_1P_2, \ldots$ be denoted by P_1, $P_2, \ldots$ we have

$$P_1 \,.\, p_1 + P_2 \,.\, p_2 + \ldots + P_n \,.\, p_n = 0\,;$$

and if the sides represent forces, each term in this equation is the virtual work of the corresponding force for the displacement OA. Since the resultant, R, of $n-1$ of the forces is $-P_n$, we have

$$R \,.\, r = P_1 \,.\, p_1 + P_2 \,.\, p_2 \ldots\,;$$

and if the displacement is small, this equation is written (as in Art. 64)

$$R\delta r = P_1\delta p_1 + P_1\delta p_2 + \ldots. \tag{1}$$

In particular, if X, Y, Z denote the rectangular components of R, we have

$$R\delta r = X\delta x + Y\delta y + Z\delta z. \tag{2}$$

260.] **Extension to any number of Connected Particles.** If two particles, m_1 and m_2, are connected by a rigid inextensible rod, and are in equilibrium under the action of forces, P_1, $Q_1, \ldots$

applied to m_1 and P_2, Q_2, ... applied to m_2, it is evident (as in Art. 105) that the force arising from the connexion acts in the line joining m_1 to m_2. If, then, this force be denoted by T, and the distance between the particles by r, we have for the equilibrium of m_1

$$P_1\delta p_1 + Q_1\delta q_1 + \ldots + T\delta_1 r = 0,$$

$\delta_1 r$ denoting the change in r arising from an arbitrary small displacement of m_1. The equation of equilibrium of m_2 is

$$P_2\delta p_2 + Q_2\delta q_2 + \ldots + T\delta_2 r = 0;$$

and if in the new positions of m_1 and m_2 the distance between them remains unaltered, $\delta_1 r + \delta_2 r = 0$. Hence, by addition, from these equations we obtain the equation

$$P_1\delta p_1 + Q_1\delta q_1 + \ldots + P_2\delta p_2 + Q_2\delta q_2 + \ldots = 0, \qquad (1)$$

which is free from the internal force T.

This is exactly the same as the investigation already given for coplanar forces in Chap. VI. The extension to any number of particles, that is, to any extended body, proceeds just as in that chapter, and the enunciation of the principle of virtual work there given applies in general without the limitation that the forces are coplanar.

If in the case of the two particles m_1 and m_2, considered above, their new positions are such that the distance between them is altered by δr, the equation of virtual work will be

$$P_1\delta p_1 + Q_1\delta q_1 \ldots + P_2\delta p_2 + Q_2\delta q_2 + \ldots + T\delta r = 0; \qquad (2)$$

and, generally, if the virtual displacement is such that the internal forces do virtual work, these forces will enter into the equation of virtual work in exactly the same manner as the applied forces. The theorem of virtual work may, therefore, be thus enunciated:—

When a material system is in equilibrium under the action of any external and internal forces, the sum of the virtual works of the external and internal forces is equal to zero for any small virtual displacement whatsoever.

Instead of saying that the total virtual work is *zero*, we should in strictness say that it is an indefinitely small quantity of the second order, the greatest of the displacements being considered as a small quantity of the first order. This has been already explained in Vol. I.

The proof of the converse proposition—namely, that when the virtual work vanishes for all imagined displacements, the system

is in equilibrium—has been already given in Art. 108 for coplanar forces; and as the proof obviously holds for non-coplanar forces, it is unnecessary to reproduce it here.

261.] **Displacements along Smooth Surfaces.** If any body or system of connected bodies be in contact with smooth curves or surfaces, and the system be imagined to receive any small displacement along these curves or surfaces, it is clear that, since the point of application of each of the geometrical forces (reactions of the curves or surfaces) moves in a plane at right angles to the corresponding force, these forces will contribute nothing to the equation of virtual work for such a displacement.

If any of the bodies of the system are connected by strings or rods whose lengths are unaltered in the virtual displacement chosen, the tensions of these strings or rods will not enter into the equation of virtual work. But, as already explained in Arts. 73 and 107, we may choose virtual displacements of the system which violate the imposed conditions at the expense of bringing into our equation the corresponding forces.

262.] **Kinematical Theorem I.** When all the points of a rigid body move parallel to a plane, the motion may be produced by a pure rotation round an axis perpendicular to this plane.

DEF. A motion of a body round an axis whereby each point in the body describes an arc of a circle having its centre on the axis and its plane perpendicular to it is called *pure rotation.*

The position of the body will evidently be known if the positions of any two points in a plane parallel to the plane of motion are known.

Let A and B be any two points in such a plane, and suppose that after the displacement of the body they occupy the positions A' and B' (Fig. 252). At the middle points of AA' and BB' erect two perpendiculars, which meet in I. Then in the triangles AIB and $A'IB'$, $AI = A'I$, $BI = B'I$, and $AB = A'B'$; therefore the triangle $A'IB'$ is nothing more than AIB turned round the point I through an angle AIA' or BIB'. Hence the line AB can be brought into its new position by a pure rotation about I, and the same is true of every point rigidly connected with A and B in the plane AIB.

If through I an axis be drawn perpendicular to the plane of motion, it is evident that the body can be brought into its new position by a pure rotation about this axis through an angle

$= AIA'$, however complicated the paths along which A and B have travelled to A' and B'.

When the motion of the body is small, this axis is called the *Instantaneous Axis*; and it is obviously constructed by *drawing two planes normal to the lines of motion of any two points in the body.* The intersection of these planes is the instantaneous axis.

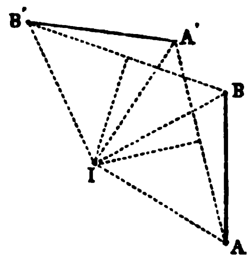

Fig. 252.

When the body is a plane figure, the point I is called the *Instantaneous Centre*; and the consideration of this point is of very extensive use in Kinematics, Statics, and Geometry.

To construct the instantaneous centre, *at any two points erect perpendiculars to the lines of motion of these points, and their intersection is the required point.*

263.] **Kinematical Theorem II.** The motion of a rigid body round a fixed point is at every instant a pure rotation round an axis.

One point, O, in the body being fixed, the position of the body will be known if the positions of any two points, A and B, not in directum with O are known.

Round O let a sphere, forming part of the body or rigidly connected with it, be described with arbitrary radius, and let A and B (Fig. 252) be any two points on the sphere. After the motion of the body let A' and B' be the positions of A and B. Imagine the lines AB, $A'B'$, AA', and BB' in this figure to be arcs of great circles on the sphere instead of right lines. Then, at the middle points of AA' and BB' draw two great circles perpendicular to AA' and BB', respectively, and let them meet in I. In exactly the same way as in the last theorem, we have the spherical triangles AIB and $A'IB'$ equal; that is, the latter triangle is the former turned round the axis OI through an angle AIA' or BIB'. Hence the whole body is brought by rotation through this angle round the axis OI from the old to the new position.

264.] **Kinematical Theorem III.** If a body has a motion of translation represented in magnitude and direction by a right line OA, and at the same time a motion of translation represented in magnitude and direction by a right line OB, the

resulting motion of translation is represented in magnitude and direction by the diagonal, OC, of the parallelogram determined by OA and OB.

This proposition has been already illustrated in Art. 11. It follows immediately that any motion of translation can be resolved by the parallelopiped law into three motions along the axes of x, y, and z, after the manner of forces.

265.] **Kinematical Theorem IV.** If a body receives a motion of rotation round an axis OA, the rotation being represented in magnitude by OA,—i.e. so many units of circular measure being represented by so many centimetres, the scale being, of course, quite arbitrary—and at the same time a motion of rotation (of the same sign as the first) round an axis OB, the rotation being represented in magnitude by OB, the resulting motion is one of rotation round the diagonal, OC, of the parallelogram determined by OA and OB, and is represented in magnitude by this diagonal.

[The signs of rotations are determined by the rule given in Art. 200. We shall, for definiteness, suppose that when a watch is held with its face perpendicular to AO, so that OA passes up through the glass, the rotation about OA takes place in a sense opposite to that of the hands; and similarly for OB.]

Let P be any point on OC, p the perpendicular from P on OA, q the perpendicular from P on OB, and $k.OA$ and $k.OB$ the angular motions round OA and OB, respectively. Then in virtue of the rotation round OA, P moves upwards from the plane of the paper through a distance equal to $kp\,.\,OA$; and in virtue of the rotation round OB, P moves downwards from the plane of the paper through a distance equal to $kq\,.\,OB$. Therefore the whole motion of P upwards is equal to

$$k(p\,.\,OA - q\,.\,OB).$$

But this is obviously zero; therefore P is at rest, and so is every point on OC. The motion is, then, a rotation round OC. Let Ω be the angular rotation of the body round OC. Then the point A moves upwards from the plane of the paper through a distance equal to $\Omega\,.\,OA \sin AOC$, since $OA \sin AOC =$ the perpendicular from A on OC. But A in turning round OB moves through a distance equal to $k\,.\,OB\,.\,OA \times \sin AOB$. Hence

$$\Omega\,.\,OA \sin AOC = k\,.\,OB\,.\,OA \sin AOB,$$

or
$$\Omega = k \,.\, OB \,.\, \frac{\sin AOB}{\sin AOC}$$
$$= k \,.\, OC.$$

Therefore the resulting angular velocity is represented by OC, if the component rotations are represented by OA and OB.

This proposition is known as the 'parallelogram of angular velocities.' It follows at once that an angular motion about any axis, OL, may be decomposed into three angular motions about three axes, Ox, Oy, and Oz. If these latter are rectangular, an angular motion ω about OL is equivalent to angular motions, $\omega \cos a$, $\omega \cos \beta$, and $\omega \cos \gamma$, of the same sign, round the axes of x, y, and z, the direction angles of OL being a, β, γ.

266.] **General Displacement of a Rigid Body.** The position of every point in a rigid body is known when the positions of any three points in it are known, provided that these points are not in one right line. The general displacement of a rigid body is, therefore, the same as that of a system of three points forming a triangle.

Let A, B, C be the positions of three points in the body before the displacement, and A', B', C' the positions occupied by these points after the displacement. Then the triangle ABC may be brought into the position $A'B'C'$ by moving A directly to A' while B and C move parallel to AA' through distances equal to AA', and then turning the triangle about A' until B and C coincide with B' and C'. But (Art. 263) this latter motion is one of rotation round some axis through A'. Hence *the general displacement of a rigid body consists of a motion of translation which is the same for all its points, and a motion of rotation round an axis through an angle which is the same for all its points.*

To find the changes produced in the co-ordinates, x, y, z of any point in the body by a general displacement, we may consider the motions of translation and of rotation separately.

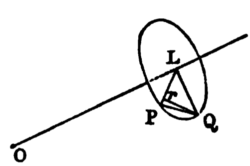

Fig. 253.

Although we shall be concerned only with small displacements, it is well to investigate the changes produced in the co-ordinates of a point by a rotation through any angle, θ, round an axis whose position is given.

Let the direction angles of the axis, OL (Fig. 253), be a, β, γ;

let P be the point (x, y, z) which, after the body has rotated through an angle θ round OL, occupies the position Q; let PL $(=p)$ be the perpendicular from P on OL, and Qr a perpendicular from Q on LP. Now the x of Q is the projection of OQ on the axis of x; therefore the change in x is the projection of PQ along Ox, or the sum of the projections of Pr and rQ. But $Pr = p\,(1-\cos\theta)$, and $Qr = p\sin\theta$.

Again, if the direction angles of PL are λ, μ, ν, since Qr is at right angles to OL and PL, the direction cosines of Qr are $\cos\beta\cos\nu - \cos\gamma\cos\mu$, &c. Hence, if the x of Q is x',

$$x' - x = p\sin\theta\,(\cos\beta\cos\nu - \cos\gamma\cos\mu) - 2p\cos\lambda\sin^2\frac{\theta}{2}. \quad (1)$$

But $p\cos\lambda$ is the projection of PL along the axis of x, or the projection of OP — the projection of OL, and since $OL = x\cos a + y\cos\beta + z\cos\gamma$,

$$p\cos\lambda = x - (x\cos a + y\cos\beta + z\cos\gamma)\cos a.$$

Similarly

$$p\cos\mu = y - (x\cos a + y\cos\beta + z\cos\gamma)\cos\beta,$$
$$p\cos\nu = z - (x\cos a + y\cos\beta + z\cos\gamma)\cos\gamma.$$

Substituting these values in (1), we have

$$x' - x = \sin\theta\,(z\cos\beta - y\cos\gamma) + 2\sin^2\frac{\theta}{2}\left[(x\cos a + y\cos\beta + z\cos\gamma)\cos a - x\right], \quad (2)$$

and similar values for the changes in y and z,— a result which follows at once from the equation (A), p. 66, by taking $S.i$ of both sides, i being a unit vector in the direction of the axis of x.

If the angular rotation θ is very small, we have

$$\delta x = (z\cos\beta - y\cos\gamma)\,\delta\theta,$$
$$\delta y = (x\cos\gamma - z\cos a)\,\delta\theta,$$
$$\delta z = (y\cos a - x\cos\beta)\,\delta\theta,$$

and if the components of the rotation $\delta\theta$ along the axes be denoted by $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, these equations give

$$\left.\begin{aligned} \delta x &= z\delta\theta_2 - y\delta\theta_3 \\ \delta y &= x\delta\theta_3 - z\delta\theta_1 \\ \delta z &= y\delta\theta_1 - x\delta\theta_2 \end{aligned}\right\}. \quad (3)$$

Of course these equations can be obtained very simply by considering the separate changes in the co-ordinates produced by

successive rotations $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ round the axes of x, y, z, respectively. (See Routh's *Rigid Dynamics*.)

If the components of the motion of translation common to all points in the body be δa, δb, δc, the complete changes in the co-ordinates for a small displacement will be

$$\left.\begin{aligned} \delta x &= \delta a + z\delta\theta_2 - y\delta\theta_3 \\ \delta y &= \delta b + x\delta\theta_3 - z\delta\theta_1 \\ \delta z &= \delta c + y\delta\theta_1 - x\delta\theta_2 \end{aligned}\right\}. \tag{4}$$

267.] **Deduction of the Six Equations of Equilibrium.** Replacing the virtual work of each force in equation (1) of Art. 260 by the virtual work of its three components, the general equation of virtual work becomes

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = 0, \tag{1}$$

and substituting in this equation the values of δx, δy, and δz given by (4), we have

$$\delta a\,.\,\Sigma X + \delta b\,.\,\Sigma Y + \delta c\,.\,\Sigma Z + \delta\theta_1\,.\,\Sigma(Zy - Yz) \\ + \delta\theta_2\,.\,\Sigma(Xz - Zx) + \delta\theta_3\,.\,\Sigma(Yx - Xy) = 0. \tag{2}$$

Now, the displacement being quite arbitrary, its components δa, δb, δc, $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, are completely independent. Hence in (2) we may consider all of them zero except one, and the equation then gives the coefficient of this one equal to zero. Thus (2) involves the six equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$
$$\Sigma(Zy - Yz) = 0, \quad \Sigma(Xz - Zx) = 0, \quad \Sigma(Yx - Xy) = 0,$$

which are the equations of equilibrium before obtained (see p. 47).

268.] **Method of Lagrange.** Lagrange based the whole of Dynamics—alike its applications to the equilibrium and motion of rigid bodies, of inextensible and extensible strings, of elastic rods and membranes, of fluids, and of elastic media propagating disturbances by undulatory motions—on the single Principle of Virtual Work. So far as the equilibrium problem is concerned, in its reference to any of the material systems just named, the idea of the method is shortly this—

Imagine the system to have taken its position or configuration of equilibrium; then imagine any small derangement whatever of the points, or infinitesimal elements, of the system; calculate the total quantity of work, both of the external forces applied to the system

and of its internal forces (forces mutually exerted by neighbouring parts of the system), and equate to zero this sum total of work.

Now the system whose equilibrium is proposed for investigation in any case may be one in which certain specified geometrical conditions have to be satisfied—as, for instance, a system of particles connected by inextensible flexible strings or inextensible and inflexible rods—and, as has been abundantly illustrated in the earlier parts of this work, we may either respect the imposed geometrical conditions (as it is often convenient to do when we merely seek for *positions* of equilibrium), or we may imagine a derangement of the parts of the system in which no regard is paid to these imposed conditions. But if we do the latter, it is at the expense of introducing into our equation of Virtual Work the work which would be done by an internal force whose existence is a necessary consequence of the particular geometrical condition under consideration. The imposition of every geometrical condition in a system establishes the existence of an internal force in the system; and the examples hitherto treated have related to the simpler cases in which such forces are due to the invariability of distances between particles or the restriction of the positions of particles to smooth surfaces.

We now proceed to consider, after the manner of Lagrange, the theory of all imposed geometrical conditions for a system of particles in a general manner.

269.] **Equations of Condition may be replaced by Forces.** Suppose a system of n particles whose co-ordinates are connected by k equations of condition,

$$L_1 = 0, \quad L_2 = 0, \dots L_k = 0, \tag{1}$$

each of these equations being of the form

$$f(x_1, y_1, z_1, x_2, y_2, z_2, \dots x_n, y_n, z_n) = 0,$$

that is, involving the co-ordinates of all the points in general. Then the equation of virtual work for the position of equilibrium of the system is

$$\Sigma (X\delta x + Y\delta y + Z\delta z) = 0,$$

which, when written at full length, is

$$X_1 \delta x_1 + Y_1 \delta y_1 + Z_1 \delta z_1 + \dots + X_n \delta x_n + Y_n \delta y_n + Z_n \delta z_n = 0. \tag{2}$$

Now *if the virtual displacements of the particles were all independent*, this equation would involve the vanishing of the coefficient of each displacement (see Art. 267); but the displacements of the particles must be such as still to satisfy the equations (1). Hence the quantities δx, &c., are connected by the k equations

$$\left.\begin{aligned}
&\frac{dL_1}{dx_1}\delta x_1 + \frac{dL_1}{dy_1}\delta y_1 + \frac{dL_1}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_1}{dx_n}\delta x_n + \frac{dL_1}{dy_n}\delta y_n + \frac{dL_1}{dz_n}\delta z_n = 0, \\
&\frac{dL_2}{dx_1}\delta x_1 + \frac{dL_2}{dy_1}\delta y_1 + \frac{dL_2}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_2}{dx_n}\delta x_n + \frac{dL_2}{dy_n}\delta y_n + \frac{dL_2}{dz_n}\delta z_n = 0, \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&\frac{dL_k}{dx_1}\delta x_1 + \frac{dL_k}{dy_1}\delta y_1 + \frac{dL_k}{dz_1}\delta z_1 + \dots \\
&+\frac{dL_k}{dx_n}\delta x_n + \frac{dL_k}{dy_n}\delta y_n + \frac{dL_k}{dz_n}\delta z_n = 0.
\end{aligned}\right\} \tag{3}$$

Solving these k equations for any k of the displacements—suppose $\delta x_1, \delta x_2, \dots \delta x_k$—and substituting their values in (2), we obtain an equation connecting the remaining $3n-k$ displacements of the form

$$\begin{aligned}
A_{k+1}\delta x_{k+1} + \dots + A_n \delta x_n \\
+B_1\delta y_1 + \dots + B_n\delta y_n \\
+C_1\delta z_1 + \dots + C_n\delta z_n = 0.
\end{aligned} \tag{4}$$

Now, the remaining quantities, δx_{k+1}, &c., are completely independent, and therefore (see Art. 267) every coefficient in this equation must $= 0$. Thus, we obtain $3n-k$ equations involving the forces, that is, statical equations of condition. Combining these statical equations with the equations of connexion (1), we have finally $3n$ equations for the $3n$ co-ordinates of the particles. The elimination of the displacements from the equations, can, however, be exhibited in a more symmetrical and useful form.

Multiply the left sides of the equations (3) by $\lambda_1, \lambda_2, \dots \lambda_k$ in

order, these multipliers being undetermined quantities; then add all the results together, and finally equate to zero the coefficient of every displacement in the resulting equation. Thus we shall have the following $3n$ equations:—

$$\left.\begin{aligned}
X_1 + \lambda_1 \frac{dL_1}{dx_1} + \lambda_2 \frac{dL_2}{dx_1} + \ldots + \lambda_k \frac{dL_k}{dx_1} &= 0,\\
Y_1 + \lambda_1 \frac{dL_1}{dy_1} + \lambda_2 \frac{dL_2}{dy_1} + \ldots + \lambda_k \frac{dL_k}{dy_1} &= 0,\\
Z_1 + \lambda_1 \frac{dL_1}{dz_1} + \lambda_2 \frac{dL_2}{dz_1} + \ldots + \lambda_k \frac{dL_k}{dz_1} &= 0,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&
\end{aligned}\right\} \quad (5)$$

If from these equations we eliminate the k undetermined multipliers, we shall have $3n-k$ statical equations of condition, as before; and we see, as an essential characteristic of the Lagrangian method, that *the variations of the co-ordinates of all the particles of the system may be treated as independent quantities after we have taken the variations of the left-hand sides of all the equations of condition, multiplied each of these by an arbitrary multiplier, and included the products in one equation of Virtual Work.*

Now, this method of elimination has the advantage of discovering the geometrical forces, or forces arising from the connexions, of the system. For, suppose that we suppress the condition $L_1 = 0$; then the system will begin to move; but it may be kept at rest by applying a special force to each particle.

Let the components of the force applied to m_1 be X_1', Y_1', Z_1', those of the force applied to m_2, X_2', Y_2', Z_2', and so on for all the others. The equations of equilibrium of m_1 will then be

$$X_1 + X_1' + \lambda_2 \frac{dL_2}{dx_1} + \ldots + \lambda_k \frac{dL_k}{dx_1} = 0,$$

$$Y_1 + Y_1' + \lambda_2 \frac{dL_2}{dy_1} + \ldots + \lambda_k \frac{dL_k}{dy_1} = 0,$$

$$Z_1 + Z_1' + \lambda_2 \frac{dL_2}{dz_1} + \ldots + \lambda_k \frac{dL_k}{dz_1} = 0,$$

similar equations holding for the other particles.

Subtracting the left side each of these from that of the corresponding equation in (5), we have

$$X_1' = \lambda_1 \frac{dL_1}{dx_1},$$

$$Y_1' = \lambda_1 \frac{dL_1}{dy_1},$$

$$Z_1' = \lambda_1 \frac{dL_1}{dz_1}.$$

Hence $$X_1' : Y_1' : Z_1' = \frac{dL_1}{dx_1} : \frac{dL_1}{dy_1} : \frac{dL_1}{dz_1},$$

and $$\sqrt{X_1'^2 + Y_1'^2 + Z_1'^2} = \lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

If, now, all the co-ordinates involved in the equation $L_1 = 0$ are considered constant except x_1, y_1, and z_1, this equation will denote a surface on which the particle m_1 is constrained to lie, and

$$\frac{dL_1}{dx_1}, \quad \frac{dL_1}{dy_1}, \quad \frac{dL_1}{dz_1},$$

each divided by $\sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}$,

will be the direction-cosines of the normal to this surface at the point (x_1, y_1, z_1). It is evident, therefore, that the force required to keep the particle m_1 at rest, when the condition $L_1 = 0$ is suppressed, is a force acting normally to this surface, its magnitude being

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_1}\right)^2 + \left(\frac{dL_1}{dy_1}\right)^2 + \left(\frac{dL_1}{dz_1}\right)^2}.$$

In the same way the force required to keep m_2 at rest acts normally to the surface denoted by $L_1 = 0$ when x_2, y_2, z_2 are considered as the only variable co-ordinates in the equation, and the magnitude of this force is

$$\lambda_1 \sqrt{\left(\frac{dL_1}{dx_2}\right)^2 + \left(\frac{dL_1}{dy_2}\right)^2 + \left(\frac{dL_1}{dz_2}\right)^2}.$$

If the condition $L_2 = 0$ were suppressed, it follows in like manner that forces

$$\lambda_2 \sqrt{\left(\frac{dL_2}{dx_1}\right)^2 + \left(\frac{dL_2}{dy_1}\right)^2 + \left(\frac{dL_2}{dz_1}\right)^2}, \text{ \&c.,}$$

should be applied to the particles m_1, &c., in directions normal to the surfaces represented by the equation $L_2 = 0$ when the sole variables in it are the co-ordinates of m_1, &c., in succession. It is easy to see that

$$\lambda_1\left(\frac{dL_1}{dx_1}\delta x_1 + \frac{dL_1}{dy_1}\delta y_1 + \frac{dL_1}{dz_1}\delta z_1\right)$$

is equal to $F_1(\cos\alpha\,.\,\delta x_1 + \cos\beta\,.\,\delta y_1 + \cos\gamma\,.\,\delta z_1)$,

where F_1 is the force of connexion acting on m_1 in virtue of the condition $L_1 = 0$, and α, β, γ the direction angles of the normal to the surface denoted by $L_1 = 0$ when the co-ordinates of m_1 are regarded as the only variables in it.

Now, the multiplier of F_1 in this expression is evidently the projection of the displacement of m_1 along the normal to this surface. If this projection be denoted by δn, n being the length of the normal at the position of m_1 measured from some fixed point on the normal, we have

$$\lambda_1 \delta L_1 = F_1 \delta n,$$

in which the variation of L_1 has reference solely to the particle m_1.

The right-hand side of this equation at once identifies the term $\lambda_1 \delta L_1$ with the virtual work of an internal force, since $F_1 \delta n$ is explicitly such; and this force acts along the direction in which the function L_1 varies most rapidly (i.e. the normal to the surface denoted by the equation $L_1 = 0$).

Hence Lagrange habitually speaks of such a term as $\lambda \delta L$ in the equation of virtual work as 'the virtual moment of a force tending to vary the function L.'

Examples.

1. A number of heavy particles are attached at given intervals to a weightless string the extremities of which are fixed; investigate the circumstances of equilibrium (Funicular Polygon).

Let (a, b) be the co-ordinates of one of the fixed extremities, (x_1, y_1), (x_2, y_2),... the co-ordinates of the particles taken in order from this extremity, l_{01}, l_{12},... the lengths of the portions of the string between these points, and W_1, W_2,... the weights of the particles.

Then the equations of connexion of the system are

$$(a-x_1)^2 + (b-y_1)^2 = l_{01}^2,$$
$$(x_1-x_2)^2 + (y_1-y_2)^2 = l_{12}^2, \text{ \&c.}$$

Hence the Lagrangian equation of virtual work is

$$W_1\delta y_1+W_2\delta y_2+\ldots-\lambda_1\{(a-x_1)\delta x_1+(b-y_1)\delta y_1\}$$
$$+\lambda_2\{(x_1-x_2)(\delta x_1-\delta x_2)+(y_1-y_2)(\delta y_1-\delta y_2)\}+\ldots=0.$$

Equating to zero the coefficients of the several displacements,

$$\lambda_1(a-x_1)-\lambda_2(x_1-x_2)=0,$$
$$\lambda_2(x_1-x_2)-\lambda_3(x_2-x_3)=0,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$W_1-\lambda_1(b-y_1)+\lambda_2(y_1-y_2)=0,$$
$$W_2-\lambda_2(y_1-y_2)+\lambda_3(y_2-y_3)=0.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The first set of these equations evidently gives

$$\lambda_1(a-x_1)=\lambda_2(x_1-x_2)=\lambda_3(x_2-x_3)=\ldots=T,\text{ suppose,}$$

and by substituting in the remaining set,

$$\frac{b-y_1}{a-x_1}=\frac{y_1-y_2}{x_1-x_2}+\frac{W_1}{T},$$
$$\frac{y_1-y_2}{x_1-x_2}=\frac{y_2-y_3}{x_2-x_3}+\frac{W_2}{T}.$$

But $\dfrac{b-y_1}{a-x_1}$ is the tangent of the inclination of the portion l_{01} of the string to the horizon. Hence we have

$$\tan\theta_{01}=\tan\theta_{12}+\frac{W_1}{T},$$
$$\tan\theta_{12}=\tan\theta_{23}+\frac{W_2}{T},$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

as in Art. 35. Also the tension of the string joining (a, b) to (x_1, y_1) is $\dfrac{\lambda_1}{l_{01}}$ acting from the first point towards the second, and so on for the other tensions.

2. Deduce by the method of Lagrange the conditions of equilibrium of a system of three particles forming a rigid triangle, each particle being acted on by given forces.

Let (x_1, y_1, z_1) be the co-ordinates of one particle, and (X_1, Y_1, Z_1) the components of the force acting on it, with similar notation for the other two particles. Then, if l_{12}, l_{23}, l_{31} denote the sides of the triangle, the equations of connexion are

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=l_{12}^2,$$
$$(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2=l_{23}^2,$$
$$(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2=l_{31}^2.$$

Hence the Lagrangian equation of equilibrium is

$$X_1\delta x_1+Y_1\delta y_1+Z_1\delta z_1+\ldots+\lambda_{12}\{(x_1-x_2)(\delta x_1-\delta x_2)$$
$$+(y_1-y_2)(\delta y_1-\delta y_2)+(z_1-z_2)(\delta z_1-\delta z_2)\}+\ldots=0,$$

the undetermined multipliers being λ_{12}, λ_{23}, and λ_{31}.

Equating to zero the coefficients of the displacements, we have

$$X_1+\lambda_{12}(x_1-x_2)-\lambda_{31}(x_3-x_1)=0, \tag{1}$$

$$Y_1+\lambda_{12}(y_1-y_2)-\lambda_{31}(y_3-y_1)=0, \tag{2}$$

$$Z_1+\lambda_{12}(z_1-z_2)-\lambda_{31}(z_3-z_1)=0, \tag{3}$$

with similar equations for the other particles.

By addition, we have at once

$$X_1+X_2+X_3=0, \text{ or } \Sigma X=0,$$

$$Y_1+Y_2+Y_3=0, \text{ or } \Sigma Y=0,$$

$$Z_1+Z_2+Z_3=0, \text{ or } \Sigma Z=0,$$

which are the ordinary equations of translation.

Again, multiplying (1) by y_1 and (2) by x_1, and subtracting,

$$Y_1x_1-X_1y_1-\lambda_{12}(x_1y_2-y_1x_2)-\lambda_{31}(x_1y_3-y_1x_3)=0,$$

and by taking the similar equations for the other particles, and adding, we get $\qquad \Sigma(Yx-Xy)=0.$

Similarly, $\qquad \Sigma(Xz-Zx)=0,$

and $\qquad \Sigma(Zy-Yz)=0.$

These last three are the equations of moments, and they constitute, with the first three, six equations of equilibrium. Now these are all the conditions that can be obtained among the forces and co-ordinates. For if n particles be connected by k equations of condition, there are (Art. 269), $3n-k$ final equations. But here $n=3$, $k=3$, therefore $3n-k=6$. It is to be observed that the equations of equilibrium of any rigid body must be the same in number as those for three particles forming a rigid triangle, because if three points of a rigid body are determined in position, the position of the body is determined.

3. Show that the equations of equilibrium of a system subject to given conditions may be expressed as the vanishing of the differential coefficients of a single function of the co-ordinates of the system.

Suppose that

$$(X_1dx_1+Y_1dy_1+Z_1dz_1)+(X_2dx_2+Y_2dy_2+Z_2dz_2)+\dots,$$

or $\Sigma(Xdx+Ydy+Zdz), \equiv -d\Pi$ where Π is a function of the co-ordinates $x_1, y_1, z_1, x_2, y_2, z_2, \dots$ Then taking

$$U\equiv -\Pi+\lambda_1L_1+\lambda_2L_2+\dots,$$

where $L_1=0$, $L_2=0,\dots$ are the equations of condition, we shall have

$$\frac{dU}{dx_1}\equiv X_1+\lambda_1\frac{dL_1}{dx_1}+\lambda_2\frac{dL_2}{dx_1}+\dots+L_1\frac{d\lambda_1}{dx_1}+L_2\frac{d\lambda_2}{dx_1}+\dots.$$

But since the co-ordinates make $L_1=L_2=\dots=0$,

$$\frac{dU}{dx_1}\equiv X_1+\lambda_1\frac{dL_1}{dx_1}+\lambda_2\frac{dL_2}{dx_1}+\dots,$$

and comparing with equations (5), we see that the equations of equilibrium are $\dfrac{dU}{dx_1}=0,\ \dfrac{dU}{dx_2}=0,\dots\dfrac{dU}{dy_1}=0,\ \dfrac{dU}{dy_2}=0$, &c.

270.] Distinctive Feature of the Lagrangian Method. If the first method of eliminating the displacements described in the last article is adopted, we arrive at an equation such as (4) of that Article, from which the conditions of equilibrium are obtained by equating to zero the coefficients of the displacements. But in proceeding thus, we fail to obtain the values of the internal and geometrical forces of the system. Now these forces are, as we have seen, intimately related to the undetermined multipliers; and as these latter are found from the Lagrangian equations, it follows that—

The method of Lagrange gives not only the conditions of equilibrium, but also the internal forces of the system.

A single very elementary example will suffice to render this clear.

Two heavy particles of weights W_1 and W_2 are connected by a rigid rod, and each particle rests on a smooth inclined plane. The inclinations of the planes are i_1 and i_2 and their intersection is horizontal; find the position of equilibrium and the internal and geometrical forces.

Let the line of intersection of the planes be taken as axis of z, let the axis of y be vertical and that of x horizontal. Also let $(x_1\ y_1\ z_1)$ $(x_2\ y_2\ z_2)$ be the co-ordinates of the particles, and l the length of the rod connecting them. Then the equations of connexion are

$$y_1 - x_1 \tan i_1 = 0,$$
$$y_2 + x_2 \tan i_2 = 0,$$
$$(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 = l^2.$$

Hence the Lagrangian equation of equilibrium is

$$-W_1\delta y_1 - W_2\delta y_2 + \lambda_1(\delta y_1 - \tan i_1 . \delta x_1) + \lambda_2(\delta y_2 + \tan i_2 . \delta x_2)$$
$$+\tau\{(x_1-x_2)(\delta x_1 - \delta x_2) + (y_1-y_2)(\delta y_1 - \delta y_2) + (z_1-z_2)(\delta z_1 - \delta z_2)\} = 0,$$

λ_1, λ_2, and τ being the undetermined multipliers.

Equating to zero the coefficients of the separate displacements,

$$-W_1 + \lambda_1 + \tau(y_1-y_2) = 0,$$
$$-W_2 + \lambda_2 - \tau(y_1-y_2) = 0,$$
$$\lambda_1 \tan i_1 - \tau(x_1-x_2) = 0,$$
$$\lambda_2 \tan i_2 - \tau(x_1-x_2) = 0,$$
$$\tau(z_1-z_2) = 0.$$

From the last equation we have $z_1 - z_2 = 0$, which shows that both particles must lie in a vertical plane perpendicular to the line of intersection of the inclined planes.

If θ be the inclination of the line joining the particles to the horizon, the other equations give

$$(W_1+W_2)\tan\theta = W_1\cot i_2 - W_2\cot i_1,$$

$$\tau l = \frac{W_1 \sin i_1}{\cos(i_1-\theta)},$$

$$\lambda_1 = \frac{W_1\cos\theta\cos i_1}{\cos(i_1-\theta)},$$

$$\lambda_2 = \frac{W_2\cos\theta\cos i_2}{\cos(i+\theta)}.$$

The reader will easily perceive that τl is the tension of the rod, and $\lambda_1 \sec i_1$ and $\lambda_2 \sec i_2$ the reactions of the smooth planes. Thus we have the same values of the inclination of the rod and of the internal forces as we should have obtained by the ordinary statical methods.

Now suppose that the equation of virtual work is employed according to the first method; that is, let us write

$$W_1\delta y_1 + W_2\delta y_2 = 0,$$

$$\delta y_1 - \tan i_1 . \delta x_1 = 0,$$

$$\delta y_2 + \tan i_2 . \delta x_2 = 0,$$

$$(x_1-x_2)(\delta x_1-\delta x_2)+(y_1-y_2)(\delta y_1-\delta y_2)+(z_1-z_2)(\delta z_1-\delta z_2)=0,$$

and eliminate the displacements without employing undetermined multipliers. Then we obtain simply the equations

$$z_1-z_2=0,$$

$$(W_1+W_2)\tan\theta = W_1\cot i_2 - W_2\cot i_1,$$

which define the position of equilibrium, without giving the values of the unknown forces of the system.

271.] **Work.** If a force, R, acts at a point (x, y, z) which, from any cause, receives a small displacement whose projections on the axes of co-ordinates are dx, dy, dz, and if the components of R are X, Y, Z, the work actually done by the force is

$$Xdx + Ydy + Zdz. \qquad (1)$$

If a force P which is constant both in magnitude and line of action acts at a point, A, which from any cause is displaced through any distance, AB, along the line of action and in the sense of P, the whole amount of work done by the force is

$$P \times AB;$$

and if the displacement takes place in the sense opposite to that of P, the work done by P is $-P\times AB$.

If the force P is constant in magnitude and direction (but not line of action) while its point, A, of application is displaced along

any curve, AB (Fig. 254), the work done by the force (which is the integral of all the elements of work done during the passage) is

$$P \times \text{projection of } AB \text{ along the direction of } P.$$

As an instance, take the case of a heavy body of weight W whose centre of gravity occupies the point A initially. If the body is displaced along any curve or surface whatever, so that its centre of gravity finally occupies the position B, the work done by W is

$$W \times h,$$

where h is the excess of the height of A over that of B; so that W does positive work if B is below A, negative work if B is above A, and no work if A and B are at the same horizontal level. Similarly in Fig. 254, the working force being constant in magnitude and direction, if AD is perpendicular to P, no work is done on the whole in the passage from A to D.

If the working force, P, is constant in magnitude and variable in direction, while its point of application is at each instant moving along the line of action of P, the work done by P from one point A to another B is the product $P.s$, where s is the whole length of the path of the point of application between A and B. For instance, a constant pressure, P, exerted on the arm of a capstan.

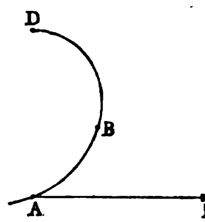

Fig. 254.

If the working force varies both in magnitude and in direction while its point of application describes any path between a point A and a point B, the total work must be obtained by taking the elementary work done by the force for a very small displacement of its point of application, and integrating this. We may at each point resolve the force into three components, so that the element of work is expressed by (1), and the total work done between A and B is

$$\int_B^A (Xdx + Ydy + Zdz), \tag{2}$$

the suffixes indicating the points between which the work is done.

The work done by a force whose point of application is displaced from any one position, A, to any other B, is often very

usefully represented graphically by means of a *Work Diagram.* If in any position P is the magnitude of the force, and dp the projection of the displacement of its point of application along the direction of P, the element of work is

$$P.dp,$$

and the whole work is the integral of this. Hence if we take two rectangular axes, Ox and Oy, and lay off, successively, along Ox the values of dp as they occur in the working of the force between A and B; and if perpendicularly to each of these elements we draw the corresponding value of P (as an ordinate), the extremities of these ordinates will trace out a curve whose area will represent the work done.

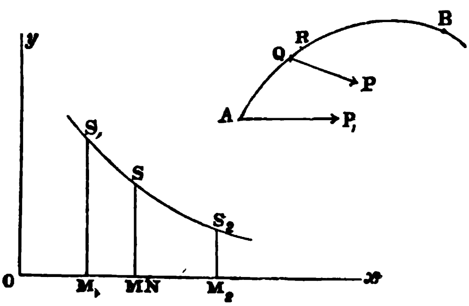

Fig. 255.

Thus, in Fig. 255, if P_1 is the magnitude of the working force at A, P_2 its magnitude at B, while P is its magnitude at any intermediate point, Q, we may take any point, M_1, on Ox at which to draw the ordinate P_1, and the distance $M_1 M$ will be the sum of the values of the projections, such as Qq, of the elements, QR, of arc along the corresponding directions of P between A and Q. We may, of course, choose the small arcs $QR,\ldots$ of such lengths that the elements, $MN,\ldots$ are all equal, i.e. dp may be taken as a constant element.

The expression $\int_B^A Pdp$ for the work done between A and B becomes then

$$\text{the area } M_1 S_1 S_2 M_2,$$

properly translated from square centimètres (suppose) into kilogramme-mètres, according to the scale of length on which force magnitude is represented in drawing the ordinates MS, and (generally) the diminished scale on which the projections Qq are represented by the elements MN.

If C.G.S. units are adopted, the unit of work is that done by a dyne in displacing its point of application through one centimètre in its own direction. This unit of work is called an *erg.*

Examples.

1. If one end of an elastic string is fixed while the other is drawn out through a given distance, find the work done by its tension, and the work diagram.

If l_0 is the natural length of the string, λ its modulus of elasticity, and l any stretched length which is productive of a tension T, we have $T = \lambda \frac{l-l_0}{l_0}$. For a small increment of length, dl, the tension does work equal to $-Tdl$; therefore disregarding the sign of the work, we may represent it by drawing the values of $l-l_0$ along Ox, so that OM is proportional to $l-l_0$; then at M we are to draw an ordinate, MS, proportional to T, and therefore proportional to OM. The locus of S is obviously a right line passing through O, and the work done by the tension for any amount of extension is represented by the area of a trapezium, affected with a negative sign.

The amount of work done by the tension in an extension from a length l_1 to a length l_2 is

$$-\frac{\lambda}{2l_0}(\overline{l_2-l_0}^2 - \overline{l_1-l_0}^2).$$

2. Another simple example of a work diagram is furnished by a gas enclosed in a cylinder fitted with a gas-tight piston, the gas expanding or contracting at a constant temperature.

In this case let us calculate the work done by the total pressure on the piston in the expansion of the gas by a given amount.

If P is the force exerted on the piston, and x, the distance of the piston, in any position, from the closed end of the cylinder, the law of Boyle and Mariotte gives

$$Px = \text{constant} = P_1 x_1,$$

where P_1 is the pressure in the first position and x_1 the distance of this position from the closed end.

The values of x being laid off along Ox, the extremities of the ordinates will trace out a rectangular hyperbola, and the area included between any portion of this curve, the ordinates at its extremities, and the axis of x, represents the work done by the pressure. The work done by the pressure from x_1 to x is

$$P_1 x_1 \log_e \frac{x}{x_1}.$$

3. In general, if a gas expands from a volume v_1 to a volume v_2, and if p is its intensity of pressure (or pressure per unit area), the work done by the gas against external resistance is

$$\int_{v_1}^{v_2} p\,dv. \qquad (a)$$

For, if at any time the gas is enclosed within a surface S, whose

element of area at any point is dS, the amount of pressure on this element is pdS; and if in a small expansion the element dS is driven out along the normal through a distance dn, the work done by the pressure on dS is $pdS.dn$; therefore for the small expansion of the whole volume enclosed by S the sum of the works done by the pressures on all its elements dS is (since p is constant throughout the gas), $p\int dSdn$; but $\int dSdn$ is the increase of volume of the whole gas for the small expansion considered, that is, dv; hence the work for this expansion is pdv, and therefore in the change from volume v_1 to volume v_2—the intensity of pressure, p, of course continuously varying—the work done is given by (a).

For example, if the gas changes *adiabatically*—i. e. so that no *heat* is conducted either into or out of it, while its *temperature* and intensity of pressure both vary—the relation between p and v is

$$pv^k = \text{constant}, \tag{b}$$

where k is about 1·408. In this case the curve whose abscissae and ordinates are the varying values of v and p is asymptotic to both axes—like the rectangular hyperbola

$$pv = \text{constant}, \tag{c}$$

which represents the relation between p and v when the expansion is unaccompanied by change of temperature—but it approaches the axis of volumes more rapidly than the hyperbola.

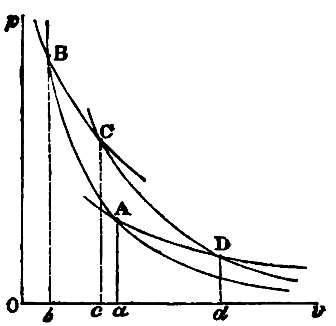

Fig. 256.

The curves obtained by varying the constant in (b) are called *adiabatics*, while those given by (c) are called *isothermals*. Thus, let A be a point whose co-ordinates Oa and aA are v_1 and p_1, respectively; then the curve whose equation is

$$pv^k = p_1 v_1^k$$

is AB, while the curve (rectangular hyperbola) whose equation is

$$pv = p_1 v_1$$

is AD. The co-ordinates of the points on AB between A and B represent the states of the gas as to volume and intensity of pressure in the adiabatic transformation from state A to state B.

A gas contained in a cylinder with a gas-tight piston can be transformed adiabatically and isothermally, successively, to any extent in the following manner. Suppose the base of the cylinder to be made of thin polished copper or silver. (Theoretically this base is to have perfect thermal conductivity, i.e. any heat applied to the outside is instantly transmitted to the inside, any difference of temperature between the outer and the inner surfaces of the base being at once annulled. Thin polished silver or copper will be an approximation. With such a base we are to imagine heat as flowing with no resistance into or out of the cylinder.)

Let the piston and all the rest of the cylinder be made of an infinitely bad thermal conductor, so that no heat can enter or leave the cylinder anywhere except through the base.

To produce adiabatic transformation. Place the cylinder with its base on a slab which is an infinitely bad thermal conductor, and do work on the gas by pressing down the piston. No heat can get into the cylinder by conduction from without, and none can leave it. Moreover, of the work thus done by the piston on the gas a portion goes to increase the energy of motion of its molecules, and the remainder is used in doing work against the (repulsive) forces existing between these molecules. From an experiment of Joule's, however, it appears that these molecular forces are non-existent; and subsequent experiments by Joule and Thomson show that, though this is not perfectly true for all gases, it is so nearly true, that the work absorbed in overcoming these molecular forces may be quite neglected.

The result, then, is that the work done on the gas goes wholly to increase its heat, and therefore its temperature. [Observe, this is not a contradiction of our supposition that no heat is communicated to it by conduction from any external source.]

If, instead of compression by means of the piston, the gas is allowed to expand and drive the piston before it, its temperature falls in an adiabatic transformation.

To produce isothermal transformation. Place the cylinder with its base on a very large reservoir of heat—so large that the volume of the gas is negligible in comparison—and let the temperature of the heat in the reservoir be the same as that of the heat of the gas. Allow the piston to be driven by the gas. The effect of even the smallest expansion would be a lowering of the temperature inside the cylinder, but as the base is an infinitely good conductor, the inequality of temperature inside and outside is instantly annulled by a flow inwards of heat from the reservoir, the temperature of which (on account of its capacity) suffers no sensible diminution. Thus the temperature inside the cylinder remains constant all through the expansion.

The piston might also be pressed down so as to compress the gas, the instantaneous effect being a rise of temperature, which is instantly annulled by the flow of heat from the gas into the reservoir.

The theoretical processes here described are those which are postulated in the working of *Carnot's Engine*, the theory of which is fundamental in Thermodynamics (see Clerk Maxwell's *Theory of Heat*, or almost any work on Physics).

Starting with the state represented in Fig. 256 by the point A, let the following cycle of operations occur:—adiabatic compression represented by the adiabatic AB, until state B is reached; isothermal expansion represented by BC, the gas receiving heat at constant temperature, and doing external work by driving the piston before it, until state C is reached; adiabatic expansion represented by CD,

the gas driving out the piston and doing external work, while its temperature falls and it receives no heat, until the temperature which it had originally (at A) is reached; finally, isothermal compression represented by DA, the piston being forcibly driven down until the original state (A) is reached.

It is required to calculate the whole amount of positive work done by the gas. This work is obviously the areal sum

$$-AabB+BCcb+CDdc-DdaA,$$

where a, b, c, d are the feet of the ordinates of A, B, C, D. Let the equation of

$$AD \text{ be } pv = m; \quad BC \text{ be } pv = m';$$
$$AB \text{ be } pv^k = n; \quad CD \text{ be } pv^k = n'.$$

Then the area $AabB = \frac{n}{k-1}\left(\frac{1}{v_2^{k-1}} - \frac{1}{v_1^{k-1}}\right)$, where v_1 and v_2 are the abscissae of A and B. But $v_2^{k-1} = \frac{n}{m'}$, and $v_1^{k-1} = \frac{n}{m}$; therefore this area $= \frac{m'-m}{k-1}$, which value is also that of $CDdc$. Hence the external work done by the gas is

$$\frac{m'-m}{k-1}\log_e\frac{n'}{n},$$

and this is also, of course, the area of the figure $ABCD$ included between the two isothermals and the two adiabatics.

272.] **Static Energy, or Potential Work of a Force System.** If the point of application of a force whose components are X, Y, Z occupies at any instant a position which we may denote by (p), and if (p_0) denotes any other position which it might occupy, the amount, Π, of work which the force can do in the displacement from (p) to (p_0) is given by the equation

$$\Pi = \int_{(p)}^{(p_0)} (Xdx + Ydy + Zdz). \tag{1}$$

The amount of work which the force can do from the present position (p) *to the supposed position* (p_0) *is called the* Potential Work *of the force.*

In the same way, if any number of forces act on any system of particles, $m_1, m_2, \ldots$, and if the present system of positions of these particles, or their present configuration, is denoted by (p), while another configuration, or system of positions which they might occupy, is denoted by (p_0), the whole amount of work

which the forces can do in the motion from the present to the contemplated position is given by the equation

$$\Pi = \Sigma \int_{(p)}^{(p_0)} (Xdx + Ydy + Zdz), \tag{2}$$

where Σ denotes a summation of the works done on all the particles. The configuration denoted by (p_0) may be taken arbitrarily. We shall speak of it as *the configuration of reference.* Here, as before, Π is *the potential work of the forces of the system.*

Defining the term *Energy* to mean *capacity for doing work*, we may speak of the Potential Work of a force system as its *Static Energy**.

If the particles do not form a rigid body, but can alter their relative distances; and if, moreover, they exert on each other forces, either of attraction or of repulsion, the work done by the internal forces in the change of configuration must, of course, be included in the Static Energy of the system; so that if Π_i and Π_e are the potential works of its internal and external forces, respectively, the total Static Energy of the system is

$$\Pi_i + \Pi_e.$$

Any material system—whether it consists of particles at finite distances from each other, each acted upon by some external force and also by attractions from neighbouring particles, or particles at infinitesimal distances (as in the case of a bent spring, a membrane, or an elastic string)—may occupy several different configurations successively and finally return to its original configuration (p). If when it does return to its original configuration, the Static Energy of its force-system (internal and external forces included) returns also to its original value, the system is said to be *Conservative.* The consideration of such a system is of the greatest importance.

Any material system will be conservative when for any small derangement of the particles the work done by the external forces is the differential of a single-valued function of the co-ordinates of the particles, and the internal forces are functions only of the mutual distances of the particles, and are directed in the lines joining them.

* It is usually spoken of as 'Potential Energy'—an illogical term which, as has been pointed out by an able writer, expresses 'a double remotion from actuality.'

For if the co-ordinates of the particles are (x_1, y_1, z_1), &c., and the external forces (X_1, Y_1, Z_1), &c., the work of the external forces for any small derangement is

$$X_1\, dx_1 + Y_1\, dy_1 + Z_1\, dz_1 + X_2\, dx_2 + Y_2\, dy_2 + Z_2\, dz_2 + \ldots,$$

and if this $= d\phi\,(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$, the Static Energy of these forces is $\phi_0 - \phi$, where ϕ_0 is the value of ϕ when the co-ordinates of the configuration (p_0) are substituted; and if ϕ is not a multiple-valued function—such as $\tan^{-1}\frac{y_1}{x_1}$—it is obvious that the Static Energy of the external forces must always be the same whenever the system has the same configuration.

Again, if the internal force between m_1 and m_2 is expressed as $f(r_{12})$, where r_{12} is the distance between them, and if it is directed in the line joining them, the element of work of this force is $\pm f(r_{12}) \,.\, dr_{12}$, according as the force is repulsive or attractive. Hence if $f(r_{12}) \,.\, dr_{12} = d\psi(r_{12})$, the Static Energy of the internal forces in any configuration is, by summation for all the particles,

$$\pm\,[\Sigma\psi(r)_0 - \Sigma\psi(r)],$$

which is manifestly the same whenever the configuration is the same.

For example, an elastic rod bent and twisted in any way, but not to such an extent as to alter sensibly its constants of elasticity, will be an example of a conservative system, if, moreover, the bending and twisting are not accompanied by heating. The effect of such heating might be to alter its various elastic constants, in such a manner that if it returns to its original configuration, the amount of work required to produce a given deformation either by bending or by twisting, would not be the same as it was originally to produce exactly the same deformation.

If the deformation is produced slowly, the heating effect is avoided, and the system is conservative.

By definition, if work, W, is done by external agency on a conservative system to change its configuration from (p) to (p'), the system will give back exactly the same amount, W, of work against external resistance in returning from (p') to (p).

A simple example of a non-conservative system is furnished by a heavy particle on a rough inclined plane of inclination i.

To raise the particle through a given vertical height, h, by an up-plane force an amount of work equal to $wh\ (1+\mu \cot i)$ must be expended; while if the particle is allowed to slide down to its original position, it will give out only the amount $wh\,(1-\mu \cot i)$, and would give out none if $\mu =$ or $> \tan i$.

In all such cases—i.e. cases in which friction comes into play—a part of the work expended on the system in changing its configuration is transformed into heat, which is speedily lost to the system; and, in general, if any machine, or combination of machines, transforms a portion of the work done on it into heat, it cannot restore even so much of the work as has not been thus transformed, i.e. it is non-conservative.

273.] **Stability and Instability of Equilibrium.** When a rigid body, or any material non-rigid system, in equilibrium under the action of given forces is slightly disturbed from its position, it will not, in general, be in equilibrium in the new position. Now the effect of all the forces in play in the new position may be either to drive it back to the original position, or to deviate it still further. In the former case the equilibrium is *stable*, and in the latter *unstable*.

As an example for the case of a rigid body, suppose a heavy bar, AB, moveable round a smooth horizontal axis fixed through the end A. If the rod is placed in a vertical position, it will be in equilibrium; but if the end B is vertically *above* A, a slight displacement will cause the rod to fall from this position; while if the end B is *below* A, and the rod is slightly displaced, it will return to its position of equilibrium.

As an example for a non-rigid system, take the case of an indiarubber ring on an umbrella handle. If the substance of the ring is rotated round the circle formed by the centres of all its normal sections through an angle which is constant all through the ring, one configuration of equilibrium is obtained when this angle of rotation is π, i.e. when the ring is turned inside-out. But this configuration is, of course, unstable, the slightest disturbance causing the ring to return to its natural state. On the other hand, the natural state of the ring on the handle is a stable configuration of equilibrium.

274.] **Universal Criterion of Stability and Instability.** The determination of the nature of the equilibrium of any system—i.e. its stability or instability—is a question belonging to

Kinetics. The conditions as regards constraints and connections of parts of the system with each other will enable us to express any possible configuration of the system in terms of a certain number of independent variables, $q_1, q_2, q_3, \ldots$, which may be described as 'co-ordinates' of the system, by an extension of the usual employment of this term. For example, suppose the system to consist of two particles, B and C, which are connected by an inextensible string, while another inextensible string, BA, is attached to B, and the system is suspended vertically by fixing the end A of the second string. In this case, supposing the displacements to be confined to a given vertical plane, if we imagine any configuration satisfying the conditions of the system, i.e. one in which the distances AB and BC are each constant, such a configuration is obtained by deviating AB from the vertical through any angle, θ, and then deviating BC from the vertical through any angle, θ', these two angles being entirely independent of each other. The configuration of the system, then, depends on the two independent variables θ and θ', which are its 'co-ordinates.'

If the displacements of the particles are not confined to any vertical plane, AB can be deviated through an angle θ from the vertical, and rotated (after the manner of a conical pendulum) round the vertical through an angle ϕ; and BC can be similarly displaced through angles θ' and ϕ'; so that there are *four* generalised co-ordinates $(\theta, \theta', \phi, \phi')$ of this system in the most general case of its displacement.

Such variables are usually called the *generalised co-ordinates* of the system, and they determine the number of degrees of freedom of the system—this being equal to the number of the generalised co-ordinates.

The kinetical process which determines whether the equilibrium of the system is stable or unstable consists in supposing each of the generalised co-ordinates, q, to receive any small increment, Δq, and then, from the equations of motion of the system, expressing each Δq as a function of the time. If the value of Δq is a periodic function of the time, the magnitude of Δq will oscillate between infinitely narrow limits, and the equilibrium of the system will be stable; while if any of the displacements Δq involves the time in a non-periodic form of the type e^t, this displacement increases indefinitely, and the equilibrium is unstable.

The result is this—*If for any possible small displacement of the system from its configuration of equilibrium there would be positive work done by the acting forces, both external and internal, the configuration is unstable; while if for every possible small displacement the sum total of the works of these forces is negative, the configuration is stable;* in other words, *the system will be in* **stable equilibrium,** *if the* **Static Energy** *of the system, i.e., the* **Potential Work** *of its forces (internal and external) is a* **minimum,** *and in unstable equilibrium if this potential work is a maximum.*

This fundamental result we shall assume, referring the reader for the proof to Lagrange's *Mécanique Analytique*, 6th section of the *Dynamique*, p. 320; to Thomson and Tait's *Natural Philosophy*, Arts. 291, &c.; and to Laurent's *Traité de Mécanique Rationnelle*, vol. ii. p. 222, where an extremely concise proof by Dirichlet is given.

We shall revert to the proof of this principle in the next Article.

275.] **Work Coefficients.** When the rectangular co-ordinates (x_1, y_1, z_1), &c., of the points of application of the forces of the system are all independent, since

$$-d\Pi = X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1 + X_2 dx_2 + \dots, \qquad (1)$$

we see that the differential coefficient of the Potential Work (with sign changed) with respect to any co-ordinate is the corresponding component of force. Thus $-\dfrac{d\Pi}{dx_1} = X_1$, &c. But if the co-ordinates are not all independent, but expressible in terms of a number, k, of independent variables, $q_1, q_2, \dots q_k$, this is no longer true. Expressing the co-ordinates $x_1, y_1, z_1, \dots$ in terms of the q's, equation (1) for the element of Potential Work assumes the form

$$-d\Pi = Q_1 dq_1 + Q_2 dq_2 + \dots + Q_k dq_k, \qquad (2)$$

in which the coefficients $Q_1, Q_2, \dots$ may be of the dimensions either of *force* or of *couple*, according to the nature of the generalised co-ordinates $q_1, q_2, \dots$. In all cases each term, $Q_1 dq_1$, in (2) is an elementary work, so that if q_1 is a *linear* co-ordinate, like x_1, the coefficient Q_1 will be of the dimension of *force*; but if q_1 is an *angle*, Q_1 will be of the dimension of *couple*.

Take, for example, the case of two coplanar forces, P_1 and P_2, acting at the ends, A and B, of a line of constant length, a, and

consider only displacements in the plane of the forces. The generalised co-ordinates of the system may be taken as the rectangular co-ordinates (x, y) of A, and the angle, θ, which AB makes with the axis of x. If (x', y') are the co-ordinates of B, we have $x' = x + a\cos\theta$; $y' = y + a\sin\theta$, and, the components of P_1 and P_2 being, respectively, (X_1, Y_1) and (X_2, Y_2), we have

$$-d\Pi = (X_1 + X_2)\,dx + (Y_1 + Y_2)\,dy + a\,(Y_2\cos\theta - X_2\sin\theta)\,d\theta,$$

in which the coefficients of dx and dy are of the dimensions of force, while that of $d\theta$ is of the dimensions of couple.

The coefficients $Q_1, Q_2, \ldots$ in (2) are sometimes spoken of as 'generalised components of force.' This expression is very objectionable on more grounds than one; but we fall into no error if we describe them as *Work Coefficients*. Thus Q_1 is the q_1- rate at which the system does work if the other independent variables, $q_2, \ldots q_k$, are all kept constant and q_1 alone allowed to vary; and it does not appear to be possible to specialise the meanings of the Q's any further—i.e. to give a rule applicable to all cases for localising $Q_1, Q_2, \ldots$ as forces or couples at particular points or round particular axes in the system.

Since in a position of equilibrium $d\Pi$ is zero for all possible displacements, in such a position we must have

$$Q_1 = 0, \quad Q_2 = 0, \ldots Q_k = 0. \tag{3}$$

Now the fundamental principle of last Article, that the Potential Work of the system of forces, both internal and external, is a minimum in a configuration of stable, and a maximum in a configuration of unstable, equilibrium cannot be inferred from the vanishing of all the first differential coefficients $Q_1, Q_2, \ldots$. For, since Π is a function of several independent variables, k in number, there are $k-1$ additional independent conditions that Π should be either a maximum or a minimum.* In a particular case, however, the truth of the principle can be seen without the general kinetical investigation. This case is that in which the material system has one degree of freedom, i.e. when its position depends on a single variable, q. Here, since $\frac{d\Pi}{dq} = 0$ in the position of equilibrium, it follows that Π is, in general, either a maximum or a minimum; and it is easy to see that the maximum belongs to instability. For, if the equilibrium is unstable, the

* Williamson's *Diff. Cal.*, Note 2.

system will require positive work to be done on it by an external agent to resist the growth of the displacement dq; that is, the forces (internal and external) of the system must during the displacement be doing positive work—resisting the positive work which the external agent is applying; in other words, in leaving the position of equilibrium, the Static Energy of the given system is diminished. Clearly, then, the maximum value of Π corresponds to instability.

276.] **Maximum or Minimum height of the Centre of Gravity.** When gravity is the only external force, besides the reactions of smooth fixed surfaces, acting on a material system, and when for any change of its configuration its internal forces (such as mutual reactions at the contacts of smooth parts) do no work, the Potential Work of the forces is simply

$$W.\bar{z},$$

where W is the total weight of the system and $\bar{z}$ is the height of its centre of gravity above any horizontal plane which is assumed as the *reference position* (Art. 272) of the centre of gravity.

For, let $w_1, w_2, \ldots$ be the weights and $z_1, z_2, \ldots$ the heights of the centres of gravity of the various separate bodies, or particles, of the system. Then the virtual work of the system for any small displacements is $-w_1 dz_1 - w_2 dz_2 - w_3 dz_3 \ldots$; hence*

$$d\Pi = w_1 dz_1 + w_2 dz_2 + \ldots = W.d\bar{z},$$

$$\therefore \quad \Pi = W.\bar{z},$$

the reference level being taken as that from which $\bar{z}$ is measured.

Now the maximum value of Π will occur when $\bar{z}$ is greatest; hence *when the centre of gravity of any system of bodies is in the lowest position that it can occupy consistently with the geometrical conditions of the system, that system is in a position of stable equilibrium; and when its centre of gravity is in the highest position, the system is in a position of unstable equilibrium.*

Unless the position of the system depends on a single variable, we cannot assert conversely that a position of equilibrium is one in which the height of the centre of gravity is either a maximum or a minimum.

If any bodies of the system rest on *rough* curves or surfaces,

---

* This assumes that none of the geometrical forces required for a position of equilibrium are infinite; for the term $\lambda \delta L$ cannot be assumed to vanish, even though $\delta L = 0$, if λ is infinite.

the equation of virtual work will involve the reactions of these curves or surfaces for displacements along them. Hence we have no longer the equation $\overline{W}.\delta\bar{z} = 0$, and the principle of maximum or minimum height of the centre of gravity does not hold.

Even when the position depends on one variable, it may happen that in a position of equilibrium the height of the centre of gravity is neither a maximum nor a minimum. Take, for example, the case of a heavy particle placed at a point of inflexion on a smooth curve in a vertical plane, the tangent at the point being horizontal. The particle is evidently in equilibrium, since for a small displacement $P\delta z$ is zero, P being the weight and z the height of the particle. But z is neither a maximum nor a minimum, and the equilibrium, accordingly, is stable for a small displacement along the upper part of the curve, and unstable for a displacement along the lower part.

When the system has only one degree of freedom, the centre of gravity describes, in all positions of the system compatible with the given conditions, a curve which is sometimes very easily found. In the position of equilibrium the centre of gravity will be the point of contact of a horizontal tangent to this curve, and in this manner we can most readily perceive the nature of the equilibrium of the body.

When the system has more than one degree of freedom, it may happen that its centre of gravity is constrained, in all displacements compatible with the connexions, to describe a fixed *surface*. In this case the position of equilibrium will be one in which the tangent plane to this surface at the centre of gravity is horizontal; and if the surface lies entirely below the tangent plane in the neighbourhood of the point of contact, the equilibrium will be unstable, as in the case of a curve; if the surface lies above the tangent plane, the equilibrium will be stable; and if the tangent plane intersects the surface in a real curve in the neighbourhood of contact, the equilibrium will be stable for some displacements and unstable for others.

277.] **Continuous Equilibrium.** If in all positions of the system, compatible with the geometrical conditions, the statical equation

$$d\Pi = 0$$

is satisfied, every position is one of equilibrium. Writing down this equation in all positions and adding, the left sides of the

equations thus obtained is evidently the same thing as integrating it. Hence if all positions of the system are positions of equilibrium, the applied forces must satisfy the equation

$$\Pi = \text{constant}.$$

In the particular case of a heavy system under the action of gravity alone, Π is $W.\bar{z}$; therefore if a system be continuously in equilibrium under the action of gravity, the centre of gravity of the system for all displacements compatible with the conditions moves in a fixed horizontal plane, or, in other words, *maintains a constant height.*

EXAMPLES.

1. A heavy beam, AB (Fig. 121, Art. 104), rests on two smooth inclined planes; find the nature of its equilibrium.

It is very easy to prove that if the right line AB moves between two fixed right lines, OA and OB, the given point G on AB describes an ellipse whose equation with reference to OA and OB as axes of x and y is

$$\frac{x^2}{b^2} + 2\frac{xy}{ab}\cos(a+\beta) + \frac{y^2}{a^2} = 1.$$

The centre of this ellipse is the point O. In the position of equilibrium G is the point of contact of a horizontal tangent to this ellipse. Now two such tangents can be drawn, one above the intersection of the inclined planes and the other below it. There are, therefore, two positions of equilibrium; that with which we were concerned in the example of Art. 104 is obviously the position in which G is at a maximum height, and it is, therefore, *unstable*; the other requires the planes to be prolonged below their line of intersection, and as it also requires the reactions of the planes to assume impossible directions, it is physically impossible. It would, however, be possible if the planes were replaced by smooth fixed rods to which the extremities of the beam are attached by rings. The second position of equilibrium would then be *stable*.

The impossibility in a certain case of any position of equilibrium, except one of continuous contact with either plane, which has been signalized in Art. 104, is now easily explained. It occurs when the point of contact of the horizontal tangent to the ellipse locus of G falls underneath the plane (a) or the plane (β), so that it is not a possible position of G.

The problem may be solved by a purely analytical method. If z is the height of the centre of gravity of the beam, it will be easily found that in the position of equilibrium

$$\frac{d^2 z}{d\theta^2} = -\frac{\sin a \sin\beta\cos\theta}{(a+b)\sin(a+\beta)}\left\{(a+b)^2 + (a\cot a - b\cot\beta)^2\right\}.$$

2. Two given points of a body rest in contact with two smooth inclined planes; show that the equilibrium of the body is unstable.

We know that if two vertices of a *given* triangle move along two fixed right lines, the locus of the third vertex is an ellipse whose centre is the intersection of the given lines.

Hence, if we consider a given triangle in the body to be formed by the centre of gravity and the two points which are in contact with the planes, we see that the locus of the centre of gravity is an ellipse whose centre is at the intersection of the inclined planes. Now in the position of equilibrium the centre of gravity is the point of contact of a horizontal tangent to this ellipse. Hence the only possible position of equilibrium is one in which the height of the centre of gravity is a maximum; therefore the equilibrium is unstable; and if, as explained in the last Example, the point of contact of the tangent falls underneath either plane, the only position of equilibrium of the body is one of continuous contact with one of the planes. The student will find several particular examples of this problem in Walton's *Mechanical Problems* (pp. 164, &c.), where the solutions are analytical.

3. A heavy body has two plane surfaces, CP and CQ (Fig. 257), which rest against two smooth fixed pegs, P and Q, the line PQ making an angle with the horizon; show that the positions of equilibrium are determined by drawing horizontal tangents to a Limaçon.

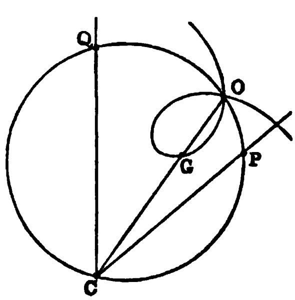

Fig. 257.

The centre of gravity and the pegs must lie in one vertical plane, which is that of the figure. Since P and Q are fixed points and the angle at C between the plane faces is constant, the circle described round the triangle PCQ is fixed in space. Again, let G be the centre of gravity of the body. Then since CG and CP are lines fixed in the body, the angle GCP is given; and if CG meet the circle in O, the point O is fixed in space; also the distance CG is given.

Hence in all positions of the body—i.e. in all positions of C on the circle—the centre of gravity is found by drawing the line OC from O to the circumference of the circle, and taking a constant length, CG, on this line. The curve deduced in this way from a circle is a Limaçon, which is, therefore, the locus of the centre of gravity.

A particular example has been already discussed in p. 149, Vol. I.

4. A heavy plane body of any shape is suspended from a smooth peg, fixed in a vertical wall, by means of a string of given length, the extremities of which are attached to two fixed points in the body. Determine the nature of the equilibrium.

This problem, so far as the positions of equilibrium are concerned, has been already discussed (Ex. 11, p. 152, Vol. I). We propose here

to show that there are two positions of stable and one position of unstable equilibrium. In the figure of the Example referred to, the point of contact of GP_3 with the evolute is between G and P_3; the point of contact of GP_1 is between G and P_1; and the point of contact of GP_2 is on P_2G produced. Now it is easy to see that GP_3 is a line of maximum length drawn from G to the ellipse. For, let Q be a point on the ellipse close to P_3, and let QC be the normal at Q. Then C is the centre of curvature, and therefore the point of contact of GP_3 and the evolute. Hence $CP_3 = CQ$, therefore $GP_3 = GC + CQ$, which is $> GQ$, therefore $GP_3 > GQ$, and GP_3 is, therefore, a maximum.

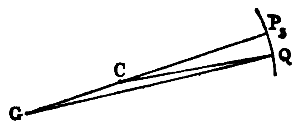

Fig. 258.

In the same way GP_1 is a maximum and GP_2 a minimum distance of G from the ellipse.

Hence, in the positions of equilibrium, GP_1 and GP_3 are maximum distances of the centre of gravity from the peg. The positions in which these lines are vertical are, therefore, positions of stable equilibrium. And since GP_2 is a minimum depth of G, the position in which GP_2 is vertical is one of unstable equilibrium.

5. To find the nature of the equilibrium of the beam in Example 7, p. 176, Vol. I.

Take any position of the beam (in which, of course, the lines GW, AR, and PS (p. 148, Vol. I) do not meet in a point). Then, if y is the ordinate of P, the point of contact of the beam and the curve, referred to a fixed horizontal axis, the ordinate of G will be

$$y + (GA - PA)\cos\theta,$$

or

$$y + a\cos\theta - x\cot\theta.$$

Denoting this by $\bar{y}$, we have

$$\frac{d\bar{y}}{d\theta} = \frac{dy}{d\theta} - a\sin\theta + \frac{x}{\sin^2\theta} - \cot\theta \cdot \frac{dx}{d\theta}.$$

Now $$\frac{dy}{dx} = \cot\theta, \quad \therefore \quad \frac{dy}{d\theta} - \cot\theta\frac{dx}{d\theta} = 0.$$

Hence $$\sin^2\theta\frac{d\bar{y}}{d\theta} = -a\sin^3\theta + x.$$

Differentiating this, and remembering that in the position of equilibrium $\frac{d\bar{y}}{d\theta} = 0$, we have

$$\sin^2\theta\frac{d^2\bar{y}}{d\theta^2} = \frac{dx}{d\theta} - 3a\sin^2\theta\cos\theta. \tag{1}$$

Again, since $\cot\theta = \frac{dy}{dx}$, we have

$$-\operatorname{cosec}^2\theta\frac{d\theta}{dx} = \frac{d^2y}{dx^2}.$$

But if ρ is the radius of curvature of the curve at P,

$$-\frac{1}{\rho}=\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\sin^3\theta\frac{d^2y}{dx^2}.$$

Therefore $\frac{d\theta}{dx}=\frac{1}{\rho\sin\theta}$, and (1) gives

$$\sin\theta\frac{d^2\bar{y}}{d\theta^2}=\rho-3a\sin\theta\cos\theta$$
$$=\rho-3PO.$$

Hence, since $\sin\theta$ is necessarily positive, $\frac{d^2\bar{y}}{d\theta^2}$, will be positive, and $\bar{y}$ therefore a minimum, if $\rho>3PO.$

The equilibrium will therefore be stable or unstable according as $\rho>$ or $<3PO.$

To arrive at this result, it would have been sufficient to demonstrate it for a circle, which is very easily done. The curve in the neighbourhood of P may be replaced by the circle of curvature at this point.

6. Prove geometrically that the equilibrium of the beam in Example 2, p. 145, Vol. I, is stable.

7. Two uniform heavy rods freely jointed together at a common extremity rest on a smooth parabola whose axis is vertical and vertex upwards; find the position of equilibrium.

Ans. Let the weights of the rods be P and Q, their lengths $2a$ and $2b$, and let them make angles θ and ϕ, respectively, with the vertical in the position of equilibrium; then these angles are determined from the equations

$$Pa\sin^3\theta+(P+Q)m\cot\phi=0,$$

$$Qb\sin^3\phi+(P+Q)m\cot\theta=0,$$

$4m$ being the latus rectum of the parabola.

[Taking the tangent at the vertex as axis of y, the abscissa of the point of intersection of two tangents, $y=tx-\frac{m}{t}$ and $y=t'x-\frac{m}{t'}$, is $-\frac{m}{tt'}$. Hence

$$(P+Q)\bar{x}=Pa\cos\theta+Qb\cos\phi+(P+Q)m\cot\theta\cos\phi.$$

Then $\bar{x}$ is to be a maximum or minimum.]

8. A heavy uniform rod, AB, moveable about a fixed horizontal axis at A, has its end B connected with a string which, passing over a smooth pulley at a point C vertically above A, sustains a given weight which rests on a smooth inclined plane passing through C. Find the positions of equilibrium, and the nature of each.

Ans. Let W and $2a$ be the weight and length of the rod; P the weight on the plane whose inclination to the horizon is i; $2c$ the distance AC, and θ the inclination of the rod to the vertical. Then,

if $(c-a)\,W < 2Pc\sin i$, there will be three positions of equilibrium defined by the equations

$$\theta = 0,\ \cos\theta = \frac{W^2(a^2+c^2)-4P^2c^2\sin^2 i}{2\,ac\,W^2},\ \text{and}\ \theta = \pi.$$

The first and last positions are stable and the intermediate one is unstable.

If $(c-a)\,W > 2Pc\sin i$, there is no intermediate position, and the first and last positions are unstable and stable respectively.

9. One end of a beam rests against a smooth vertical plane, and the other on a smooth curve in a vertical plane; find the nature of the curve so that the beam may rest in all positions.

Ans. An ellipse whose axis major is the horizontal line described by the centre of gravity of the beam, the axis minor lying in the vertical plane.

10. A uniform heavy rod rests inside a smooth fixed sphere whose diameter is equal to the length of the rod. In all positions of the rod its centre of gravity is fixed; hence the rod should rest in all positions; but, except in the vertical position, it is impossible that the acting forces can give equilibrium. Explain this.

(See note, p. 127.)

11. A uniform rod rests in all positions with its extremities on two smooth curves in a vertical plane; given the equation of one, find that of the other.

Ans. Let the axis of y be vertical, $2a$ the length of the rod, h the constant height of the centre of the rod, and $x = \phi\,(y)$ the equation of one curve; then the equation of the other will be

$$x = \phi\,(2h-y) - 2\sqrt{a^2-(h-y)^2}.$$

12. Find the general equation of a smooth curve (in a vertical plane) on which if the ends of a uniform rod are placed, the rod will rest in all positions.

Ans. If the line described by the centre of gravity is axis of x, the equation is the form $[\phi(y^2)+x]^2+y^2=a^2$, where $2a =$ length of rod, and $\phi(y^2)$ is a function which does not change sign with y.

13. Investigate the equilibrium of the sphere and cone each resting on a smooth inclined plane, they being also in contact with each other, as in Example 5, p. 206, Vol. I.

Their positions being varied in any way, subject to the condition of contact, it is easy to prove that the locus of their common centre of gravity is a right line. If this line is not horizontal, it is impossible to have $d\bar{y} = 0$, and therefore, *in general*, there is no position of equilibrium in which each body is in contact with only one plane. If the line is horizontal, all positions are positions of equilibrium.

Taking horizontal and vertical lines through O as axes of x and y, respectively, and taking OA $(=\xi)$ as the single variable which determines the configuration of the system, we find that $(W+W')\,\bar{y}$ is the sum of a constant and the term

$$[W \sin a - W' \frac{\sin a'}{\cos \gamma} \cos(a + a' - \gamma)] \times \xi;$$

so that $\bar{y}$ will be constant if equation (3), in the example referred to, is satisfied.

14. Of all curves that can be drawn through two given points, A and B, and having the same length, determine that one whose revolution round the line AB generates a surface of maximum area.

Ans. A Common Catenary. For, imagine AB to be placed in a horizontal position, and let heavy uniform inextensible strings, all of the same length, coincide with various curves that can be drawn through A and B. These strings will one and all abandon their given configurations and become Catenaries. And since the equilibrium of the Catenary is stable, negative work would be done by all the forces acting on its particles for any imagined displacement of these particles which is consistent with the geometrical conditions of the figure. These conditions are simply that the two ends of the curve are fixed, and that there is perfect flexibility but no extensibility. Hence any change of figure consistent with these would *raise* the centre of gravity of the string; and therefore the centre of gravity of the Catenary is lower than the centre of gravity of any of the given curves; and since, by the Theorems of Pappus (Vol. I, p. 301), the surface generated by revolution is equal to the length of the revolving curve multiplied by the circumference of the circle described by its centre of gravity, the surface generated by the Catenary is greatest.

278.] **Expansion of the Abscissa and Ordinate of a Curve in Powers of the Arc.** Let A and B (Fig. 259) be any points on a curve, and let Am and An be the tangent and normal at A. Also let ψ be the angle between the normals at A and B, and let $Am (= x)$ and $Bm (= y)$ be the co-ordinates of B with reference to the tangent and normal at A as axes.

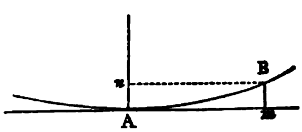

Fig. 259.

Then, by Maclaurin's Theorem, we have

$$\psi = \psi_0 + s\left(\frac{d\psi}{ds}\right)_0 + \frac{s^2}{1.2}\left(\frac{d^2\psi}{ds^2}\right)_0 + \dots$$

s denoting the arc AB, and $\psi_0, \left(\frac{d\psi}{ds}\right)_0, \dots$, the values of ψ and its differential coefficients at A.

Now $\psi_0 = 0$, and $\frac{d\psi}{ds} = \frac{1}{\rho}$, where ρ is the radius of curvature.

Hence

$$\psi = \frac{s}{\rho} + \frac{s^2}{1.2} \frac{d\left(\frac{1}{\rho}\right)}{ds} + \frac{s^3}{1.2.3} \frac{d^2\left(\frac{1}{\rho}\right)}{ds^2} + \dots, \tag{1}$$

the suffix being omitted, it being understood that ρ is the radius of curvature at A.

Again, we have

$$x = x_0 + s\left(\frac{dx}{ds}\right)_0 + \frac{s^2}{1.2}\left(\frac{d^2x}{ds^2}\right)_0 + \dots;$$

also
$$\frac{d^2x}{ds^2} = -\frac{1}{\rho}\frac{dy}{ds}, \text{ and } \frac{d^2y}{ds^2} = \frac{1}{\rho}\frac{dx}{ds}.$$

But

$$\left(\frac{dx}{ds}\right)_0 = 1, \text{ and } \left(\frac{dy}{ds}\right)_0 = 0; \text{ therefore } \left(\frac{d^2x}{ds^2}\right)_0 = 0, \left(\frac{d^2y}{ds^2}\right)_0 = \frac{1}{\rho},$$

and the successive differential coefficients are calculated with ease.

We thus obtain

$$Bn = x = s - \frac{s^3}{6\rho^2} + \frac{s^4}{8\rho^3}\frac{d\rho}{ds} + \dots; \tag{2}$$

$$An = y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds} - \frac{s^4}{24}\left\{\frac{1}{\rho^3} - \frac{2}{\rho^3}\left(\frac{d\rho}{ds}\right)^2 + \frac{1}{\rho^2}\frac{d^2\rho}{ds^2}\right\} + \dots. \tag{3}$$

279.] **Equilibrium of a Heavy Body resting on a Fixed Rough Surface.** Let AD (Fig. 260) be a fixed rough surface on which a heavy body, AC, rests, under the action of gravity, at a single given point A; and let this body receive a slight displacement of rolling on the fixed surface.

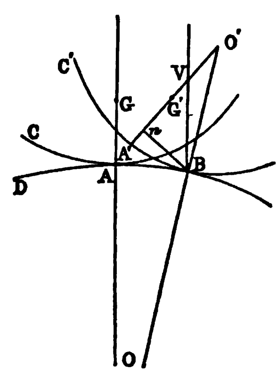

Fig. 260.

We propose to investigate the nature of the equilibrium. The figure represents a section of the bodies made by the vertical plane through their common normal, AO, in which the rolling takes place. We suppose the normal AO to be vertical.

Then, since, in the position of equilibrium the body AC is acted on by only two forces—namely, its own weight and the total resistance of the fixed surface—its centre of gravity, G, must be vertically over the point of contact.

Let the point A of the rolling body come to A', and G to G', the new point of contact being B, and the new common normal OO'. Draw the vertical line BV, meeting $A'O'$ in V.

Then, if $A'V$ is $> A'G'$, the weight of the body acting through G' will produce a rotation round B which will send the body back to its original position; while, if $A'V$ is $< A'G'$, the rotation produced by the weight will be in the opposite direction, and the body will deviate still further from its original position. For stability, therefore,
$$A'V > A'G'. \qquad (1)$$

Let ρ and ρ' be the radii of curvature of the curves AD and AC at A, and let ψ and ψ' be the angles AOB and $A'O'B$. Then drawing Bn perpendicular to $A'O'$, we have
$$A'V = A'n + nV = A'n + Bn \cot A'VB;$$
but $\angle A'VB = \psi + \psi'$; therefore the condition for stability is
$$A'n + Bn \cot(\psi + \psi') > A'G',$$
or, denoting $A'G'$ (or AG) by h,
$$Bn > (h - A'n) \tan(\psi + \psi'). \qquad (2)$$

Now, carrying approximations as far as s^3, it will be found from equation (1) of last Article that
$$\tan(\psi+\psi') = \left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right) + \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\},$$
s being the common length of the arcs AB and $A'B$.

Substituting this, and the values of Bn and $A'n$ from last Article, in (2), the condition for stability is
$$s - \frac{s^2}{6\rho'^2} > \left(h - \frac{s^2}{2\rho'} + \frac{s^3}{6\rho'^2}\frac{d\rho'}{ds'}\right)\left[\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)s + \frac{s^2}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right) + \frac{s^3}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\}\right],$$

or
$$1 - \frac{s^2}{6\rho'^2} > h\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) + h\frac{s}{2}\left(\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}\right) + h\frac{s^2}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\} - \frac{s^2}{2\rho'}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right) \ldots \qquad (3)$$

Neglecting all powers of s, the first condition for stability is

$$1 > h\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

or

$$h < \frac{\rho\rho'}{\rho+\rho'}. \tag{4}$$

If $h > \frac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be unstable.

A special case occurs when $h = \frac{\rho\rho'}{\rho+\rho'}$, and this is commonly called the 'neutral' case, or the equilibrium is said to be neutral. We shall, however, call this the *critical* case.

To find the real nature of the equilibrium in this case, we revert to the general condition (3), and neglect all powers of s beyond the first. The condition for stability now is

$$0 > \frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'}.$$

Hence when $h = \frac{\rho\rho'}{\rho+\rho'}$, the equilibrium will be stable or unstable according as

$$\frac{d\frac{1}{\rho}}{ds} + \frac{d\frac{1}{\rho'}}{ds'} \text{ is negative or positive.} \tag{5}$$

The bodies are, however, frequently in contact at *vertices*, or points of maximum or minimum curvature, and then

$$\frac{d\frac{1}{\rho}}{ds} \text{ and } \frac{d\frac{1}{\rho'}}{ds'}$$

are both zero. Hence the condition (5) fails to determine the nature of equilibrium. Reverting to the condition (3), the terms as far as s^2 destroying each other on both sides, we see that equilibrium will be stable if

$$-\frac{1}{6\rho'^2} > \frac{h}{6}\left\{\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} + 2\left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^3\right\} - \frac{1}{2\rho'}\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

or, substituting $\frac{\rho\rho'}{\rho+\rho'}$ for h, if

$$\frac{d^2\frac{1}{\rho}}{ds^2} + \frac{d^2\frac{1}{\rho'}}{ds'^2} < -\frac{(\rho+\rho')(\rho+2\rho')}{\rho^3\rho'^2}; \tag{6}$$

and in the contrary case the equilibrium will be unstable.

If the lower surface is concave, instead of convex, to the upper, the conditions are obtained by changing the sign of ρ. Thus, the equilibrium will be stable or unstable, according as

$$h < \text{ or } > \frac{\rho\rho'}{\rho-\rho'},$$

and in the critical case, the equilibrium will be stable or unstable, according as

$$\frac{d\frac{1}{\rho'}}{ds'} - \frac{d\frac{1}{\rho}}{ds}$$

is negative or positive; and in case of contact at vertices, the condition (6) is to be similarly modified.

If the body rest on a *plane* surface, $\rho = \infty$, and the differential coefficients of $\frac{1}{\rho}$ are all zero. Hence the limiting value of h for stability is ρ'; but if $h = \rho'$, the equilibrium will be stable or unstable according as $\frac{d\rho'}{ds'}$ is positive or negative; and if the point of contact is a vertex, equilibrium will be stable or unstable, according as

$$\frac{d^2\frac{1}{\rho'}}{ds'^2}$$

is negative or positive*.

---

* Different methods of arriving at the conditions for stability have been published in the *Quarterly Journal of Pure and Applied Mathematics* by Professor Curtis (vol. ix., p. 41), and Mr. Routh (vol. xi., p. 102). The kinetical method of treatment adopted by the latter is very exhaustive. The method in the text was employed independently by Professor Wolstenholme and the author.

It may be well to caution the student against the error of replacing the sections, AD and AC, of the surfaces in contact by their osculating circles at A. For, if we do this, the condition (5) necessarily disappears, and the application of (6) is not allowable, since, to the third power of the arc, the value of $A'n$ is not the same for the circle of curvature as for the curve AC, as at once appears from the expression for $A'n$ given by equation (3) of last Article. The nature of the equilibrium, therefore, as determined from the osculating circles is erroneous.

Examples.

1. If a cone of the same substance and of equal base with a hemisphere be fixed to the latter, so that their bases coincide, find the greatest height of the cone in order that the equilibrium may be stable, when the hemisphere rests symmetrically on a horizontal plane. (Walton's *Mechanical Problems*, p. 185.)

Ans. The height of the cone must be $< r\sqrt{3}$, r being the radius of the hemisphere.

2. Prove that any body with a plane base, resting on a fixed rough spherical surface, will, when the height of its centre of gravity has the critical value, be in unstable equilibrium.

3. A heavy body whose section in the plane of displacement is a catenary, resting on a rough horizontal plane, has its centre of gravity at the critical height; prove that the equilibrium is really stable.

(The condition (6) reduces in this case to $\dfrac{d^2\frac{1}{\rho'}}{ds'^2} < 0$ for stability.)

4. A heavy body in the shape of a paraboloid of revolution, placed on a rough horizontal plane, has its centre of gravity at the critical height; determine this height, and find the real nature of the equilibrium.

Ans. The critical height = the radius of curvature of the generating parabola at the vertex, and the equilibrium is really stable.

5. In the critical case, if both of the conditions (5) and (6) fail, prove that the equilibrium will be stable or unstable, according as

$$\frac{d^3\frac{1}{\rho}}{ds^3} + \frac{d^3\frac{1}{\rho'}}{ds'^3} - \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)\left(\frac{4}{\rho} + \frac{1}{\rho'}\right)\frac{d\frac{1}{\rho'}}{ds'}$$

is negative or positive, the surfaces being convex towards each other.

6. A uniform heavy bar, AB, moveable in a vertical plane round a fixed smooth axis passing through A has a string attached to the end B; this string passes over a fixed pulley C vertically over A. Find the positions of equilibrium, and determine whether they are stable or unstable.

Ans. Let W = weight of bar, $2a$ its length, P = suspended weight, $AC = h$, $\theta = \angle CAB$. Then the positions of equilibrium are given by the equations

$$\theta = 0, \quad \cos\theta = \frac{a}{h} + \left(\frac{1}{4} - \frac{P^2}{W^2}\right)\frac{h}{a}, \quad \text{and} \quad \theta = \pi.$$

The first will be stable if $\dfrac{2h}{h-2a} > \dfrac{W}{P}$, and then the second (when it exists) will necessarily be unstable and the third stable. If the second does not exist, the third will be opposite in nature to the first.

[To find the condition for stability in this problem, we may either take any position of the bar and calculate the moment of force tending to turn it round A, or find the positions of the system for which the common centre of gravity of the bar and weight is highest or lowest. Employing the first method, if $M =$ the restoring moment, and $\phi = \angle ACB$,

$$M = Ph \sin\phi - Wa \sin\theta. \tag{1}$$

Also

$$h \sin\phi = 2a \sin(\theta + \phi). \tag{2}$$

Now $M = 0$ in a position of equilibrium; and if $\frac{dM}{d\theta}$ is positive, a slight increase of θ will call into play a moment tending to restore equilibrium.

In the position $\theta = 0$, we have from (2)

$$\frac{d\phi}{d\theta} = \frac{2a}{h - 2a};$$

and from (1)

$$\frac{dM}{d\theta} = Ph\frac{d\phi}{d\theta} - Wa.$$

Therefore, &c.]

7. If the equilibrium in the first position is critical, find its real nature.

Ans. It is really unstable.

[In the position $\theta = 0$, it will be found from (2) that $\frac{d^2\phi}{d\theta^2} = 0$,

$$\frac{d^3\phi}{d\theta^3} = -\frac{2ah(h+2a)}{(h-2a)^3}; \quad \frac{d^2M}{d\theta^2} = 0, \quad \frac{d^3M}{d\theta^3} = - \cdot]$$

8. Determine whether the equilibrium of the beam in example 12, p. 138, Vol. I, is stable or unstable.

Ans. Unstable. [Either by taking the restoring moment about O, or by the maximum or minimum value of the static energy.

If we imagine the position in which the beam lies horizontal as the reference position, the acting forces, W and P, could do an amount of work equal to

$$Wa \sin\theta - P\{a + b - (a+b)\cos\theta\}$$

in reaching this position by a slipping of the ends of the beam along the planes. This is therefore the value of Π, the static energy—in which, if we please, we may discard the constant term $P(a+b)$. Therefore, &c.]

9. Four bars, AB, BC, CD, DA (p. 177, Vol. I), forming a plane quadrilateral, and freely jointed at the vertices, are kept in equilibrium by an elastic string stretched between the middle points of BC and DA, and an elastic strut compressed between the middle points of AB and CD, the string and the strut both following Hooke's Law. Show that there are always two, and there may be four, configurations of equilibrium.

280.] **General Properties of Static Energy [Potential Work].** If the generalised co-ordinates which determine the configuration of the system are $q_1, q_2, \ldots, q_k$, we shall have

$$\Pi = f(q_1, q_2, \ldots q_k).$$

If the configuration is one of equilibrium, and if we imagine any small displacements of the system for which q_1 becomes $q_1 + h_1$, &c., the value of Π in the new configuration will be

$$P + \frac{1}{1\,.\,2}\left\{\frac{d^2 P}{dq_1^{\,2}} \cdot h_1^{\,2} + \frac{d^2 P}{dq_2^{\,2}} \cdot h_2^{\,2} + \ldots + 2\frac{d^2 P}{dq_1\, dq_2} \cdot h_1 h_2 + \ldots\right\}, \tag{1}$$

where P is the value of Π in the equilibrium position. This follows by Taylor's Theorem, observing that $Q_1, Q_2, \ldots$, or $\frac{dP}{dq_1}, \frac{dP}{dq_2}, \ldots$ all vanish since the configuration $(q_1, q_2, \ldots q_k)$ is one of equilibrium.

It will be convenient to denote the coefficients of $h_1^{\,2}, h_2^{\,2}, \ldots$ inside the bracket above by the notation $(1, 1), (2, 2), \ldots$, and those of $h_1 h_2, h_2 h_3, \ldots$ by $2(1,2), 2(2,3), \ldots$, while denoting the whole function inside the brackets by $2H$; so that in the new configuration we have

$$\Pi = P + H, \tag{2}$$

where

$$2H = (1, 1)h_1^{\,2} + (2, 2)h_2^{\,2} + (3, 3)h_3^{\,2} + \ldots$$
$$+ 2(1,2)h_1 h_2 + 2(1,3)h_1 h_3 + \ldots + 2(2, 3)h_2 h_3 + \ldots, \tag{3}$$

a homogeneous quadratic function of the increments of the generalised co-ordinates.

Since in a configuration of stable equilibrium the Static Energy of the system is a minimum, it follows that *in the neighbourhood of such a configuration H is positive, whatever may be the values of the displacements h_1, h_2.*

In any other possible displacement of the system let $h_1', h_2', \ldots$ be the small increments of $q_1, q_2, \ldots$; then, if Q_r is the work coefficient corresponding to h_r in the first displacement, and Q_r' the work coefficient corresponding to h_r' in the second, it is obvious that

$$\frac{dQ_r}{dh_s} = \frac{dQ_s}{dh_r}, \tag{4}$$

$$\text{and} \quad \Sigma\, Q_r h_r' = \Sigma\, Q_r' h_r. \tag{5}$$

Again, if we take a displacement in which the increments of $q_1, q_2, \ldots$ are $h_1 + h_1', h_2 + h_2' \ldots$ the work coefficients in this

displacement—which is a superposition of the two former displacements—are the sums, $Q_1 + Q_1'$, $Q_2 + Q_2'$,... of those belonging to the two constituent displacements.

This is obvious since

$$Q_1 = (1,1)\,h_1 + (1,2)\,h_2 + \ldots + (1,k)\,h_k,$$
$$Q_1' = (1,1)h_1' + (1,2)h_2' + \ldots + (1,k)\,h_k',$$

the work coefficients being (for small displacements from a position of equilibrium) *linear* functions of the displacements, with constant coefficients.

Similarly $h_1 - h_1'$, &c., will give rise to $Q_1 - Q_1'$, &c.

Also $$H_{h \pm h'} = H_h + H_{h'} \pm \Sigma Q_r h_r', \tag{6}$$

where $H_{h \pm h'}$ denotes the increment of the Static Energy of the system when the two displacements $(h_1, h_2, \ldots)$ and $(h_1', h_2', \ldots)$ are superposed by addition or by subtraction, H_h and $H_{h'}$ being the increments of Static Energy corresponding to them separately. This is at once obvious since to get $H_{h \pm h'}$ we write $h_1 \pm h_1'$, ... for h_1,... in (3); and (6) is merely the expression of the result by Taylor's Theorem.

The properties just given are mere analytical properties of a homogeneous quadratic function. We now proceed to prove a general physical property which belongs to any configuration of stable equilibrium.

281.] **General Property of Stable Configurations.** *If any material system is, under the influence of external forces and its own internal forces, in a configuration of stable equilibrium, and if a new set of external forces be applied, each acting with a given magnitude and line of action, so that the system assumes a new configuration of equilibrium slightly differing from the former one, the newly applied forces will, when the new configuration is reached, have done more work if, when they were about to be applied, no degree of freedom was taken away from the system than they would have done if its freedom was in any way reduced; and, moreover, the Potential Work of the original forces of the system is greater in the former than in the latter case.*

Let the system have originally k degrees of freedom, so that its configuration is determined by the generalised co-ordinates $q_1, q_2, \ldots, q_k$. Let the newly applied external forces have for components (X_1, Y_1, Z_1), &c., acting at given points (x_1, y_1, z_1), &c. in the system; and let $h_1, h_2, \ldots h_k$ be the increments of the

co-ordinates produced by the new forces at any instant during the passage of the system from its original (stable) to its new configuration of equilibrium.

Then in all cases—i. e. whether the q's remain perfectly independent or not during the application of the new forces—if (x_1, y_1, z_1) are the co-ordinates of the point of application of the force (X_1, Y_1, Z_1) at any instant during the passage to the new configuration of equilibrium, we have

$$x_1 = \phi_1(h_1, h_2, \ldots), y_1 = \psi_1(h_1, h_2, \ldots), z_1 = \chi_1(h_1, h_2, \ldots);$$

where the forms of the functions ϕ_1, ψ_1, χ_1 are given.

Any interference with the freedom of the system does not alter the forms of these functions; it merely causes the h's to be no longer all *independent*.

Moreover, since the h's are all very small quantities, $\phi_1, \ldots$ are linear functions, so that

$$x_1 = a_1 + a_1 h_1 + a_2 h_2 + \ldots,$$

in which the coefficients are all given constants depending on the geometry of the given system in its original (stable) configuration.

Now in any configuration intermediate to the original and the new one of equilibrium the work done by the newly applied forces in a further small displacement is

$$X_1 dx_1 + Y_1 dy_1 + Z_1 dz_1 + \ldots;$$

$$\text{or} \quad A_1 dh_1 + A_2 dh_2 + \ldots + A_k dh_k, \tag{1}$$

where $A_1, A_2, \ldots$ are all given constants.

At the same time, the element of work done by the forces of the given system is $-d\Pi$, i. e.

$$-[(1,1)h_1 + (1,2)h_2 + \ldots]dh_1$$
$$-[(1,2)h_1 + (2,2)h_2 + \ldots]dh_2 - \ldots; \tag{2}$$

and since the configuration of equilibrium is defined by the vanishing of the total work done in any elementary displacement (Art. 268), the new configuration is defined by equating to zero the sum of (1) and (2).

Forming this sum, we should, if the h's were all independent, equate to zero separately the coefficients of $dh_1, dh_2, \ldots$; but if new hampering conditions are introduced, we cannot do this.

Suppose that, just as the newly applied external forces are about to act, a single hampering condition is introduced. This

(since the h's are all very small) will be expressed by an equation of the form

$$\lambda_1 h_1 + \lambda_2 h_2 + \ldots + \lambda_k h_k = 0, \tag{3}$$

where $\lambda_1, \lambda_2, \ldots$ are all assigned.

Following then the method of Lagrange (Art. 269), we take the differential of (3), multiply the result by an undetermined multipler, θ, add it to the sum of (1) and (2), and then, treating $dh_1, dh_2, \ldots$ as independent, equate to zero their separate coefficients. Thus we have

$$(1,1)h_1 + (1,2)h_2 + \ldots + (1,k)h_k = A_1 + \theta\lambda_1, \tag{a_1}$$

$$(1,2)h_1 + (2,2)h_2 + \ldots + (2,k)h_k = A_2 + \theta\lambda_2, \tag{a_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(1,k)h_1 + (2,k)h_2 + \ldots + (k,k)h_k = A_k + \theta\lambda_k, \tag{a_k}$$

$$\lambda_1 h_1 + \lambda_2 h_2 + \ldots + \lambda_k h_k = 0. \tag{l}$$

From these $k+1$ equations we eliminate the k quantities $h_1, \ldots h_k$, and obtain the value of θ.

Now let Δ denote the determinant

$$\begin{vmatrix} (1,1), & (1,2), \ldots & (1,k) \\ (1,2), & (2,2), \ldots & (2,k) \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ (1,k), & (2,k), \ldots & (k,k) \end{vmatrix},$$

let its minors be denoted by $[1,1], [1,2], \ldots [2,2], \ldots$, and let

$$2\phi_A \equiv [1,1]A_1^2 + [2,2]A_2^2 + \ldots [k,k]A_k^2 + 2[1,2]A_1A_2 + 2[1,3]A_1A_3 + \ldots, \tag{4}$$

while $2\phi_\lambda$ denotes the function at the right-hand side when $\lambda_1, \lambda_2, \ldots$ are substituted for $A_1, A_2, \ldots$.

Then, solving the first k equations for $h_1, h_2, \ldots h_k$, we have

$$\Delta . h_1 = \frac{d\phi_A}{dA_1} + \theta\frac{d\phi_\lambda}{d\lambda_1}, \tag{b_1}$$

$$\Delta . h_2 = \frac{d\phi_A}{dA_2} + \theta\frac{d\phi_\lambda}{d\lambda_2}, \tag{b_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and substituting these values in (l), we have

$$\theta = -\frac{\Sigma A\frac{d\phi_\lambda}{d\lambda}}{2\phi_\lambda}. \tag{5}$$

Now the increase, H, of the Static Energy of the system

being given by the expression (3) of last Article, this quantity is obtained by multiplying (a_1), (a_2), ... by h_1, h_2, ... and adding; so that in virtue of (l), we have

$$2H = A_1h_1 + A_2h_2 + \ldots + A_kh_k, \qquad (6)$$

which shows that in reaching the new configuration of equilibrium, the work done by the newly applied forces is double the gain of Potential Work of the forces of the system.

[NOTE—This is not a contradiction of the Principle of Work and Energy in Kinetics; for in this statical problem we suppose the system to be, by any proper means, gradually guided to its new configuration, during the action of the newly applied forces. If this is not done, the system would rush through the new configuration.]

Substituting in (6) the values just found for θ, h_1, h_2, ..., we have

$$2\Delta . H = 2\phi_A + \theta . \Sigma A \frac{d\phi_\lambda}{d\lambda},$$

or, by (5),

$$\Delta . H = \phi_A - \frac{\left\{\Sigma A \frac{d\phi_\lambda}{d\lambda}\right\}^2}{4\phi_\lambda}. \qquad (c)$$

It is to be noted that if the λ's are all zero, i.e. if no hampering condition is introduced, the second term on the right-hand side of this equation assumes the form $\frac{0}{0}$. But reverting to equations (a_1), ..., we are no longer in this case to make use of (l), and the values of h_1, h_2, ... are those given by (b_1), ... when the terms in θ are rejected.

And in this case we have simply

$$H = \frac{\phi_A}{\Delta}. \qquad (d)$$

Now (Williamson's *Differential Calculus*, Note 2) the determinant Δ is positive, since in the configuration of stable equilibrium Π is an absolute minimum. Also, by last Article, for any displacements whatever, H is positive; therefore ϕ_A is positive; and this is true whatever be the values of A_1, A_2, ...; so that ϕ_λ is also positive; therefore the right-hand side of (c) is a maximum when $\Sigma A \frac{d\phi_\lambda}{d\lambda} = 0$, i.e. when

$$\lambda_1 \frac{d\phi_A}{dA_1} + \lambda_2 \frac{d\phi_A}{dA_2} + \ldots = 0. \qquad (e)$$

But if the λ's satisfy this equation, the value of θ is zero, and the values of the displacements, $h_1, \ldots$ obtained from the equations $(b_1), \ldots$ are what they would be if no restriction were imposed; so that an equation of the form (3) in which the λ's are any system satisfying (e) is not a *restricting* equation at all, but one which is satisfied by the unrestricted displacements of the system.

Consequently the imposition of any relation of the form (3) in which the multipliers $\lambda_1, \ldots$ are not consistent with the unrestricted displacements of the system involves a diminution of the Potential Work of its forces in the new configuration of equilibrium.

Example.

As a simple illustration of this theorem, take the case of a rod, AB, lying in a smooth horizontal plane, its extremities, A and B, being each attached to an elastic string in a state of tension, these strings, OA and $O'B$, being attached to two fixed points, O and O', in the horizontal plane, and their lengths being equal.

The position of stable equilibrium is that in which the points O, A, B, O' are (in this order of succession) in one right line. The rod being in this position, suppose that at a given point, M, on it a small force P is applied perpendicularly to AB, so that the rod is displaced.

Now any position of the rod may be defined by three co-ordinates, viz. those determining the position of the end A and the angle θ, through which the rod has turned. Taking A as origin, and AB as axis of x, and supposing $A'B'$ to be any displaced position of the rod, let the co-ordinates of A' be (x, y). Thus the generalised co-ordinates are x, y, θ.

Let $OA = O'B = c$; $AB = 2a$; $b =$ natural length of each string.

Then we easily find $OA' = c + x + \dfrac{y^2}{2c}$, and

$$O'B' = c - x + \frac{y^2 + 2a(2a+c)\theta^2 + 4a\theta y}{2c}.$$

Again, the tension of an elastic string, given by Hooke's Law, being $\mu(l-b)$, if it is stretched from a length c to a length l the potential work of its tension is $\frac{1}{2}\mu(l-c)(l+c-2b)$. Hence the sum of the potential works of the tensions of OA' and $O'B'$ is

$$\mu\frac{c-b}{c}\left\{\frac{c}{c-b}x^2 + y^2 + a(2a+c)\theta^2 + 2a\theta y\right\}. \tag{7}$$

This, then, is H, the gain of potential work of the forces of the system due to displacement. Hence $(1, 1) = 2\mu$, &c.

We have now to determine A_1, A_2, and A_3 from the expression

$$A_1 dx + A_2 dy + A_3 d\theta \tag{8}$$

for the elementary work of the disturbing force.

If $AM = p$, the ordinate of the point M in the displaced position is $y + p\theta$, and the elementary work of P is

$$P\,(dy + p\,d\theta),$$

showing that $$A_1 = 0,\ A_2 = P,\ A_3 = Pp. \tag{9}$$

Assuming any restricting condition,

$$\lambda_1 x + \lambda_2 y + \lambda_3 \theta = 0, \tag{10}$$

the equations (a_1)..., which are

$$\frac{dH}{dx} = \lambda_1 x;\quad \frac{dH}{dy} = P + \lambda_2 y;\quad \frac{dH}{d\theta} = Pp + \lambda_3 \theta, \tag{11}$$

give for the displacements

$$2\mu x = \lambda_1 \xi, \tag{12}$$

$$2\mu \frac{c-b}{c}(y + a\theta) = P + \lambda_2 \xi, \tag{13}$$

$$2\mu \frac{c-b}{c}(ay + a\,.\,\overline{2a + c}\,\theta) = Pp + \lambda_3 \xi, \tag{14}$$

in which we have used ξ as the undetermined multiplier instead of the θ of equations (a_1)....

If no hampering condition is introduced, we have

$$x = 0;\quad y = \frac{P}{2\mu}\frac{c}{c-b}\left(1 - \frac{p-a}{a+c}\right);\quad \theta = \frac{P}{2\mu}\frac{c(p-a)}{a(a+c)(c-b)},$$

which obviously verify in the simple case in which $p = a$, or the rod is pulled at its middle point.

Of course these values can be at once obtained by the elements of Statics.

The potential work of the tensions in the new position of equilibrium is

$$\tfrac{1}{2} P(y + p\theta),\ \text{i.e.}$$

$$\frac{P^2 c}{4\mu(c-b)}\left[1 + \frac{(p-a)^2}{a(a+c)}\right], \tag{15}$$

which, of course, is also given by (d), since

$$\Delta = \begin{vmatrix} 2\mu, & 0\ \ , & 0 \\ 0\ , & 2\mu r\ , & 2\mu ra \\ 0\ , & 2\mu ra, & 2\mu ra(2a+c) \end{vmatrix},$$

in which, for shortness, we have used r for $\dfrac{c-b}{c}$.

Multiplying these by h_1', h_2',... and adding, the terms in θ, θ',... disappear in virtue of the restricting equations (3), (16), &c. Hence putting $\frac{dH}{dh_1}$ for A_1, &c., we have

$$\Sigma\left(\frac{dH'}{dh'}\cdot h'\right) = \Sigma\left(\frac{dH}{dh}\cdot h'\right). \tag{24}$$

But by last Article the right-hand side $= \Sigma\left(\frac{dH'}{dh'}\cdot h\right)$; hence

$$\Sigma\cdot\frac{dH'}{dh'}(h-h') = 0. \tag{25}$$

Now (24) can be written

$$\begin{aligned} 2H' &= \Sigma\left(\frac{dH}{dh}\cdot h'\right) \\ &= -\Sigma\cdot\frac{dH}{dh}(h-h') + 2H \end{aligned}$$

$$\begin{aligned} \therefore \quad 2(H-H') &= \Sigma\cdot\frac{dH}{dh}(h-h') \\ &= \Sigma\cdot\left(\frac{dH}{dh} - \frac{dH'}{dh'}\right)(h-h'), \text{ by (25)}; \\ &= 2H_{h-h'}, \end{aligned} \tag{26}$$

by last Article; that is, the Potential Work lost by restriction is that obtained by substituting the differences $h_1 - h_1'$, $h_2 - h_2'$... of the displacements in the general value of H given in (3) of last Article.

This result may also be obtained by showing that the term $\theta^2 \frac{\phi_\lambda}{\Delta}$ in equation (*c*), p. 145, which is the loss of work, results from substituting the differences, $\frac{\theta}{\Delta}\frac{d\phi_\lambda}{d\lambda_1}$, ... of the displacements in the general value of H. We thus get a number of algebraic identities.

A particular case of the application of this result is (see Watson and Burbury's *Generalised Co-ordinates*, p. 52) that if the stiffness in any part or parts of a material system is diminished, the geometrical connections remaining unchanged, the Potential Work of its forces, internal and external, due to a change of configuration produced by given disturbing forces will be increased.

282.] **Equations of Condition of Continuous Systems.** If the system of particles whose equilibrium is under consideration is continuous—as, for instance, an inextensible string, an inextensible membrane, or a rigid solid—the equations of condition will express the invariability of an infinitesimal element, such as the distance between two indefinitely close points.

Take, for example, the case of an inextensible string of which PQ (Fig. 261) is an elementary length, equal to ds. The equations $L_1 = 0$, $L_2 = 0, \ldots$ which express the invariable connections of the particles of the system, will be $ds_1 = \text{constant}$, $ds_2 = \text{constant}, \ldots$, where $ds_1, ds_2, \ldots$ are the distances between successive points on the curve; and the typical term $\lambda\delta L$ which enters into the equation of Virtual Work will be the typical term $\lambda\delta ds$.

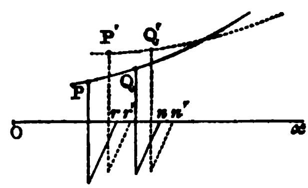

Fig. 261.

Let us enquire more particularly into the meaning of the expression δds. If we contemplate any small displacement whatever of the string, such that the element PQ comes into the position $P'Q'$, the new length $P'Q'$ being either greater or less than PQ, the meaning of the expression δds is $P'Q'-PQ$; and the condition that no change of length of the element takes place in the displacement is

$$\delta ds = 0.$$

Now (x, y, z) being the co-ordinates of P, we imagine these to receive, respectively, increments δx, δy, δz; i.e. the co-ordinates of P' are $(x+\delta x, y+\delta y, z+\delta z)$; while those of Q are

$$(x+dx,\ y+dy,\ z+dz).$$

The co-ordinates, therefore, of Q' (to which Q is imagined to be displaced) are represented by

$$x+dx+\delta(x+dx);\ y+dy+\delta(y+dy);\ z+dz+\delta(z+dz).$$

The excesses of the co-ordinates of Q' over those of P' are therefore $\quad dx+\delta(dx);\ dy+\delta(dy);\ dz+\delta(dz);$

and the length of $P'Q'$ being $PQ+\delta(ds)$, or $ds+\delta ds$, we have

$$(ds+\delta ds)^2 = (\delta x+\delta dx)^2+(dy+\delta dy)^2+(dz+\delta dz)^2;$$

or
$$\delta ds = \left(\frac{dx}{ds}\frac{\delta dx}{ds}+\frac{dy}{ds}\frac{\delta dy}{ds}+\frac{d}{ds}\frac{\delta dz}{ds}\right) ds, \tag{a}$$

neglecting infinitesimals of the fourth order, such as $(\delta ds)^2$, &c.

But the increments d and δ being completely independent and essentially distinguished as above explained, it is easy to see that the order in which the double operation $d\delta$ is performed on any function is indifferent; i.e. $d(\delta V)$ is precisely the same as $\delta(dV)$, where V is any function. In fact an inspection of the figure (Fig. 261) at once shows that $\delta(dx) = d(\delta x)$, the line Ox being the axis of x. For, let the abscissæ of P and Q be Or and On, those of P' and Q' being Or' and On', measured along Ox.

Then if x is the co-ordinate of P, $dx = rn$, and $\delta x = rr'$.

Also $\delta(dx) =$ value of dx in the new position $-$ value of dx in old position $= r'n' - rn$; and $d(\delta x) =$ value of δx for $Q -$ value of δx for $P = nn' - rr'$. But obviously

$$r'n' - rn = nn' - rr';$$

therefore $\delta(dx) = d(\delta x)$.

In virtue, then, of this commutative property of d and δ, (a) may be written

$$\delta ds = \left(\frac{dx}{ds}\frac{d\delta x}{ds} + \frac{dy}{ds}\frac{d\delta y}{ds} + \frac{dz}{ds}\frac{d\delta z}{ds}\right) ds. \qquad (\beta)$$

283.] **Variation of any Function. Particular Cases.** Since a variation of any function of the co-ordinates of a point consists in making infinitesimal increments to the several co-ordinates, it is clear that all the resulting changes are subject to the ordinary rules of the Differential Calculus. To fix ideas by means of an elementary example, suppose that we have a series of points lying on a circle whose equation is

$$x^2 + y^2 - a^2 = 0.$$

If now we imagine each point (x, y) on the circle displaced to an infinitely near position which is defined by adding to the abscissa a quantity equal to $\epsilon . y \sin\frac{x}{b}$, and to the ordinate a quantity $\epsilon . x \sin\frac{y}{c}$, where ϵ is an infinitely small quantity, we shall obtain a new curve differing infinitely little in position and shape from the original. In this particular case the increments which we have denoted by δx and δy are given by the equations $\delta x = \epsilon . y \sin\frac{x}{b}$, $\delta y = \epsilon . x \sin\frac{y}{c}$; and so in general, whatever be the laws according to which the variations are made.

It is obvious, then, that if u and v are any two functions of the co-ordinates of a point,

$$\delta\frac{u}{v}=\frac{v\,\delta u-u\,\delta v}{v^2}.$$

So, again, if V is any function of x, we have

$$\delta V=\frac{dV}{dx}\delta x,$$

any arbitrary change, δx, being made in x; and in passing to an adjacent point on a given curve or surface,

$$d(\delta V)=\frac{d^2V}{dx^2}\delta x\,dx+\frac{dV}{dx}d(\delta x).$$

Also in an integration along a curve or surface, since this integration consists merely in a summation with respect to all the points on the curve or surface, we have

$$\delta\int V dx=\int\delta(V dx).$$

If, in particular, an integration, $\int V ds$, is performed along a curve, and all the points of the curve receive displacements such that the distance, ds, between two consecutive points remains unaltered, we shall have

$$\delta\int V ds=\int(\delta V).ds;$$

and the same equation holds, in like case, if the integration is performed over a surface or throughout a solid if for ds we put the element of superficial area or the element of volume.

In this case also

$$\delta\frac{dx}{ds}=\frac{d\,\delta x}{ds};\quad\delta\frac{d^2x}{ds^2}=\frac{d^2\delta x}{ds^2};\quad\delta\frac{d^n x}{ds^n}=\frac{d^n\delta x}{ds^n}.$$

EXAMPLE.

Every element of mass of a solid is multiplied by the product of two of its co-ordinates, xy, and the sum of all such products ("product of inertia") taken. If the body receives a small displacement of rotation round the axis of z, find the variation of this sum.

Let dm be the element of mass at the point x, y, z; then the sum $=\int xy\,dm$. Now $\delta\int xy\,dm=\int\delta(xy).dm=\int(x\delta y+y\delta x)\,dm$. But if the angle of rotation is $\delta\theta$, we have $\delta x=-y\delta\theta$, $\delta y=x\delta\theta$. Hence the variation of the sum is

$$\delta\theta\times\int(x^2-y^2)\,dm.$$

To determine the variation of the angle between two consecutive tangents to any curve.

Let the tangents be at points, P, Q, separated by an arc of length ds, and let $d\theta$ be the angle between them. Then

$$d\theta = \frac{ds}{\rho}, \tag{1}$$

where ρ is the radius of absolute curvature of the curve. Now $\delta d\theta$ is what we have to find; and we shall suppose for generality that in the displacements of P and Q the length ds is altered. We have then

$$\delta d\theta = \frac{1}{\rho}\delta ds - \frac{1}{\rho^2}ds\delta\rho. \tag{2}$$

But
$$\frac{1}{\rho^2} = \frac{(d^2x)^2+(d^2y)^2+(d^2z)^2}{ds^4}; \tag{3}$$

$$\therefore\ -\frac{1}{\rho^3}\delta\rho = \frac{d^2x d^2\delta x + d^2y d^2\delta y + d^2z d^2\delta z}{ds^4} - 2\frac{(d^2x)^2+(d^2y)^2+(d^2z)^2}{ds^5}\delta ds;$$

$$\therefore\ -\frac{1}{\rho^2}\delta\rho = \rho\left(\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\right) - \frac{2}{\rho}\frac{d\delta s}{ds}. \tag{γ}$$

Hence

$$\delta d\theta = \rho\left(\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\right)ds - \frac{1}{\rho}\delta ds \tag{δ}$$

$$= \Big[-\frac{1}{\rho}\frac{dx}{ds}\frac{d\delta x}{ds} - \frac{1}{\rho}\frac{dy}{ds}\frac{d\delta y}{ds} - \frac{1}{\rho}\frac{dz}{ds}\frac{d\delta z}{ds} + \rho\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \rho\frac{d^2y}{ds^2}\frac{d^2\delta y}{ds^2} + \rho\frac{d^2z}{ds^2}\frac{d^2\delta z}{ds^2}\Big]ds. \tag{ϵ}$$

To find the variation of the angle between two consecutive osculating planes of any tortuous curve.

[A *tortuous curve*, called also a 'curve of double curvature,' is one whose osculating plane varies from point to point.]

If l, m, n are the direction-cosines of the binormal, i.e. the perpendicular to the osculating plane, at any point of the curve, we have

$$l = \rho\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right);\ m = \ldots;\ n = \ldots,$$

and if $d\phi$ is the angle between two consecutive osculating planes,

since the tangent line to the curve is perpendicular to two consecutive binormals, we have $\frac{dx}{ds} = \frac{mdn - ndm}{d\phi}$. Hence

$$d\phi = \rho\left(l\frac{d^3x}{ds^3} + m\frac{d^3y}{ds^3} + n\frac{d^3z}{ds^3}\right)\cdot ds,$$

and we shall find that

$$\delta d\phi = \Big[A_1\frac{d\delta x}{ds} + B_1\frac{d\delta y}{ds} + C_1\frac{d\delta z}{ds} + A_2\frac{d^2\delta x}{ds^2} + B_2\frac{d^2\delta y}{ds^2}$$
$$+ C_2\frac{d^2\delta z}{ds^2} + A_3\frac{d^3\delta x}{ds^3} + B_3\frac{d^3\delta y}{ds^3} + C_3\frac{d^3\delta z}{ds^3}\Big]\cdot ds, \qquad (\zeta)$$

where A_1, &c., are certain functions of the differential coefficients $\frac{dx}{ds}$, &c.

For any arbitrary displacement of a surface, $z = \phi(x, y)$, to find the variations of the partial differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$. The arbitrary changes in x, y, z which we have hitherto denoted by $\delta x, \delta y, \delta z$ we shall now find it convenient to denote by u, v, w, respectively.

Let P be any point (x, y, z) on a given surface—which surface we may, to fix ideas, imagine to be a thin sheet of india-rubber—whose points may receive, or be imagined to receive, any small displacements whatever. If these displacements are completely unhampered, any small element of area described round P on the undisplaced surface will be found on the displaced surface in a distorted form, and with its area altered in magnitude.

Suppose that Q is any point on the undisplaced surface indefinitely close to P, the co-ordinates of Q being $(x+\xi, y+\eta, z+\zeta)$. Then since z is determined when x and y are given (which would not be the case if instead of a *surface* we had a *solid* to deal with), the displacements u, v, w will each be some assigned function of x and y, i.e.

$$u = f_1(x, y); \quad v = f_2(x, y); \quad w = f_3(x, y). \qquad (\eta)$$

Let P' and Q' be the displaced positions of P and Q; and observe that $\frac{dz}{dx}$ means the increment of z divided by that of x, as we pass from a point P to a close point, R, such that P

and R have the same y. Imagine, then, Q to be so chosen that Q' and P' have the same y, so that the new

$$\frac{dz}{dx} = \frac{z \text{ of } Q' - z \text{ of } P'}{x \text{ of } Q' - x \text{ of } P'}.$$

Now the x of Q' is $x+u+\xi+f_1(x+\xi,\ y+\eta)$, according to the law expressed by equations (η); and this is

$$x+u+\xi+\xi\frac{du}{dx}+\eta\frac{du}{dy}.$$

Similarly the z of Q' is

$$z+w+\zeta+\xi\frac{dw}{dx}+\eta\frac{dw}{dy};$$

and since the y of Q' = the y of P', we have

$$\xi\frac{dv}{dx}+\eta\left(1+\frac{dv}{dy}\right) = 0. \tag{θ}$$

Denoting, as is usual, $\frac{dz}{dx}$ by p and $\frac{dz}{dy}$ by q, and the values of these at P' by $p+\Delta p$ and $q+\Delta q$, we have

$$p+\Delta p = \frac{\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta}{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}}. \tag{ι}$$

Now since on the undisplaced surface $dz = pdx+qdy$, we have $\zeta = p\xi+q\eta$. Substitute this value in (ι), and then for $\xi:\eta$ put the value given by (θ), and we have, by neglecting such infinitesimals of the second order as the products $\frac{du}{dx}\frac{dv}{dy}$, &c.,

$$p+\Delta p = \frac{p\left(1+\frac{dv}{dy}\right)+\frac{dw}{dx}-q\frac{dv}{dx}}{1+\frac{du}{dx}+\frac{dv}{dy}};$$

$$\therefore\ \Delta p = \frac{dw}{dx}-p\frac{du}{dx}-q\frac{dv}{dx}. \tag{κ}$$

Similarly, $$\Delta q = \frac{dw}{dy}-p\frac{du}{dy}-q\frac{dv}{dy}. \tag{λ}$$

EXAMPLES.

1. Find the conditions to be satisfied by the displacements of all points on a perfectly inextensible surface.

The length of the line PQ must be unaltered whatever point Q may be. Now from the preceding we have

$$P'Q'^2 = [(1+\frac{du}{dx})\xi + \frac{du}{dy}.\eta]^2 + [\frac{dv}{dx}.\xi + (1+\frac{dv}{dy})\eta]^2$$
$$+[(p+\frac{dw}{dx})\xi + (q+\frac{dw}{dy})\eta]^2.$$

Hence the conditions for perfect inextensibility are

$$\frac{du}{dx} + p\frac{dw}{dx} = 0, \quad \frac{dv}{dy} + q\frac{dw}{dy} = 0,$$
$$\frac{du}{dy} + \frac{dv}{dx} + p\frac{dw}{dy} + q\frac{dw}{dx} = 0.$$

2. From these conditions find equations for the separate components of displacement.

Ans. The value of w is to be obtained from the partial differential equation

$$t\frac{d^2w}{dx^2} - 2s\frac{d^2w}{dx\,dy} + r\frac{d^2w}{dy^2} = 0,$$

where
$$r = \frac{dp}{dx}, \quad t = \frac{dq}{dy}, \quad s = \frac{dp}{dy} = \frac{dq}{dx}.$$

In the case of a plane surface, $z = ax + by + c$, we find

$$u + aw = my + n; \quad v + bw = -mx + n',$$

where m, n, n' are arbitrary constants.

284.] **Equilibrium of an Inextensible String.** We now apply the method of Lagrange to determine the equations of equilibrium of an inextensible string acted on by any system of forces. Let, as previously, m denote the mass per unit length at any point of the string, and X, Y, Z the components of the external force, per unit mass, at the point.

Now the equations, $L_1 = 0$, $L_2 = 0$, ... of condition are in this case $ds_1 =$ const., $ds_2 =$ const., ... and the general equation of equilibrium of Art. 260 becomes

$$m_1(X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1)ds_1 + m_2(X_2\delta x_2 + Y_2\delta y_2 + Z_2\delta z_2)ds_2 + \ldots$$
$$+\lambda_1\delta ds_1 + \lambda_2\delta ds_2 + \ldots \qquad = 0, \qquad (1)$$

the string being supposed to have assumed its position of equilibrium; for it is when the equilibrium position is assumed that the forces satisfy the above equation of Virtual Work.

Now the particles being infinitely numerous, we may write the above equation simply

$$\int m(X\delta x+Y\delta y+Z\delta z)\,ds+\int\lambda\,\delta ds=0. \qquad (2)$$

Reducing all the variations to variations of x, y, z, or, in other words, substituting here the value of δds given in equation (β) of Art. 282, we have

$$\int\Big[m(X\delta x+Y\delta y+Z\delta z)\,ds+\lambda\Big(\frac{dx}{ds}d\delta x+\frac{dy}{ds}d\delta y+\frac{dz}{ds}d\delta z\Big)\Big]=0. \qquad (3)$$

Now $\displaystyle\int\lambda\frac{dx}{ds}d\delta x=\Big(\lambda\frac{dx}{ds}\delta x\Big)_1-\Big(\lambda\frac{dx}{ds}\delta x\Big)_0-\int\delta x\frac{d}{ds}\Big(\lambda\frac{dx}{ds}\Big)\cdot ds,$

by integration by parts, the term $\Big(\lambda\dfrac{dx}{ds}\delta x\Big)_1$ being the value of $\lambda\dfrac{dx}{ds}\delta x$ at one of the limits of integration, i.e. at one extremity of the string; and $\Big(\lambda\dfrac{dx}{ds}\delta x\Big)_0$ being its value at the other extremity.

Performing similar integrations for the other terms, (3) becomes

$$\lambda_1\Big(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\Big)_1-\lambda_0\Big(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\Big)_0$$
$$+\int\Big[\Big\{mX-\frac{d}{ds}\Big(\lambda\frac{dx}{ds}\Big)\Big\}\delta x+\Big\{mY-\frac{d}{ds}\Big(\lambda\frac{dy}{ds}\Big)\Big\}\delta y$$
$$+\Big\{mZ-\frac{d}{ds}\Big(\lambda\frac{dz}{ds}\Big)\Big\}\delta z\Big]=0. \qquad (4)$$

Now, as in the equation of Art. 269 we equated to zero the coefficients of δx_1, δy_1, δz_1, ..., so here we have to put the coefficients of δx, δy, and δz equal to zero for each particle of the string; that is, we put the coefficients of these quantities under the sign of integration equal to zero. Hence we have at all points

$$\left.\begin{aligned} mX-\frac{d}{ds}\Big(\lambda\frac{dx}{ds}\Big)&=0,\\ mY-\frac{d}{ds}\Big(\lambda\frac{dy}{ds}\Big)&=0,\\ mZ-\frac{d}{ds}\Big(\lambda\frac{dz}{ds}\Big)&=0,\end{aligned}\right\} \qquad (A)$$

which equations are precisely the same as those of Art. 184; and it appears either by comparison of both sets of equations,

or by the end of Art. 269, that λ in these equations is minus the tension of the string.

The conditions of equilibrium, then, as expressed in (4), consist of two parts—namely, terms which relate to the extremities of the string (which are the terms outside the sign of integration), and terms which relate to every intermediate point in the string (which give the general equations of equilibrium above).

Equating to zero the terms outside the integral sign, we have

$$\lambda_1\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_1-\lambda_0\left(\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z\right)_0=0. \quad (5)$$

Now, if the extremities of the string are fixed, they will be fixed in the displaced string, and every term of (5) vanishes since

$$\delta x_1=\delta y_1=\delta z_1=\delta x_0=\delta y_0=\delta z_0=0.$$

But if each end is perfectly free, since δx_1, δy_1, ... are quite arbitrary and independent, we must have

$$\lambda_1=0 \text{ and } \lambda_0=0,$$

i.e. each terminal tension must be zero.

If the extremity $(x_1\ y_1\ z_1)$ is constrained to lie on a fixed surface, whose equation is $u=0$, we have the displacements of this extremity connected by the equations

$$\left(\frac{dx}{ds}\right)_1\delta x_1+\left(\frac{dy}{ds}\right)_1\delta y_1+\left(\frac{dz}{ds}\right)\delta z_1=0,$$

$$\left(\frac{du}{dx}\right)_1\delta x_1+\left(\frac{du}{dy}\right)_1\delta y_1+\left(\frac{du}{dz}\right)_1\delta z_1=0,$$

which give by the method of undetermined multipliers

$$\frac{\left(\frac{dx}{ds}\right)_1}{\left(\frac{du}{dx}\right)_1}=\frac{\left(\frac{dy}{ds}\right)_1}{\left(\frac{du}{dy}\right)_1}=\frac{\left(\frac{dz}{ds}\right)_1}{\left(\frac{du}{dz}\right)_1},$$

the geometrical meaning of which is that the direction of the string at this extremity is normal to the surface of constraint.

If the extremity is constrained to a curve whose equations are $u=0$, $v=0$, we find in the same way that at this extremity the direction of the string must be at right angles to the curve.

The method which we have just employed is the second method of Art. 182, and expresses that *the variation of the whole*

potential work of the external forces is zero, consistently with the geometrical condition that the distance between every two indefinitely close points in the string remains absolutely unchanged in the displaced position.

285.] **Equilibrium of an Extensible String.** In this case there are no geometrical conditions to be satisfied in the displacement (or deformation) of the string. Then the equation of equilibrium will simply express the condition that in the position of equilibrium the variation of the whole potential work of applied and internal forces is zero.

Now if we consider any elementary mass, $m\,ds$, whose length is ds, and whose internal force (the tension) is T, the work done by this force for a variation δds of the elementary length is (see Art. 70)

$$-T\delta ds.$$

Adding together the similar terms for all the elementary masses, the variation of the potential work of the applied and internal forces is

$$\int m(X\delta x + Y\delta y + Z\delta z)\,ds - \int T\delta ds,$$

which differs from (2) only in having $-T$ instead of λ. Hence the whole discussion is exactly the same as before, and the results are those arrived at in Chap. XII.

There is, however, this distinction between the case of the elastic and that of the inelastic string—that in the second case the value of m, the density, is known at each point, since it can alter only in virtue of extension, and it is therefore the same after the position of equilibrium is assumed as it was before; while in the first case the value of m at each point is not at once known, since in taking the position of equilibrium (*to which our equation of Virtual Work always refers*) extension has taken place at each point. In this case m at each point depends on T according to some law which can be known only by experiment—e.g. Hooke's Law,

$$m = \frac{m_0}{1 + \frac{T}{\lambda}},$$

as in Art. 196.

The equations of the extensible and of the inextensible system are therefore the same only *in form*, since the above constitutes a vital distinction between them.

286.] Property of Minimum. *If a uniform inextensible string, in equilibrium under the action of a given conservative system of forces, joins two fixed points, A and B, the variation of the integral*

$$\int T ds$$

will be zero when we pass from the curve of the string to any indefinitely close curve which passes through A and B.

Let us calculate the variation of this integral.

$$\delta \int T ds = \int (\delta T . ds + T \delta ds)$$

$$= \int \left\{ \delta T . ds + T \left(\frac{dx}{ds} d\delta x + \frac{dy}{ds} d\delta y + \frac{dz}{ds} d\delta z \right) \right\}.$$

Now, from (6) of Art. 184, taking $k\sigma$, or m, as unity,

$$\delta T = -\delta V = -(X\delta x + Y\delta y + Z\delta z).$$

Hence by integration by parts (as in Art. 284), we have

$$\delta \int T ds = T_1 \left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \right)_1 - T_0 \left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z \right)_0$$

$$- \int \left\{ \left[X + \frac{d}{ds}\left(T\frac{dx}{ds} \right) \right] \delta x + \left[Y + \frac{d}{ds}\left(T\frac{dy}{ds} \right) \right] \delta y + \left[Z + \frac{d}{ds}\left(T\frac{dz}{ds} \right) \right] \delta z \right\} ds.$$

Now the right-hand side of this equation is zero, since, the extreme points of the curve being fixed, the coefficients of T_0 and T_1 both vanish, and the coefficients of δx, δy, δz under the sign of integration vanish by the general equations of Art. 284, the mass of a unit length of the string being here taken as unity. Hence the proposition.

This theorem leads to a remarkable property of the common catenary. *Of all curves of the same length joining two given points in a vertical plane, the common catenary is that whose centre of gravity is lowest.* For if $\bar{y}$ be the depth of the centre of gravity of this curve, whose length is L, we have

$$\bar{y} = \frac{\int y ds}{L}.$$

But (Art. 186) $T = mgy$; therefore $\bar{y} = \dfrac{\int T ds}{mgL}$; therefore, by the theorem of this Article, we have

$$\delta \bar{y} = 0.$$

That $\bar{y}$ is in this case a minimum in the true sense of the word does not, of course, appear from this; the proof that it is

so depends on the criterion for maxima and minima furnished by the Calculus of Variations, for which see Jellett's *Calculus of Variations*, p. 80. It is there proved, that when the variation of any integral of the form $\int_{x_0}^{x_1} U dx$ vanishes (the limits being fixed) the value will be, in general, an algebraic maximum or minimum according as $\frac{d^2 U}{dp_n^2}$ is continually $-$ or continually $+$ between the limits of integration, $\frac{d^n y}{dx^n}$ being denoted by p_n, and U being any function of $x, y, p_1, p_2, \dots p_n$. In the present case $U \equiv y ds = y\sqrt{1+p_1^2}\, dx$, a change of the independent variable from s to x being necessary since it is the limits of x that are assigned. The application of the criterion is then obvious.

287.] **Observations on the Method of Lagrange.** The application of the method of Lagrange is attended by a risk of error, which must be guarded against. In applying the equation of Virtual Work to any continuous material system—e. g. a string, a membrane, a fluid—we imagine every point to receive a small displacement from the position which it occupies in the equilibrium configuration of the system. These displacements we have expressed by increments (δx, δy, δz), or (u, v, w) of the co-ordinates of the point; and, according to the nature of the system, there will be various relations between the u, v, w belonging to each point. Thus in the case of an absolutely inextensible string, these quantities have to satisfy at each point the equation

$$\delta(ds) = 0, \text{ or } \frac{dx}{ds}\frac{du}{ds} + \frac{dy}{ds}\frac{dv}{ds} + \frac{dz}{ds}\frac{dw}{ds} = 0.$$

In an absolutely free and unconnected system of particles, they have to satisfy no condition whatever.

Suppose that in any case they have to satisfy the condition of rendering a certain element—e. g. a length, an area, a volume—invariable. Suppose that this element is a function

$$\phi(dx, dy, dz, d^2x, \dots),$$

which we may briefly denote by ϕ. Then *Lagrange's method consists in reducing the problem to a case in which we may treat u, v, w at each point in the system as absolutely independent, so that* (as in Art. 284, for example) *we may equate their coefficients separately to zero.* This is done by taking the variation, $\delta\phi$, of the function which is to remain unaltered in the imagined dis-

placement, multiplying it by an undetermined multiplier, λ, and then adding it under the sign of integration to the variation of the Potential Work of the system; so that our equation, in which u, v, w (or $\delta x, \delta y, \delta z$) are all independent, is

$$-\int \delta \Pi . dm + \int \lambda \delta \phi . dm = 0. \qquad \text{(A)}$$

Now let the case be different. Suppose that the condition $\phi =$ constant has not to be satisfied in the displaced configuration, but that the alteration of ϕ is accompanied by internal forces (or *stresses*) in the system. In this case Lagrange makes no change in his mode of procedure. True, we have no longer the equation $\delta\phi = 0$, but Lagrange, recognising the fact that we have internal work, or work done by the stresses, due to the displacement which alters ϕ, *assumes that the amount of this internal work is fully represented by a term of the form*

$$\lambda \delta \phi,$$

so that our equation of virtual work is still of the form (A).

It is this last assumption which is so liable to mislead, and which is, in more instances than one, a cause of error in Lagrange's own investigations. As a marked instance in which Lagrange has fallen into an error of this kind, we may cite his discussion of the equilibrium of a perfectly flexible surface, which may be (1) perfectly inextensible, or (2) extensible, like a sheet of indiarubber (see the *Mécanique Analytique*, p. 140).

Taking case (1), if dS is the *area* of the superficial element at any point P of the surface, Lagrange assumes that the only equation which u, v, w have to satisfy is $\delta dS = 0$; in other words, that perfect inextensibility is fully provided for if every element of *area* remains unaltered in the (imagined) displaced configuration. But it is clear that perfect inextensibility requires that there shall be no alteration in the length of any line on the surface connecting P with a neighbouring point; and this characteristic is, therefore, expressed by *three* equations between u, v, w instead of one—as in Example 1, p. 157.

Again, when the condition of inextensibility is removed, and the surface is extensible, Lagrange assumes that the internal work of deformation of the element dS is fully represented by $\lambda \delta dS$, i.e. that the work of deformation is simply proportional to the change in the *area* deformed—an assumption which is true for membranes of few known materials.

On the other hand, the similar treatment of a string, whether inextensible or extensible, is perfectly valid, because $\delta ds = 0$ is a perfect expression of inextensibility; and when the string is elastic, the internal work of deformation is perfectly expressed by a term of the form $\lambda \delta ds$.

288.] **Elastic Wire.** As another example of the method of Lagrange, let us take the case of a thin wire, or rod, in the form of any tortuous curve, acted upon continuously throughout its length by a distribution of force (but not of couple), and also by special forces at its two extremities, the wire when unstrained forming any given curve.

Supposing the configuration of equilibrium assumed, write down the equation of Virtual Work for any imagined small derangement of the various points of the wire. It is to be observed that, as we are not treating the case of a *rigid* wire, we have not the conditions $\delta ds = 0$, $\delta d\theta = 0$, which, of course, would hold for the rigid wire, and partially express the condition of rigidity.

Now if T is the longitudinal tension at any point, the increment δds of the length of the element ds is resisted by an amount of work equal to $T\delta ds$.

Again, the alteration of curvature produced by the derangement of parts and depending on the term $\delta d\theta$ will be resisted by an internal couple L, and the alteration will be made against an amount of work equal to $L\delta d\theta$.

Finally, the distortion of the curve denoted by $\delta d\phi$ will be resisted by an internal couple M, and will require an amount of work equal to $M\delta d\phi$.

This last distortion ($\delta d\phi$) is overlooked by Lagrange, whose investigation of the problem is in consequence erroneous—as pointed out by Bertrand (Bertrand's edition of the *Mécanique Analytique*, pp. 143, 148, 401).

Let the special force applied at one extremity, A, have components (X_1, Y_1, Z_1), and that applied at the other, O, (X_0, Y_0, Z_0), the co-ordinates of these extremities being $(x_1 y_1 z_1)$, $(x_0 y_0 z_0)$.

The equation of Virtual Work, then, is

$$X_1\delta x_1 + Y_1\delta y_1 + Z_1\delta z_1 + X_0\delta x_0 + Y_0\delta y_0 + Z_0\delta z_0$$
$$+\int[m(X\delta x + Y\delta y + Z\delta z)\,ds - T\delta ds - L\delta d\theta - M\delta d\phi] = 0. \quad (1)$$

To obtain the three general differential equations which determine the curve—and which answer to equations (A) of Art. 284—it would be necessary to substitute in this equation the values of δds, $\delta d\theta$, $\delta d\phi$ given in (β), Art. 282, and in (ϵ) and (ζ) Art. 283, and to integrate the terms in $\frac{d\delta x}{ds}$, $\frac{d^2\delta x}{ds^2}$, $\frac{d^3\delta x}{ds^3}$, &c., as in Art. 284; finally, equating to zero the separate coefficients of δx, δy, δz under the integral sign.

The equations thus obtained are of great complexity, and would not repay any labour bestowed upon them. Moreover, as we shall see, the general problem can be more simply treated otherwise by a method which is free from false assumptions.

We shall here confine our illustration of the method of Lagrange to the simple case* in which the wire forms a plane curve, in whose plane are the forces and the displacements; so that we may neglect the terms in M—while not assuming that M is zero. If we simply put $M = 0$, while assuming the curve to be tortuous, as Lagrange does, its constitution would be like that of a continuous chain of smooth beads strung freely on a string.

Under this condition, equation (1) becomes simply

$$X_1\delta x_1 + Y_1\delta y_1 + X_0\delta x_0 + Y_0\delta y_0 + \int[m(X\delta x + Y\delta y)ds - T\delta ds - L\delta d\theta] = 0. \quad (2)$$

Substituting the values of δds and $\delta d\theta$, the integral term becomes

$$\int\left[mX\delta x - \left(T - \frac{L}{\rho}\right)\frac{dx}{ds}\frac{d\delta x}{ds} - \rho L\frac{d^2x}{ds^2}\frac{d^2\delta x}{ds^2} + \ldots\right]ds, \quad (3)$$

writing down, for simplicity, only the terms relating to x.

Integrating the second term once by parts, and the third twice, this becomes

$$X_1\delta x_1 + X_0\delta x_0 - \left[\left\{\left(T - \frac{L}{\rho}\right)\frac{dx}{ds} - \frac{d}{ds}\left(\rho L\frac{d^2x}{ds^2}\right)\right\}\delta x\right]_0^1$$

$$+\text{similar term in } y - \left[\rho L\frac{d^2x}{ds^2}\frac{d\delta x}{ds}\right]_0^1 + \text{ similar term in } y$$

$$+\int\left[\left\{mX + \frac{d}{ds}\cdot\left(T - \frac{L}{\rho}\right)\frac{dx}{ds} - \frac{d^2}{ds^2}\left(\rho L\frac{d^2x}{ds^2}\right)\right\}\delta x\right.$$

$$\left. + \text{similar term in } y\right]ds = 0, \quad (4)$$

* The following discussion, be it observed, is given merely as an exercise in the Principle of Virtual Work. The equations obtained can be arrived at with much greater rapidity by direct elementary methods.

the suffixes in the terms outside the integral sign having reference to the extremities of the wire.

As the variations δx and δy may now be considered independent at all points of the curve, we have at all points

$$\frac{d}{ds}\cdot\left(T-\frac{L}{\rho}\right)\frac{dx}{ds}-\frac{d^2}{ds^2}\left(\rho L\frac{d^2x}{ds^2}\right)+mX=0, \tag{5}$$

$$\frac{d}{ds}\cdot\left(T-\frac{L}{\rho}\right)\frac{dy}{ds}-\frac{d^2}{ds^2}\left(\rho L\frac{d^2y}{ds^2}\right)+mY=0, \tag{6}$$

which determine the form of the curve.

Assuming, for definiteness, the figure of the curve to be concave towards the axis of x, if θ is the acute angle made with this axis by the tangent at any point, we have

$$\frac{d\theta}{ds}=-\frac{1}{\rho},\quad \frac{dx}{ds}=\cos\theta,\quad \frac{dy}{ds}=\sin\theta,\quad \frac{d^2x}{ds^2}=\frac{\sin\theta}{\rho},\ \text{and so on.}$$

Hence (5) and (6) become

$$\frac{d}{ds}\cdot\left(T-\frac{L}{\rho}\right)\cos\theta-\frac{d^2}{ds^2}(L\sin\theta)+mX=0\,; \tag{7}$$

$$\frac{d}{ds}\cdot\left(T-\frac{L}{\rho}\right)\sin\theta+\frac{d^2}{ds^2}(L\cos\theta)+mY=0. \tag{8}$$

Performing the differentiations, multiplying the first by $\cos\theta$, the second by $\sin\theta$, adding, and putting $S=m(X\cos\theta+Y\sin\theta)$ = tangential component of applied force, we have

$$\frac{dT}{ds}+\frac{1}{\rho}\frac{dL}{ds}+S=0. \tag{9}$$

Again, multiplying the second by $\cos\theta$, the first by $\sin\theta$, subtracting, and putting $N=m(Y\cos\theta-X\sin\theta)$ = normal component of applied force, we have

$$\frac{d^2L}{ds^2}-\frac{T}{\rho}+N=0. \tag{10}$$

These two equations can be arrived at much more rapidly by the direct elementary process. Integrating (7) and (8), we have

$$\left(T-\frac{L}{\rho}\right)\cos\theta-\frac{d}{ds}(L\sin\theta)=a-\int mXds\equiv P; \tag{11}$$

$$\left(T-\frac{L}{\rho}\right)\sin\theta+\frac{d}{ds}(L\cos\theta)=b-\int mYds\equiv Q; \tag{12}$$

a and b being constants which we shall presently determine.

Equations (11) and (12) are easily integrable once more. For,

multiplying the first by $\sin\theta$, the second by $\cos\theta$, and subtracting,

$$\frac{dL}{ds} = -P\sin\theta + Q\cos\theta; \tag{13}$$

$$\therefore \quad L = -\int P dy + \int Q dx + k, \tag{14}$$

where k is a constant.

It is usual to assume that the value of L at any point is proportional to the change of curvature at the point; so that if r is the radius of curvature at the point before strain, we have

$$L = A\left(\frac{1}{\rho} - \frac{1}{r}\right), \tag{15}$$

where A is a constant depending on the rigidity of the wire. Hence we have for the determination of the form of the curve the equation

$$A\left(\frac{1}{\rho} - \frac{1}{r}\right) = -\int P dy + \int Q dx + k. \tag{16}$$

Equating to zero, in (4), the terms outside the integral sign, we have, so far as one extremity, A, of the rod is concerned,

$$\left[X_1 - \left(T - \frac{L}{\rho}\right)\cos\theta - \frac{d}{ds}(L\sin\theta)\right]\delta x - L\sin\theta \cdot \frac{d\delta x}{ds}$$

$$+\left[Y_1 - \left(T - \frac{L}{\rho}\right)\sin\theta + \frac{d}{ds}(L\cos\theta)\right]\delta y + L\cos\theta \cdot \frac{d\delta y}{ds} = 0, \tag{17}$$

omitting the suffix 1 (for convenience) which may be understood to be attached to every letter. The terms relating to the other extremity, O, equated to zero, give a precisely similar equation, with $-X_0$, $-Y_0$ written instead of X_1, Y_1.

Now we may have any of the following circumstances relating to the end—

(*a*) the end may be perfectly free;
(*b*) the end may be fixed, but not the tangent at it;
(*c*) the end may be tangentially fixed.

If (*a*) happens, the variations δx, $\delta\frac{dx}{ds}$, ... are all perfectly arbitrary, so that their coefficients must be severally zero.

If (*b*) happens, $\delta x = 0$ and $\delta y = 0$, therefore the first and third terms in (17) disappear without furnishing any equations; but $\delta\frac{dx}{ds}$ and $\delta\frac{dy}{ds}$ are quite arbitrary, since the direction of the tangent may be varied at pleasure. Hence the coefficients of these terms must be equated to zero.

If (*c*) happens, there is no arbitrary displacement, and each term in (17) disappears without furnishing any equation.

In addition to these cases, we might, of course, have that in which the end (by means of a small ring) is constrained to move along a given line, $y = px + q$, so that we should have $\delta y = p\delta x$, with the result that the coefficient of δy is to be equated to p times that of δx in (17).

The following results are at once obvious.

If the end is perfectly free, the change of curvature at it is zero, and the tension is equal to the component of the applied force along the tangent; for we have $\rho L = 0$, or, by (15), $1 - \frac{\rho}{\rho_0} = 0$; and then equating to zero the coefficients of δx and δy in (17), multiplying the first by $\cos\theta$ and the second by $\sin\theta$ and adding, we get $T = X\cos\theta + Y\sin\theta$.

If the end is fixed, but not tangentially, the change of curvature is zero. These results are, of course, at once perfectly obvious from elementary considerations.

The constants of integration in (11), (12), (14), are to be determined by the circumstances of the extremities. Thus, if both ends are free, and we suppose the integrations to commence at the extremity, O, we have, by substituting the co-ordinates of this extremity in (11) and (12),

$$-X_0 = a; \quad -Y_0 = b.$$

Also substituting those of the other extremity,

$$X_1 = -X_0 - \Big[\int mXds\Big]_0^1; \qquad Y_1 = -Y_0 - \Big[\int mYds\Big]_0^1,$$

results which are at once obvious from the most elementary principles.

Substituting these in (14), and observing that $L = 0$ at the end, we have $k = 0$.

Again, if the end O is tangentially fixed, substituting the co-ordinates of the other end in (11) and (12), we have, with the aid of (17), $a = X_1 + \int mXds$; $b = Y_1 + \int mYds$, the co-ordinates of A being understood to be those in the general integrals in these expressions. Also since $L = 0$ at A, we have from (14)

$$k = \int Pdy - \int Qdx,$$

in which the co-ordinates are those of A.

If *both* ends are tangentially fixed, we derive no assistance

from the terms outside the integral sign, and the course to be pursued for the determination of the constants will be at once rendered clear to the student if he considers the simpler problem of an inextensible string acted upon by gravity, its two extremities being fixed at any two given points. The process of the determination of constants is this—the forms of X and Y as functions of the co-ordinates being assumed as given, imagine the equation (16) to be completely integrated, taking x as the independent variable.

It is a differential equation of the second order, since ρ involves $\frac{d^2y}{dx^2}$; and its integration will introduce *two* more constants, m and n; so that we should finally have an integral of the form

$$\phi(x, y, a, b, k, m, n) = 0.$$

If, for simplicity, we take the end O for origin and the tangent at it for axis of x, while the tangent at A makes a given angle α with this axis, we have the following equations—

$$\phi(0, 0, a, b, k, m, n) = 0,$$

$$\phi(x, y, a, b, k, m, n) = 0,$$

$$\left(\frac{d\phi}{dx}\right)_0 = 0,$$

$$\tan\alpha\cdot\left(\frac{d\phi}{dy}\right)_1 + \left(\frac{d\phi}{dx}\right)_1 = 0,$$

and, in addition, the length of the curve is given, so that

$$\int \frac{\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right]^{\frac{1}{2}}}{\frac{d\phi}{dy}}dx = l = \text{length of curve}.$$

Hence we have five equations to determine the five constants.

289.] **Flexible Inextensible Surface.** As another illustration of the method of Lagrange, we shall now consider the equilibrium of a perfectly flexible and perfectly inextensible surface.

The equations which have to be satisfied by the component displacements, u, v, w, at every point, when there is perfect inextensibility and no other condition, are given in p. 157.

Let the components of externally applied force per unit area

at any point of the surface be X, Y, Z. Then if ϵ denotes $\sqrt{1+p^2+q^2}$, the elementary area dS at any point of the surface is $\epsilon\, dx\, dy$, and we shall have

$$\iint \epsilon (Xu + Yv + Zw)\, dx\, dy = 0, \tag{1}$$

subject to the relations referred to between u, v, w—viz.

$$\left.\begin{aligned} \frac{du}{dx} + p\frac{dw}{dx} = 0; \quad \frac{dv}{dy} + q\frac{dw}{dy} = 0; \\ \frac{du}{dy} + \frac{dv}{dx} + p\frac{dw}{dy} + q\frac{dw}{dx} = 0. \end{aligned}\right\} \tag{a}$$

Multiplying the left-hand sides of these equations by the undetermined multipliers λ, μ, ρ, respectively, and adding the results under the signs of integration in (1), we have

$$\iint \Big[\epsilon X . u + \epsilon Y . v + \epsilon Z . w + \lambda \left(\frac{du}{dx} + p\frac{dw}{dx}\right) + \mu \left(\frac{dv}{dy} + q\frac{dw}{dy}\right)$$
$$+ \rho \left(\frac{du}{dy} + \frac{dv}{dx} + p\frac{dw}{dy} + q\frac{dw}{dx}\right)\Big] dx\, dy = 0, \tag{2}$$

in which, after the method of Lagrange, we may now treat u, v, w as completely independent.

Now take the term $\iint \lambda \frac{du}{dx}\, dx\, dy$, and first perform the integration with respect to x, considering y constant.

Thus $$\int \lambda \frac{du}{dx}\, dx = \lambda'' u'' - \lambda' u' - \int u \frac{d\lambda}{dx}\, dx, \tag{3}$$

in which the term $\lambda'' u''$ relates to the point which has one extreme value of x as abscissa (with the supposed constant value of y), and $\lambda' u'$ to the point which has the other extreme value of x as abscissa. These points, occurring at the end and the beginning, respectively, of the integration with respect to x (y being constant) are points on the bounding edge of the surface—points in which it is cut by the plane $y =$ constant.

Hence $$\iint \lambda \frac{du}{dx}\, dx\, dy = \int (\lambda'' u'' - \lambda' u')\, dy - \iint u \frac{d\lambda}{dx}\, dx\, dy, \tag{4}$$

in which the single integration is one relating only to the edge, and the double is one carried over the whole surface.

Now, instead of the single integral we may evidently write $\int \lambda u\, dy$; for this last, when carried *continuously* round the edge, will include both the terms $\lambda'' u''\, dy$ and $-\lambda' u'\, dy$, which belong to points having the same y and different extreme values of x.

Treating the other double integrals in the same way, we have as the equivalent of (1)

$$\int[(\rho dx+\lambda dy)(u+pw)+(\mu dx+\rho dy)(v+qw)]$$
$$+\iint\Big[\Big(\epsilon X-\frac{d\lambda}{dx}-\frac{d\rho}{dy}\Big)u+\Big(\epsilon Y-\frac{d\rho}{dx}-\frac{d\mu}{dy}\Big)v$$
$$+\Big(\epsilon Z-\frac{d(p\lambda+q\rho)}{dx}-\frac{d(p\rho+q\mu)}{dy}\Big)w\Big]dx\,dy=0. \qquad (5)$$

Equating to zero the coefficients of u, v, w under the double integral, the general equations of equilibrium are

$$\frac{d\lambda}{dx}+\frac{d\rho}{dy}=\epsilon X, \qquad (6)$$

$$\frac{d\mu}{dy}+\frac{d\rho}{dx}=\epsilon Y, \qquad (7)$$

$$\lambda r+\mu t+2s\rho=\epsilon(Z-pX-qY), \qquad (8)$$

in which, as usual,

$$r=\frac{dp}{dx}, \qquad t=\frac{dq}{dy}, \qquad s=\frac{dp}{dy}=\frac{dq}{dx}.$$

Now it will be observed that these three equations serve merely to determine λ, μ, ρ, and furnish us with no equation for the determination of the form of the surface.

If the bounding edge is fixed at all points, u, v, and w under the sign of single integration in (5) are severally zero at all points; so that the terms relating to the edge furnish us with no equations. In this case, therefore, we must conclude that the form of the surface is geometrically determinate, quite irrespective of the forces acting on it; in other words—*if every point on the bounding edge of a perfectly inextensible surface is fixed, the surface can take only one figure,* no displacement being possible at any point on it.

If the bounding edge is completely free, we must equate to zero the coefficients of the displacements under the sign of single integration, so that the differential equations of the edge are

$$\rho\frac{dx}{ds}+\lambda\frac{dy}{ds}=0, \qquad (9)$$

$$\mu\frac{dx}{ds}+\rho\frac{dy}{ds}=0, \qquad (10)$$

(where ds is an element of length of the edge) the values of λ, μ, ρ

in these equations being obtained from the general equations (6), (7), (8).

As a very simple example, take the case of a uniform rectangular sheet $ABCD$, two of whose sides, AB and CD, are fixed horizontally and parallel to each other, gravity being the only external force. We know from elementary considerations that the sections of this surface by vertical planes perpendicular to the lines AB and CD are catenaries; and this result follows from the above equations. For, taking the axis of z vertically upwards, and the axis of x horizontal and perpendicular to the direction of AB and CD, all differential coefficients with respect to y vanish, while $X = Y = 0$, $Z = -g$; so that the general equations give

$$\frac{d\lambda}{dx} = 0, \quad \frac{d\rho}{dx} = 0,$$

$$\lambda \frac{dp}{dx} = -g\sqrt{1+p^2}, \tag{11}$$

while the terms relating to the two free sides give $\rho = 0$. Now the integral of (11) gives at once the Catenary equation between z and x.

If the bounding edge, instead of being fixed at all points, has external force applied along it, and if X_0, Y_0, Z_0 be the components of this force at any point of the edge, per unit length, the left-hand side of (5) will require the addition of the virtual work of this boundary force; that is, we must add to it the term $\int (X_0 u + Y_0 v + Z_0 w)\,ds$; so that the boundary equations (9) and (10) must be replaced by

$$\rho \frac{dx}{ds} + \lambda \frac{dy}{ds} + X_0 = 0, \tag{12}$$

$$\mu \frac{dx}{ds} + \rho \frac{dy}{ds} + Y_0 = 0, \tag{13}$$

$$pX_0 + qY_0 - Z_0 = 0, \tag{14}$$

the last of which shows that the boundary force must at every point lie in the tangent plane to the surface at the point. The general equations (6), (7), (8) serve merely to determine λ, μ, ρ; and in all cases the form of the surface is known from the equations of the boundary.

290.] **Jellett's Results.** The dynamical treatment of the equilibrium of an inextensible surface has conducted us to a conclusion with respect to the effect of fixing its bounding edge which, although holding good in general, admits of exceptions when the bounding edge is selected in a particular manner.

The properties of inextensible surfaces, with regard to dis-

placement, have been very fully treated by Jellett in a paper to which the attention of the student is directed—*On the Properties of Inextensible Surfaces*, in the Transactions of the Royal Irish Academy, vol. xxii.

We cannot do more than summarize the results arrived at in this paper. If, for brevity, we put $\xi = u + pw$, $\eta = v + qw$ in equations (a) of last Article, the equations of inextensibility become

$$\frac{d\xi}{dx} - wr = 0, \tag{1}$$

$$\frac{d\eta}{dy} - wt = 0, \tag{2}$$

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} - 2ws = 0; \tag{3}$$

and it is clear that the immobility of a point is fully expressed by the conditions $\xi = 0$, $\eta = 0$, $w = 0$.

If now we consider the effect of fixing any curve on the surface, the determination of the displacement (if any) of a point on the surface is identical with the solution of the following problem:—To find three functions, ξ, η, w, which satisfy equations (1), (2), (3), and which vanish at all points on the given curve.

Let the differential equation of the projection of the given bounding edge on the plane of x, y be

$$dy = m\,dx.$$

It is then shown that the vanishing of ξ, η, w at all points on this bounding edge will necessitate their vanishing at all points on the surface, unless m is such as to satisfy the equation

$$r + 2sm + tm^2 = 0. \tag{a}$$

Now for a whole class of surfaces it is impossible to satisfy this equation with any real value of m—viz. the class of surfaces for which

$$rt - s^2 \text{ is } + \text{ at all points,}$$

i.e. for surfaces whose two principal curvatures are of the same sign at all points; so that if a bounding edge of any form is fixed on any such surface, no displacement of any point on it is possible.

If the surface is such that at all points

$$rt - s^2 = 0,$$

i.e. if it is a developable, motion will be possible if the fixed edge is any one of its rectilinear sections, or its *edge of regression*. If on such a surface a portion of a curve, AB, not coinciding with either of these is fixed, the whole portion of the surface included between the rectilinear sections drawn through A and B, unlimited in one direction and bounded by the intercepted portion of the edge of regression in the other, is immoveable.

If the surface is such that at all points

$$rt - s^2 \text{ is negative,}$$

i.e. if its principal curvatures are of opposite signs, at each point there are two directions satisfying (a), and if the bounding edge coincides with a curve satisfying this equation at all points, displacement is possible.

291.] **Particular Case of Flexible Surface.** The only case to which the investigation given by Lagrange in the *Mécanique Analytique* applies without error is that of a flexible extensible surface of such a nature that the work done at any point by the internal forces exerted over the contour of any element of area, dS, when this area receives a small change, is proportional solely to the amount of increase δdS, of the area, and not dependent in any way on its change of *shape*.

Let P be any point on the surface, and at P draw any line PQ, of infinitesimal length, in the surface. Then, in *general*, the force exerted over the length PQ by the portion of the surface at one side on the portion of the surface on the other side, of PQ, will be oblique to PQ; but whatever be the nature of the surface, there are *two* directions, PQ_1 and PQ_2, of PQ such that this force is perpendicular to PQ (as will be shown in the chapter on Strain and Stress); the magnitude of this normal force divided by the length PQ over which it is exerted is called *surface-tension*. If the surface-tensions on PQ_1 and PQ_2 are equal, it follows (as will be subsequently proved) that the stress exerted on every elementary length PQ drawn on the surface near P is a normal force, and the surface-tension is constant all round P. If we denote this constant value at P by N, it will be easily seen, by taking for dS any small closed surface, that for a small increase of its area the work done by the (normal) stress all round it is

$$N \times \delta dS.$$

(Apply exactly the reasoning in Example 3, p. 117, by which

it is shown that the element of work done by the pressure of a gas is $p\,dv$.)

If the surface has attained its equilibrium configuration, and if we imagine displacements (u, v, w) at each point as before, the equation of Virtual Work will be

$$\int\int\epsilon\,(Xu+Yv+Zw)\,dx\,dy+\int N\delta\,dS=0. \tag{1}$$

Now, if the element of superficial area, dS, is that cut off by two very close planes perpendicular to the axis of x and two very close planes perpendicular to the axis of y, we have $dS=\epsilon\,dx\,dy$; and the changes in the co-ordinates will cause $dx\,dy$ to become $dx\,dy\left(1+\frac{du}{dx}+\frac{dv}{dy}\right)$, as is at once found. Hence

$$\begin{aligned}\delta dS &= \left(\frac{du}{dx}+\frac{dv}{dy}\right)\epsilon\,dx\,dy+dx\,dy\,.\,\delta\epsilon\\ &= \left[\left(\frac{du}{dx}+\frac{dv}{dy}\right)\epsilon+\frac{p\,\delta p+q\,\delta q}{\epsilon}\right]dx\,dy.\end{aligned}$$

Substituting in this expression the values of δp and δq given in Art. 283, and then integrating (1) in the usual way, and equating to zero the coefficients of u, v, w under the sign of double integration, we have the equations

$$\begin{aligned}\epsilon X-\frac{d\,.\,\epsilon N}{dx}+\frac{d}{dx}\frac{p^2N}{\epsilon}+\frac{d}{dy}\frac{pqN}{\epsilon}&=0,\\ \epsilon Y-\frac{d\,.\,\epsilon N}{dy}+\frac{d}{dx}\frac{pqN}{\epsilon}+\frac{d}{dy}\frac{q^2N}{\epsilon}&=0,\\ \epsilon Z-\frac{d}{dx}\frac{pN}{\epsilon}-\frac{d}{dy}\frac{qN}{\epsilon}&=0,\end{aligned}$$

which hold at all points on the surface.

Denoting $\frac{d}{dx}\frac{pN}{\epsilon}+\frac{d}{dy}\frac{qN}{\epsilon}$ by U, these equations may be written.

$$\epsilon X-\epsilon\frac{dN}{dx}+pU=0, \tag{2}$$

$$\epsilon Y-\epsilon\frac{dN}{dy}+qU=0, \tag{3}$$

$$\epsilon Z-U=0. \tag{4}$$

Substituting in (2) and (3) the value of U given by (4), we have

$$\frac{dN}{dx}=X+pZ;\quad \frac{dN}{dy}=Y+qZ; \tag{5}$$

and if $Xdx+Ydy+Zdz=-d\Pi$, the differential of single function, Π, these last equations give

$$N+\Pi=a, \tag{6}$$

where a is a constant.

Now N *may be regarded indifferently as the surface-tension or as the potential work of the stress (static energy) per unit area on the surface*, because the potential work of the stress for an increment of area equal to δdS is $N\times\delta dS$. Hence (6) asserts that in the equilibrium configuration the sum of the potential works of stress and of external forces per unit area is the same at all points on the surface.

The values of $\frac{dN}{dx}$ and $\frac{dN}{dy}$ given in (5) when substituted in U give

$$U=\frac{pX+qY+(p^2+q^2)Z}{\epsilon}+N\frac{(1+q^2)r-2pqs+(1+p^2)t}{\epsilon^3}.$$

Now if R_1 and R_2 are the principal radii of curvature of the surface at any point, we have (Salmon's *Geometry of Three Dimensions*, Chap. XI.),

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{(1+q^2)r-2pqs+(1+p^2)t}{\epsilon^3},$$

so that $$\epsilon U=pX+qY+(\epsilon^2-1)Z+N\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

and therefore (4) becomes

$$\frac{1}{\epsilon}(Z-pX-qY)=N\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

But the left-hand side of this equation is the component, F_n, of the external forces along the normal to the surface; hence we have

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{F_n}{N}=\frac{F_n}{-\Pi+a}, \tag{a}$$

as the equation of the surface.

This equation can be at once obtained by a direct elementary process, as will be shown in a subsequent chapter.

If the surface is one acted upon by gravity only, and if it lies very nearly in a horizontal plane, from which z is measured, we may neglect such products as pq, pz, pr, &c., and equation (4) gives

$$\varpi-N(r+t)=0,$$

where ϖ is the weight per unit area of the surface; or

$$\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}=\frac{\varpi}{a}=\text{const.}$$

292.] **Surface-Tension of a Liquid.** At each point, P, of the surface of contact of a liquid with a gas, another liquid, or a solid, there exists a tension, the magnitude of which across a given elementary length PQ in the tangent plane at P is the same for all directions of PQ; and the amount of this tension divided by PQ is, as just stated, the surface-tension at P.

The main laws to which this surface-tension is subject are the following:—

1. For a given liquid in contact with any given substance it is the same at all points on the surface of contact.

2. It varies with the temperature, becoming less as the temperature becomes greater.

3. It varies if the substance with which the liquid is in contact is varied. Thus, it is not the same on the surface of water in contact with air as on the surface of water in contact with mercury.

4. It is independent of the curvature of the surface of contact. Thus, if a soap bubble is blown out through the end of a tube, the surface-tension of the film (in contact with air) is the same when the diameter of the bubble is 6 inches as it was when the diameter was 1 inch.

This curious fact at once distinguishes the nature of the distention of a liquid film from that of the distention of an elastic string, because for the latter the magnitude of the tension increases with the extension, while the tension of the liquid film is independent of the extension of its surface.

A probable explanation of this result for a liquid (suggested to the Author by Mr. W. G. Gregory) may be found in the fact that the molecules of a liquid are moveable on each other with very great ease, so that when, by a distention of the extreme surface layer of molecules, the molecules which surrounded any molecule P have their distances from P increased, their vacant places on the surface are taken by others which come up to the surface by the thinning of the film, thus leaving the molecular arrangement round P practically unaltered—unless the surface is reduced below a thickness less than the diameter of the 'sphere of molecular activity' (Art. 293), in which case the constancy of surface-tension ceases.

The existence of surface-tension may be demonstrated experimentally in several ways. One of the simplest methods consists

in taking a rectangle formed of brass strips or wires, *AB*, *BC* *CD*, and *EF* (Fig. 262), of which the first three are in one rigid piece, while the last, *EF*, is capable of sliding up and down on the bars *AB* and *DC*.

The space, *abcd*, enclosed by the four bars being vacant, dip the system into a soap solution, thus forming a film (represented by the shading) in this space; this film attaches itself to all the bars; and if the moveable bar *EF* is not restrained by the fingers, it will be drawn along the others by the film until it reaches *BC*. If *EF* is not too heavy, and the plane of the bars is held vertical, *BC* being above *EF*, this latter will be lifted by the tension of the film.

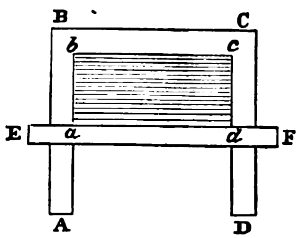

Fig. 262.

Another very striking illustration of the existence of surface-tension is obtained thus:—Take a circular brass wire, A (Fig. 263): dip it into the soap solution, thus covering its area on withdrawal with a thin film (represented by the shading); now form a loop of a piece of thread and place it gently on the surface of the film. This loop is represented by *ab* in the figure. Now perforate the film inside the loop by a pin or fine wire, and instantly the loop of thread is drawn out into a circle, *c*, by the contracting film.

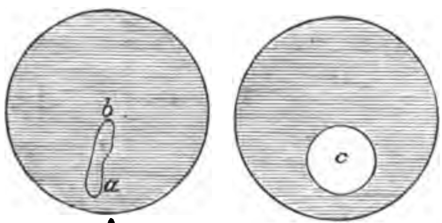

Fig. 263.

This experiment illustrates not only the existence of surface-tension, but also another property to which we shall presently refer.*

At the temperature 20°C. the surface-tension of water in contact with air is 81 dynes per centimètre; for mercury in contact with air it is 540 dynes per centimètre; and for mercury in contact with water 418 dynes per centimètre (see Everett's *Units and Physical Constants*, p. 42).

For the best method of measuring surface-tension, and verifying its independence of the curvature of the surface, the reader may consult Plateau's *Statique Expérimentale et Théorique des*

* Namely, the property of minimum area, Art. 295.

Liquides soumis aux seules Forces Moléculaires, vol. i., pp. 272, &c. This work abounds in beautiful illustrations of the forms assumed by liquid films, and contains precise information on all the details necessary for experiments.

293.] **Forms of Liquid Surfaces.** The molecules of a liquid which lie on its bounding surface—that is, its surface of contact with a solid or with a fluid—experience attraction from other molecules of the liquid which are at infinitely small distances from them. Thus, a molecule at any point, P, on the bounding surface will experience attraction from all the liquid molecules which lie in a hemisphere having P as centre and an extremely small radius, PQ. The distance PQ is called the radius of molecular activity, the attraction of two molecules separated by a distance greater than PQ being insensible.* If at each point P on the surface we measure off the distance PQ along the normal, the Q points form a layer parallel to the bounding surface such that the surface molecules are unacted upon by the layers below this one.

If also we produce the normal at P into the surrounding medium—i.e. the solid body, or air, or any superincumbent fluid—the liquid molecule at P will be acted upon by a molecule of this medium at a certain distance PQ' along this normal, and by all molecules whose distances from P are less than PQ'. We shall thus have another layer of molecules in the medium defining the limits within which the surface molecules of the liquid are acted upon by those of the medium.

Now, without any *à priori* knowledge of the magnitude of the radius of molecular activity either for the liquid or for the body in contact with it, it can be shown (as was first shown by Laplace, *Méc. Cél.*, supplement to Book X.) that at any point, P, on the liquid surface the force per unit area due to the effective molecular attractions of the liquid itself and of the medium with which the liquid is in contact is of the form

$$A + N\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{1}$$

where A and N are constants for the same liquid and same surrounding medium, provided the temperature is constant, and R_1, R_2 are the principal radii of curvature of the liquid surface at

* The fact that a liquid rises to the same height in a capillary tube, whatever its *thickness*, provided its internal diameter is constant, is supposed to justify this assumption.

P, both principal sections being supposed to be *convex* towards the medium at P. If both are *concave* towards the medium, the molecular force per unit area is

$$A - N\left(\frac{1}{R_1} + \frac{1}{R_2}\right); \tag{2}$$

and if one is concave, the corresponding radius of curvature is to be taken negatively in (1).

The quantity N is the surface-tension of the liquid when in contact with the given medium. With regard to the constant A nothing appears to have been determined (Clerk Maxwell's Article on Capillary Action in the *Encyclopædia Britannica*), except that it is much greater than the term depending on curvature.

From the expression (1) it appears that even when the liquid surface is plane, there is normal pressure at each point due to molecular action. This is due to the fact that about any point on the surface only one-half of the sphere of influencing molecules can be described. A molecule at a finite distance from the surface would be subject, on the contrary, to molecular attractions in all directions round it.

Suppose, then, a liquid mass not acted upon by any external forces. For such a mass the expression (1) must be constant all over its surface, i.e.

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{a}, \tag{3}$$

at all points on the surface.* A drop of olive oil in a mixture of water and alcohol, which is made to have the same specific gravity as the oil, will be an approximation to such a liquid mass; and by seizing this drop between two wires in the shapes of any closed curves, or by allowing the drop to form round a solid of any shape held in the suspending liquid mixture by a very fine wire, we can obtain as many figures of liquid surfaces as we please, each satisfying equation (3). A full description of beautiful experiments of this kind will be found in M. Plateau's work just quoted.

If in (a) of last Article we have F_n constant and also N constant, we have a surface satisfying equation (3), i.e. its form will be one of those assumed by a liquid surface under the conditions just described.

Instead of obtaining such surfaces by means of drops of oil, we may very easily obtain beautiful illustrations of them by means of soap bubbles. A soap bubble is a thin liquid film

* This result will be deduced in a note at the end of the Volume.

which is in contact with air at both sides of the surface. When it is blown out through the end of a tube, its surface is spherical, and the normal intensity of external force, F_n, is the excess of the air pressure per unit area inside over that outside—assuming that the film is so thin that its weight per unit area is wholly negligible. For this case, if R is the radius of the sphere, the intensity of pressure on the outer (convex) surface due to molecular attraction is

$$A+\frac{2N}{R},$$

while on the inner surface the intensity of pressure due to the same cause is

$$A-\frac{2N}{R}.$$

Hence the resultant intensity of pressure due to this cause is

$$\frac{4N}{R};$$

and if p is the *excess* of internal air pressure over external air pressure per unit area, we have

$$\frac{4N}{R}=p. \tag{4}$$

Hence for all sizes of the bubble we have $p \,.\, R$ constant.

For any shape of the film, the resultant of the molecular pressure intensities exerted at any point on both sides is

$$2N\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \tag{5}$$

and if the excess of air pressure is, as before, p, we have

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{p}{2N}. \tag{6}$$

If $p=0$, i.e. if the inner and outer surfaces are both in contact with the atmosphere, we must have

$$\frac{1}{R_1}+\frac{1}{R_2}=0, \tag{7}$$

that is, the two principal curvatures are equal and opposite at all points on the surface.

Several possible forms of equilibrium of films are at once obvious. Thus, a closed surface consisting of a cylinder terminated by two spherical caps is obviously possible; and if r is the radius of the cylinder and r' that of the sphere, the left-hand side of (6) is $\frac{1}{r}$ all over the cylindrical portion, and $\frac{2}{r'}$ over the spherical ends; therefore $r'=2r$.

This figure is very easily produced thus. Take two wires, each formed into a circle (two or three inches in diameter); support one circle horizontally on three legs (each about two inches high); by means of a thread fastened at three points on the circumference suspend the other circle with its plane parallel to that of the first circle, vertically over this circle, the distance between the planes of the circles being about three inches; then dip the end of a glass tube in the soap solution, and through this end blow a bubble between the two wires, enlarging the bubble until it attains complete contact with each circle. The surface of the bubble will then form a film between the two circles, and each wire will be covered with a spherical cap.

By raising the upper circle, or lowering it, the radii of the spheres can be diminished or increased, and the cylindrical form obtained.

The persistence of these films is greatly increased by adding glycerine to the soap and water, and by this means M. Plateau has obtained films which lasted for 18 hours.

Again, having obtained the cylinder with two spherical ends, rupture the spherical caps by driving a wire or a glass rod down through them in a direction parallel to the axis of the cylinder. The portion of surface connecting the circular wires will remain, but its shape instantly alters from that of a cylinder, becoming a surface generated by the revolution of a catenary round its axis. For in this case, as $p = 0$, the surface satisfies equation (7); and as the surface is one of revolution, its two principal radii of curvature at any point are the radius of curvature of the *meridian*, or revolving curve, and the normal terminated by the axis of revolution. Now it is well known that the catenary is the curve in which these are equal and opposite.

The figure of the film thus formed is called a *Catenoid*.

Of films whose surfaces are surfaces of revolution there are three classes which have been experimentally investigated by Plateau. We proceed to deduce their forms analytically.

Let ρ be the radius of curvature at any point, P, of the meridian, and n the length of the normal between P and the axis of revolution. Then (6) becomes

$$\frac{1}{\rho} + \frac{1}{n} = \frac{p}{2N}. \tag{8}$$

Let A (Fig. 264) be a plane circular iron wire, two or three

inches in diameter, lying horizontally over a table; suspended vertically over it let there be another circular wire, B, of the same or different diameter, both having been previously moistened with the liquid, and blow through one end of a tube a soap bubble until its surface comes into contact with both wires. Then the film will assume a figure of revolution between the wires terminating in spherical caps (not represented in the figure) covering the wires.

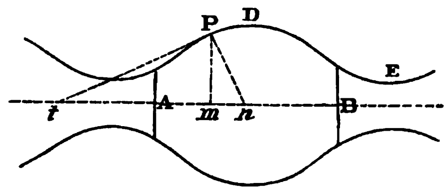

Fig. 264.

The air inside it will have no communication with the outside air. By causing B to approach or to recede from A, the figure of the film will be altered.

Let AB be the axis of revolution, Pt the tangent to the meridian at P, and Pm and Pn the ordinate and normal. Let s be the length of the arc of the meridian measured from some fixed point at the left of P up to P; let $\angle Ptm = \theta$, and $Pm = y$.

Then (8) becomes
$$-\frac{d\theta}{ds} + \frac{\cos\theta}{y} = \frac{1}{a}, \tag{9}$$

where a is put for $\frac{2N}{p}$. Again, $\frac{dy}{ds} = \sin\theta$, so that (9) becomes

$$-y\sin\theta\, d\theta + \cos\theta\, dy = \frac{y}{a}dy.$$

Integrating this (since it holds at all points), we have

$$y\cos\theta = \frac{y^2}{2a} + h,$$

where h is a constant; or, finally, if $Pn = n$,

$$y^2\left(\frac{1}{n} - \frac{1}{2a}\right) = h. \tag{10}$$

Now if p is the perpendicular from the focus of an ellipse on the tangent at any point, and r the distance of this point from the focus, we have

$$p^2\left(\frac{1}{r} - \frac{1}{2a}\right) = \frac{b^2}{2a}, \tag{11}$$

where a and b are the semiaxes. Comparing (10) and (11), we see that P is the focus of an ellipse touching the line AB

at n, the semiaxes of this ellipse being a $\left(\text{or } \frac{2N}{p}\right)$ and $\sqrt{2ah}$, or $2\sqrt{\frac{hN}{p}}$, and this ellipse is therefore invariable whatever be the position of P on the meridian.

Hence* the meridian curve of the film is the locus of the focus of a given ellipse which rolls along the line AB. The conic being an ellipse, the locus is called an *Unduloid*, and is the curve actually represented in the figure.

If the rolling conic is a parabola, the locus of its focus is a catenary, and the surface of the film is the catenoid, already discussed.

If the rolling conic is a hyperbola, the locus of the focus is a curve having a series of loops, and the corresponding shape of the film is called a *Nodoid*.

The Nodoid is produced in the same way as the Unduloid, the greater or less distance between the parallel rings between which the bubble is blown determining the character of the surface; but it is obvious that only the portion between two nodes can be actually obtained. This curve resembles that of the third class of elastic curves (see Fig. 269, next chapter). The names of these curves have been given them by M. Plateau.

294.] **Connection with Elliptic Integrals.** The abscissa and ordinate in the Unduloid and Nodoid are readily expressible in terms of Elliptic Functions. If in (10) of last Article for n we put $y\frac{ds}{dx}$, we obtain

$$(y^2+2ah)\frac{ds}{dx} = 2ay,$$

$$\therefore \quad (y^2+2ah)^2(dx^2+dy^2) = 4a^2y^2dx^2,$$

$$\therefore \quad \frac{(y^2+2ah)dy}{\sqrt{-y^4+4a(a-h)y^2-4a^2h^2}} = \pm dx,$$

or

$$\frac{y^2 \pm \alpha\beta}{\sqrt{(\alpha^2-y^2)(y^2-\beta^2)}}dy = \pm dx, \tag{1}$$

by putting $\alpha^2+\beta^2 = 4a^2-4ah$, and $\alpha\beta = \pm 2ah$; so that α and β are the greatest and least values of the ordinate. Make the usual substitution, $y^2 = \alpha^2\cos^2\phi+\beta^2\sin^2\phi$. (2)

* This Theorem is due to Delaunay.

Then, putting $k^2 = \frac{a^2-\beta^2}{a^2}$, we have

$$y = a\Delta(\phi), \tag{3}$$

$$dx = \left(a\Delta\phi \pm \frac{\beta}{\Delta\phi}\right)d\phi. \tag{4}$$

When $\phi = 0$, $y = a$, and when $\phi = \frac{\pi}{2}$, $y = \beta$. Hence in Fig. 262 if D and E are the points of maximum and minimum ordinate, all values of ϕ between 0 and $\frac{\pi}{2}$ correspond to points on the curve between D and E; and as x increases with ϕ, we take a positive sign outside the brackets in (4).

The abscissa is given by the equation

$$x = aE(\phi) \pm \beta F(\phi). \tag{5}$$

Since in the Unduloid the tangent can never be parallel to the axis of y, we can never have $\frac{dx}{dy} = 0$, therefore the $+$ sign in the numerator of the left-hand side of (1) belongs to the Unduloid and the $-$ sign to the Nodoid. Hence also in (5) the $+$ and $-$ signs correspond to these curves.

Again, we have in the Unduloid

$$-ak^2\frac{dx}{dy} = \frac{a+\beta}{\sin\phi\cos\phi} - ak^2\tan\phi,$$

so that $\frac{d^2x}{dy^2}$ will vanish when

$$\tan\phi = \sqrt{\frac{a}{\beta}}, \text{ or } y = \sqrt{a\beta},$$

and there is a point of inflexion at the corresponding point.

If s is the length of the arc between D and any point, P, on the curve, we find

$$s = (a+\beta)\,.\,\phi\,;$$

and the area of the surface generated by the revolution of DP about the axis of x is

$$2\pi a(a+\beta)\,.\,E(\phi).$$

If $a = \beta$, the surface becomes a cylinder.

When a is slightly different from β (i.e. when k^2 is small), the form of the meridian curve is that of a *curve of sines*, as is easily proved.

295.] **Minimum Property of Films.** If we seek to determine the general equation of a surface which, subject to enclosing a given volume, has a minimum area, we obtain the equation

$$\delta\int\sqrt{1+p^2+q^2}\,.\,dx\,dy+\frac{1}{a}\delta\int z\,dx\,dy=0$$

(a being a constant), which gives us the general property

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{a}, \qquad (a)$$

at every point on the surface. Now this is, by (3), Art. 293, precisely the general property of the surfaces of films and of liquids unacted upon by external bodily force.

The connection between the two problems might have been foreseen by the principle of minimum or maximum Static Energy, combined with the fact that the surface-tension is a constant for all forms of the film. For if N and S are the surface-tension and area of the film, since (Art. 291) N can be regarded as the Static Energy per unit area, the product

$$N\times S$$

is the total potential work of the forces of the system, and this is simply proportional to the area of the surface. (See p. 178.)

Hence the question of the stability or instability of any of the forms of liquid surfaces can be exhibited in the following form: Subject to its bounding conditions, is the area of the surface greater or less than that of any surface differing infinitely little from it and satisfying the same differential equation,

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{a}?$$

The determination of the nature of the equilibrium in this way will often, however, be very troublesome, inasmuch as the comparison of the areas will involve the retention of small quantities to at least the *second* order; for the deduction of the general equation (a) satisfied by all the surfaces under comparison has been made by the retention of small quantities of the *first* order.

In this way M. Mathieu (*Théorie de la Capillarité*, pp. 73, &c.) has discussed the stability of a cylinder of oil formed between two equal circular plates in Plateau's glycerine mixture, and he arrives at the result that the figure is stable only for distances

between the plates greater than the semicircumference and less than the circumference of either plate.

296.] **Stability of the Catenoid.** Clerk Maxwell (Article on Capillary Action, *Encyclop. Brit.*) has applied a simple and ingenious method to the determination of the Stability of the Catenoid, without the direct consideration of minimum area.

Thus, let B and C (Fig. 222, Art. 186) be two fixed points, and Ox a fixed right line parallel to BC.

Then there are *two* catenaries which pass through B and C and have Ox for their common axis. For, if $BC = 2h$, and the distances of B and C from Ox are each equal to r, we have

$$2r = c\left(e^{\frac{h}{c}} + e^{-\frac{h}{c}}\right),$$

and it is very easy to prove that only two (if any) real values of c can be found to satisfy this equation; i.e. there can be drawn only two catenaries. Let these catenaries be BAC and $BA'C$ (the latter not represented in the figure), the vertex, A', of the second being, suppose, between O and A. Every catenary lying above BAC and every catenary lying below $BA'C$ has its horizontal axis lying *below* Ox; and every catenary lying between BAC and $BA'C$ has its axis *above* Ox.

Now, supposing Ox to be the line joining the centres of the two parallel circular wires between which is formed a soap film, these wires passing through B and C, the catenoids generated by the revolution of BAC and of $BA'C$ are possible figures of equilibrium.

Draw a catenary BaC slightly above BAC; let it revolve round Ox, and consider whether positive or negative work should be done on a film coinciding with the catenoid BAC in order to make it coincide with the quasi-catenoid BaC.

Let P be any point on BaC; draw the normal at P meeting Ox in n and the horizontal axis of BaC in m. Then the radius of curvature of this curve at P is equal and opposite to Pm, and therefore the principal radii of the surface generated by the revolution of the curve round Ox are Pn and $-Pm$, so that the sum of its curvatures measured towards the interior of this quasi-catenoid is

$$\frac{1}{Pn} - \frac{1}{Pm}, \qquad (a)$$

which is positive. Now if p is the intensity of the pressure

excess which must be applied to the surface at P to keep the film in equilibrium in the figure of the quasi-catenoid, we have

$$\frac{1}{Pn} - \frac{1}{Pm} = \frac{p}{2N}, \qquad (\beta)$$

(Art. 293), and p must therefore be applied as an *outward* pressure, i.e. in the sense nP. This holds at all points on the quasi-catenoid AaC, since (a) is everywhere positive. Hence to change the film from the catenoid BAC to the quasi-catenoid BaC, requires positive work, and there is therefore no tendency to such a displacement (Art. 274).

Again, draw a catenary $B\beta C$ very slightly below BAC. Since its axis is higher than Ox, Pm is now $< Pn$, and the sum of its principal curvatures at *every* point is positive towards the *exterior* of the film, so that this involves a pressure excess, p, directed towards the interior; i.e. positive work would be required to change the catenoid BAC into the quasi-catenoid $B\beta C$. The catenoid BAC is, therefore, stable.

In precisely the same way, if we consider the work which would be required to change the catenoid $BA'C$ into a very close quasi-catenoid *above* it, we see that this work is negative, so that there is a tendency to this displacement; and if the slightest motion in this direction is given to the film, it will move continually up to coincidence with the stable catenoid BAC. To change the film $BA'C$ into a quasi-catenoid *below* it requires, in the same way, negative work. Hence a displacement in this direction would increase indefinitely, and the catenoid $BA'C$ is, therefore, unstable.

It is easily seen that the tangents at B and C to the upper (stable) catenoid intersect above the axis Ox of the catenary, while those drawn to the unstable catenoid intersect below it. This form of the criterion of the stability or instability is given by Clerk Maxwell in the Article referred to.

It is understood, of course, that the ends of the catenoid are open to the external air and not closed by plates. If they are closed, the equilibrium will in all cases be necessarily stable—the film being presumed strong enough to resist rupture from slight motions.

CHAPTER XVI.

EQUILIBRIUM OF STRINGS AND SPRINGS.

297.] **Tangential and Normal Resolutions.** We now propose to complete the discussion of the equilibrium of flexible strings by considering the case in which the external forces are not coplanar.

Reverting to Fig. 221 of Chapter XII, consider the equilibrium of the element PQ apart from the rest of the string. Then the external force per unit mass at P will be, as before, of the form $\phi(x, y, z)$, where (x, y, z) are the co-ordinates of P; and the external force exerted on PQ will be

$$\phi(x, y, z) \times k\sigma ds, \text{ or } k\sigma F ds,$$

where k and σ are the density and area of normal section at P.

Now, the element PQ is kept in equilibrium by three forces —namely, the tension (T) at P, the tension $(T+dT)$ at Q, and the external force $(k\sigma F ds)$, which acts at the middle point of PQ.

These three forces must be coplanar and meet in a point. Now, the two tensions act along two consecutive tangents to the string, and as the plane of two consecutive tangents to any curve in space is the *osculating plane*, we see that—

The resultant applied force at any point of a flexible string acts in the osculating plane of the string at the point.

If the string is stretched over any smooth surface by means of two forces applied at its extremities, the only applied force which is *continuously* distributed throughout the string is the reaction of the surface; and as this reaction is everywhere normal to the surface, we see that—

A string which is stretched along any smooth surface, and acted on by no external forces, except the reaction of the surface and two terminal tensions, has its osculating plane at every point normal to the surface.

The string in this case assumes the form of a shortest line, or *geodesic*, on the surface.

Let Pt be the tangent and Pn the normal at P; let $d\theta$ be the angle between the tangents at P and Q; and let ϕ be the angle between Fdm and Pt.

Then, resolving along Pt the forces acting on the element, we have
$$(T+dT)\cos d\theta + k\sigma F\cos\phi\, ds - T = 0;$$
but $\cos d\theta = 1$, neglecting $(d\theta)^2$; therefore this equation gives
$$\frac{dT}{ds} + k\sigma F\cos\phi = 0, \tag{1}$$
which asserts that the rate of variation of the tension per unit of length along the string is numerically equal to the tangential component of the applied force per unit of length.

Again, resolving the forces along Pn, the normal, we have
$$(T+dT)\sin d\theta - k\sigma F\sin\phi\, ds = 0,$$
or since ρ, the radius of curvature at P, is equal to $\frac{ds}{d\theta}$,
$$\frac{T}{\rho} - k\sigma F\sin\phi = 0, \tag{2}$$
which asserts that the curvature of the string at any point is equal to the normal force per unit of length divided by the tension.

From (1) we have $T = C - \int k\sigma F\cos\phi\, ds$,

where C is an arbitrary constant. Now, $\cos\phi\, ds$ is the projection of ds on the direction of F. Denoting this projection by df,
$$T = C - \int k\sigma F df. \tag{3}$$
But $\int k\sigma F df$ is evidently the potential of the applied forces if they are a conservative system. Hence, if V and V_0 denote the potentials at two points in the string at which the tension are T and T_0, we have
$$T = T_0 - (V - V_0), \tag{4}$$
or *the difference of the tensions at any two points is equal to the difference of the potentials*—a result which we shall find to be true also in the case in which the string rests on a smooth surface.

298.] **Equations of Equilibrium.** Let the force F acting on the unit mass at any point P whose co-ordinates are x, y, z be resolved into three components, X, Y, Z parallel to three fixed

rectangular axes. Then the components acting on the element PQ are $k\sigma X ds$, $k\sigma Y ds$, $k\sigma Z ds$. Also the components of the tension acting on the extremity P are

$$-T\frac{dx}{ds}, \quad -T\frac{dy}{ds}, \quad -T\frac{dz}{ds};$$

the components of this tension are affected with negative signs, since, when the element PQ is considered apart, the tension at P will be directed towards the left-hand side of Fig. 221, where the origin of co-ordinates is supposed to be.

These components of the tension will at any point be functions of the length of the arc measured from some origin point, A, of the string up to the point considered. Thus, if $AP = s$, we shall have

$$T\frac{dx}{ds} = f(s),$$

and the component of the tension at Q is therefore $f(s+ds)$, or

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right)\cdot ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right)\cdot\frac{ds^2}{1.2} + \dots.$$

Hence, for the equilibrium of PQ, resolving forces parallel to the axis of x, we have

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right)\cdot ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right)\cdot\frac{ds^2}{1.2} + \dots$$
$$+ k\sigma X ds - T\frac{dx}{ds} = 0,$$

or, rejecting the terms which cancel, dividing out by ds, and diminishing ds indefinitely, and denoting $k\sigma$ by m, the mass per unit length,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0. \tag{1}$$

Similarly,

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0, \tag{2}$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right) + mZ = 0. \tag{3}$$

Performing the differentiations, we obtain

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\frac{dx}{ds} + mX = 0, \tag{4}$$

$$T\frac{d^2y}{ds^2} + \frac{dT}{ds}\frac{dy}{ds} + mY = 0, \tag{5}$$

$$T\frac{d^2z}{ds^2} + \frac{dT}{ds}\frac{dz}{ds} + mZ = 0. \tag{6}$$

For the future we shall systematically use (α, β, γ) for the direction-angles of the tangent at any point P of a curve, the positive sense of this tangent being that in which the arc, s, measured up to P from some origin point on the curve, receives a positive increase.

Also by (ξ, η, ζ) we shall denote the direction-angles of the radius of absolute curvature at P, *taken in positive sense from P towards the centre of curvature.*

We may suppose these angles to be measured from lines drawn at P parallel to the positive directions of the axes of co-ordinates.

Equations (4), (5), (6) may then be written

$$\frac{T}{\rho}\cos\xi + \frac{dT}{ds}\cos\alpha + mX = 0, \tag{7}$$

$$\frac{T}{\rho}\cos\eta + \frac{dT}{ds}\cos\beta + mY = 0, \tag{8}$$

$$\frac{T}{\rho}\cos\zeta + \frac{dT}{ds}\cos\gamma + mZ = 0. \tag{9}$$

Multiplying these by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and adding, we have

$$\frac{dT}{ds} + mS = 0, \tag{10}$$

where $S =$ the component of the forces along the positive sense of the tangent. This equation gives

$$T = C - \int S\,ds = C - \int m(X\,dx + Y\,dy + Z\,dz),$$

which is obviously the same as (3) of last Article.

Again, eliminating T and $\dfrac{dT}{ds}$ from (7), (8), (9), we have

$$\begin{vmatrix} \cos\xi, & \cos\alpha, & X \\ \cos\eta, & \cos\beta, & Y \\ \cos\zeta, & \cos\gamma, & Z \end{vmatrix} = 0,$$

or $$X\cos\theta + Y\cos\phi + Z\cos\psi = 0, \tag{11}$$

where (θ, ϕ, ψ) are the direction-angles of the normal to the osculating plane. This equation asserts—what is evident from first principles—that the resultant external force at any point lies in the osculating plane.

Another form of the value of T is obtained by integrating (1), (2), (3) separately, and squaring and adding. Thus

$$T^2 = \left(A - \int mX\,ds\right)^2 + \left(B - \int mY\,ds\right)^2 + \left(C - \int mZ\,ds\right)^2, \tag{12}$$

A, B, C being constants which must be determined after each integration by knowing the values of $T\frac{dx}{ds}$, ... at the point from which s is measured.

Again, by multiplying (7), (8), (9) by $\cos\xi$, $\cos\eta$, $\cos\zeta$ and adding,

$$\frac{T}{\rho}+mP = 0, \tag{13}$$

where P is the component force along the radius of curvature in the positive sense (i.e. towards the centre of curvature).

The equations of the curve formed by the string are obtained from (1), (2), (3) thus, by elimination of T,

$$\frac{A-\int mX\,ds}{\frac{dx}{ds}} = \frac{B-\int mY\,ds}{\frac{dy}{ds}} = \frac{C-\int mZ\,ds}{\frac{dz}{ds}}. \tag{14}$$

From (10) it follows that *if at no point of the string is there any component force along the tangent, the tension will be constant throughout.*

299.] **String on a Smooth Surface.** Now suppose that the string, while acted upon continuously by any forces, is placed on a smooth surface, which produces at each point a normal reaction, equal to $R\,ds$ on the element of length ds at the point, P.

We shall denote by (l, m, n) the direction-angles of the normal to the surface in the sense in which R acts along the normal. Then we have simply to add the components $R\cos l$, $R\cos m$, $R\cos n$ to X, Y, and Z, respectively in equations (1), (2), (3), or (7), (8), (9) of last Article, so that our equations are now

$$\frac{T}{\rho}\cos\xi+\frac{dT}{ds}\cos\alpha+mX+R\cos l = 0, \tag{1}$$

$$\frac{T}{\rho}\cos\eta+\frac{dT}{ds}\cos\beta+mY+R\cos m = 0, \tag{2}$$

$$\frac{T}{\rho}\cos\zeta+\frac{dT}{ds}\cos\gamma+mZ+R\cos n = 0. \tag{3}$$

If ω is the angle between the radius of curvature and the *inward* drawn normal to the surface at P (i.e. the normal drawn in the *sense opposite to that of R*), we have by multiplying these by $\cos l$, $\cos m$, $\cos n$, and adding,

$$R+mN-\frac{T}{\rho}\cos\omega = 0, \tag{4}$$

N being the normal force per unit mass in the sense of R.

By multiplying by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and adding, we have

$$\frac{dT}{ds}+mS=0, \tag{5}$$

just as in the case of a free string.

When the applied forces have a potential, V, the integral of this equation, as in Art. 192, is

$$T=T_0-(V-V_0). \tag{6}$$

In the particular case in which the string rests on any smooth surface under the influence of gravity, this equation gives

$$T=T_0-mg\,(y-y_0), \tag{7}$$

the axis of y being a vertical line. From this it follows that all points at which the tension is the same lie in the same horizontal plane.

The curve of equilibrium of the string is found by eliminating T and R from the equations (1), (2), (3). Thus, if we eliminate first $\frac{dT}{ds}$ and R, we have

$$\begin{vmatrix} T\frac{d^2x}{ds^2}+mX, & \frac{dx}{ds}, & \frac{du}{dx} \\ T\frac{d^2y}{ds^2}+mY, & \frac{dy}{ds}, & \frac{du}{dy} \\ T\frac{d^2z}{ds^2}+mZ, & \frac{dz}{ds}, & \frac{du}{dz} \end{vmatrix}=0, \tag{8}$$

in which $u=0$ is the equation of the given surface, so that $\cos l:\cos m:\cos n=\frac{du}{dx}:\frac{du}{dy}:\frac{du}{dz}$. The value of T derived by integrating (5) must be substituted in (8), and we then get a differential equation which, with $u=0$, determines the curve.

300.] **String on a Rough Surface.** If a string, acted on by no forces, is stretched over a rough surface it need not, as in the case of a smooth surface, assume the form of a geodesic or shortest line. One simple case in which it will be a geodesic is that in which it is about to slip on the surface at every point in the direction of the tangent to the string at this point.

Geodesic. Consider the equilibrium of an element, PQ, of the string, whose length is ds, and suppose that it is about to slip in the direction QP. The element is acted upon by three forces—namely, a tension T, at P, a tension $T+dT$, at Q, and the total

resistance of the rough surface, which must pass through the intersection of the tangents at P and Q.

It is evident that we may consider this total resistance as acting at P, ultimately, since it is of the form $R_1 ds$, R_1 being a finite quantity, and if it be assumed to act at any point between P and Q, its components in any directions will differ from those of the total resistance supposed to act at P by infinitesimals of the order of $(ds)^2$. Resolve the total resistance at P into a normal force, Rds, and a force in the tangent plane, μRds, μ being the coefficient of friction between the string and the surface.

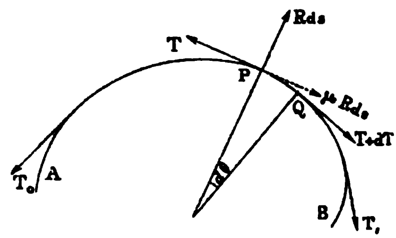

Fig. 265.

Now the component μRds must act along the tangent at P, since, by hypothesis, slipping is about to take place along this tangent. Hence the three forces T, $T+dT$, and μRds being all in the osculating plane of the curve at P, the remaining force, Rds, must also lie in this plane; that is, the osculating plane at every point of the curve contains the normal to the surface. Hence the string assumes the form of a geodesic.

Denoting the angle between the tangents at P and Q by $d\theta$, we have, by resolving along the tangent at P,

$$dT + \mu Rds = 0. \tag{1}$$

Again, resolving along the normal at P,

$$Td\theta - Rds = 0. \tag{2}$$

From (1) and (2) we have

$$\frac{dT}{T} + \mu d\theta = 0, \qquad \therefore \quad T = Ce^{-\mu\theta},$$

C being the constant of integration, and θ the sum of the *angles of contingence*, or angles between successive tangents to the string from any chosen point, A, to the point, P. Let T_0 be the tension at A. Then $T = T_0$ when $\theta = 0$; therefore

$$T = T_0 e^{-\mu\theta}. \tag{3}$$

General Case. Suppose now that the string is acted upon by any forces, and that F is the force of friction per unit length at any point P, the direction of this force being in the tangent

plane, but otherwise unknown. Let its direction-angles be $(\alpha', \beta', \gamma')$. Then with the same notation as before,

$$\frac{T}{\rho}\cos\xi + \frac{dT}{ds}\cos\alpha + mX + R\cos l + F\cos\alpha' = 0, \tag{4}$$

$$\frac{T}{\rho}\cos\eta + \frac{dT}{ds}\cos\beta + mY + R\cos m + F\cos\beta' = 0, \tag{5}$$

$$\frac{T}{\rho}\cos\zeta + \frac{dT}{ds}\cos\gamma + mZ + R\cos n + F\cos\gamma' = 0. \tag{6}$$

Intrinsic equations, completely equivalent to the above, can be obtained by taking the axes of z, y, and x, respectively, parallel to the normal (direction of R), the tangent (direction of T), and a line drawn perpendicular to both, so that

$$\alpha = \gamma = \frac{\pi}{2}, \quad \beta = 0; \quad l = m = \frac{\pi}{2}, \quad n = 0;$$

$$\xi = \frac{\pi}{2} - \omega, \quad \eta = \frac{\pi}{2}, \quad \zeta = \pi - \omega.$$

If Q denotes the component force per unit mass at P along the new axis of x, and θ is the angle which the direction of F makes with the tangent, these equations become

$$\frac{T}{\rho}\sin\omega + mQ + F\sin\theta = 0, \tag{7}$$

$$\frac{dT}{ds} + mS + F\cos\theta = 0, \tag{8}$$

$$-\frac{T}{\rho}\cos\omega + mN + R = 0, \tag{9}$$

the last of which, therefore, holds both for a rough and for a smooth surface.

Consider the particular case in which there is no continuously applied external force, i. e. let $N = S = Q = 0$, and suppose that slipping is about to take place at a point. Then at this point $F = \mu R$, and we have

$$\frac{dT}{ds} = \frac{T}{\rho}\sqrt{\mu^2\cos^2\omega - \sin^2\omega}. \tag{10}$$

At a point, therefore, at which the osculating plane is inclined at the angle of friction to the normal to the surface, the tension is a maximum or minimum; and if slipping is about to take place at all points, the tension will be constant throughout if the osculating plane of the curve in which the

string is placed makes throughout the angle of friction with the normal.

If the osculating plane is everywhere normal to the surface, $\omega = 0$, and therefore $\sin\theta = 0$, i.e. the force of friction acts along the tangent—as is evident from the fact that of the four forces, T, $T+\frac{dT}{ds}ds$, R, and F, which keep an element in equilibrium, the first three are then coplanar, so that F must lie in the tangent.

Example.

A string whose weight may be neglected is placed along a circular section of a rough right cone and is pulled at its extremities by two given forces, P and Q; find the relation between these forces when the whole string is about to slip, and the direction of slipping at each point.

Ans. If a = semivertical angle of cone, μ = coefficient of friction, l = length of string, r = the radius of the circle, and if P is about to overcome Q,

$$Q = P\,e^{-\frac{l}{r}\sqrt{\mu^2\cos^2 a - \sin^2 a}},$$

and the direction of slipping makes at each point with the tangent the angle whose cotangent is $\sqrt{\mu^2\cot^2 a - 1}$.

301.] Equilibrium of an Extensible String. With the same notation as that employed in Art. 196, the equations of equilibrium of a flexible extensible string in the general case will be

$$m\,ds = m_0\,ds_0, \tag{1}$$

$$ds = \left(1+\frac{T}{\lambda}\right)ds_0, \tag{2}$$

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right)+mX &= 0,\\ \frac{d}{ds}\left(T\frac{dy}{ds}\right)+mY &= 0,\\ \frac{d}{ds}\left(T\frac{dz}{ds}\right)+mZ &= 0. \end{aligned}\right\} \tag{3}$$

$$ds = \sqrt{dx^2+dy^2+dz^2}, \tag{4}$$

$$m_0 = f(s_0). \tag{5}$$

In general, then, the two equations of the curve of equilibrium are found by eliminating m, m_0, s, s_0, T from these seven equations.

As before, we take only two cases, viz. that in which m_0 is constant, and that in which X, Y, Z are constant.

Firstly, consider m_0 constant.

Then, multiplying the left-hand sides of (3) by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, and adding,

$$\frac{dT}{ds}+m\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right) = 0\,; \tag{6}$$

while from (1) and (2) we have $m = \dfrac{m_0}{1+\frac{T}{\lambda}}$; so that (6) gives

$$\left(1+\frac{T}{\lambda}\right)dT+m_0(Xdx+Ydy+Zdz) = 0. \tag{7}$$

Integrating this and putting V for the potential of the external forces, per mass m_0 at the point (x, y, z), viz.

$$m_0\int(Xdx+Ydy+Zdz),$$

we have
$$\frac{\lambda}{2}\left(1+\frac{T}{\lambda}\right)^2 = A-V \tag{8}$$

where A is a constant.

Hence by (2)
$$\frac{ds}{\sqrt{A-V}} = \sqrt{\frac{2}{\lambda}}\,ds_0, \tag{9}$$

which gives s in terms of s_0, and therefore the extension.

If V and V' are the potentials at two points at which the tensions are T and T', respectively,

$$(T-T')\left(1+\frac{T+T'}{2\lambda}\right) = V'-V. \tag{10}$$

The equations of the curve of equilibrium are obtained by substituting the value of T given by (8) in any two of the equations

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dx}{ds}\right)+m_0X = 0,$$

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dy}{ds}\right)+m_0Y = 0,$$

$$\left(1+\frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dz}{ds}\right)+m_0Z = 0.$$

Secondly, suppose the external forces X, Y, Z to be constant.

Then, integrating equations (3), we have

$$\left.\begin{aligned} T\frac{dx}{ds} &= A - X\int m_0 ds_0, \\ T\frac{dy}{ds} &= B - Y\int m_0 ds_0, \\ T\frac{dz}{ds} &= C - Z\int m_0 ds_0, \end{aligned}\right\} \qquad (11)$$

A, B, C being constants. Squaring and adding these,

$$T^2 = (A - X\int m_0 ds_0)^2 + (B - Y\int m_0 ds_0)^2 + (C - Z\int m_0 ds_0)^2, \quad (12)$$

which gives T in terms of s_0. Suppose

$$T = \phi(s_0). \qquad (13)$$

Hence from (2) $$s = \int\left\{1 + \frac{\phi(s_0)}{\lambda}\right\} ds_0,$$

from which the extension is known.

The equations of the curve are obtained from equations (11) by substituting for ds in terms of ds_0. Thus,

$$dx = (A - X\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0,$$

$$dy = (B - Y\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0,$$

$$dz = (C - Z\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\} ds_0.$$

Integrating these and eliminating s_0, we obtain the two equations of the curve of equilibrium.

302.] **Equilibrium of a Plane Elastic Rod.** The equilibrium of a string has been investigated, in Art. 297, on the supposition that if we take a normal section of it at any point, P, the action exerted on the portion PB by the remaining portion PA consists simply of a force directed along the tangent. The rod differs from the string in this—that the internal action exerted on any normal section is much more complicated, being equivalent to a force, I, acting at some point

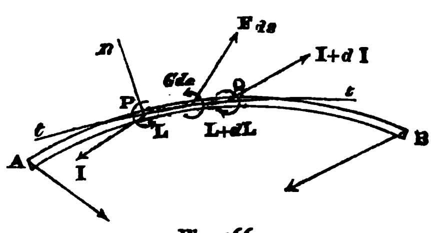

Fig. 266.

of the section, oblique to the tangent, together with a couple L. In the case, now before us, of a rod lying wholly in one plane and acted upon by external forces and couples, also confined to this plane, the axis of the couple, L, will at every point be perpendicular to the plane of the rod. Indeed, the remarks in Art. 103 on the nature of internal action, or stress, prepare us for seeing this.

In the above figure (Fig. 266), consider the nature of the action exerted over the normal section at P on the part PB by the part PA. Near the upper, or convex, side the bending has the effect of making the part PA try to tear away from PB, so that, on the whole, there will be in this neighbourhood forces on PB directed towards the *left*; while near the lower, or concave, side of the rod at P, the bending causes the portion PA to push into PB, and consequently the particles of PB in this neighbourhood will experience forces directed towards the *right* of the figure.

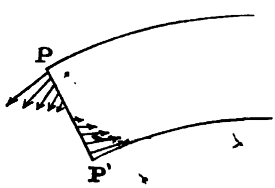

Fig. 267.

This state of stress is roughly represented in Fig. 267. If the arrows in it represent the forces experienced by the individual molecules of the portion PB, it is clear how such forces might reduce to a single force acting below P', perhaps a long way off from the rod; and how this single force, again, could (Art. 79) be replaced by the force I (Fig. 266) acting at an arbitrary point of the section, together with the counter-clockwise couple L.

A remark, to which we shall return in the Chapter on Strain and Stress, may now be made with reference to the system of stress (Fig. 267) on the section of the rod. It is this—that even though the normal section may be extremely small, as in the case of a very narrow wire, the forces experienced by the successive molecules lying in the section vary in both magnitude and sense with enormous rapidity. On an *infinitely* small area of this normal section—such as the surface of a single atom—the stress action consists necessarily of a force simply—without any couple; but the normal section of even a very thin wire contains an infinitely great number of such infinitely small surfaces, and therefore furnishes abundant possibility for the enormously rapid change in the magnitude and sense of the separate internal forces,

and hence for that force and couple to which, if the wire be very stiff, these individual forces must reduce.

The rod, whose equilibrium we are considering, is supposed, for generality, to be acted upon continuously throughout its length by applied force, whose amount per unit length at any point is F, together with applied couple, whose amount per unit length is G. A magnetized spring acted upon, in addition to any other forces, by the earth's magnetic attraction, gives an instance of continuously distributed external couple.

303.] **Conditions of the Extremities.** Our figure represents the rod as kept in equilibrium by continuously distributed force (Fds) and couple, together with two terminal forces at A and B. These terminal forces may be produced either by direct pulls or by fixing smooth pins through the extremities, since (Art. 103) the pressures all round the surface of a smooth cylindrical axis are equivalent to a single force acting through the centre of the axis.

Another, and essentially different, state of affairs at the ends is produced by fixing not only the end itself but also the tangent at it. In this case we shall speak of the end as *tangentially fixed*. It is clear that this mode of fixture could not be produced by the application of a *single force* at the end so fixed; it would require the application of a *force and a couple* to the end. In the case of coplanar forces this force and couple are equivalent to a single force acting at a distance from the end (Art. 79).

Pivoting at an extremity is, then, productive of a single force acting at the extremity; and *tangential fixture* is productive of a force and a couple acting at it.

304.] **Equations of Equilibrium.** We now proceed to obtain the equations connecting the stress with the external forces and couples, exactly as in the case of a string. Consider the separate equilibrium of the element PQ. The stress which it experiences at P has been already described; over the normal section at Q, the stress will consist of a slightly different force, $I+dI$, and a slightly different couple, $L+dL$; while the externally applied force is Fds, and the externally applied couple is Gds. (The force $I+dI$ and the couple $L+dL$ are exerted by the portion QB on QA, and are therefore in the senses represented in the figure.)

Suppose the arc s to be measured from A, so that $AP = s$;

let Sds and Nds be the components of Fds along the tangent tP and the normal Pn drawn towards the convex side of the curve; let θ be the angle which the tangent at P makes with some fixed line (axis of x) which we may, for definiteness, suppose drawn at the lower side of the figure, so that the radius of curvature, ρ, at P is $-\dfrac{ds}{d\theta}\cdot$ Also, let the internal force, I, be resolved into components, T, along the tangent and, U, along the normal. The second component is called the *shearing stress* at P; the first is, of course, the *tension* of the rod. Let the tension and shearing stress at Q be $T+dT$ and $U+dU$, respectively.

Then, for the equilibrium of PQ, resolving along the tangent, we have
$$-T+T+dT-(U+dU)\,d\theta+Sds=0,$$
observing that $d\theta$ is negative. Hence, proceeding to the limit,
$$\frac{dT}{ds}+\frac{U}{\rho}+S=0. \tag{1}$$

Similarly, resolving along the normal,
$$U+dU-U+(T+dT)\,d\theta+Nds=0.$$
$$\therefore\quad \frac{dU}{ds}-\frac{T}{\rho}+N=0. \tag{2}$$

Finally, taking moments about an axis through P perpendicular to the plane of the figure, and observing that the moment of the external force would give a term of the order ds^2, we have
$$L-L-dL+(U+dU)ds+Gds=0,$$
$$\therefore\quad \frac{dL}{ds}-U-G=0. \tag{3}$$

From this last we see that when there is no *continuously* distributed external couple, *the shearing stress at any point is equal to the differential coefficient of the stress couple with respect to the arc.*

With regard to the sense of the stress couple L, observe, in general, that the couple exerted on any portion PA by the remaining portion PB is in the sense in which the tangent at P revolves as we move from P along PB; and in (3) the shearing stress exerted on PA by PB is measured along the normal drawn towards the *convex* side of the curve. If U is measured towards the centre of curvature we have simply to change its sign in the equations.

Sometimes it is of more advantage to obtain equations from the consideration of the equilibrium of a portion of finite length of the curve.

Thus, consider the equilibrium of the whole length AP. Take any two fixed axes of x and y; let Xds and Yds be the components of the external force, Fds, parallel to these axes; let α and β be the components of the force (arising from any such cause as fixture) at A, and let λ be the special couple (if any) applied at A. Then, by resolution, we have

$$T\cos\theta - U\sin\theta = \alpha - \int X ds, \tag{4}$$

$$T\sin\theta + U\cos\theta = \beta + \int Y ds, \tag{5}$$

which give T and U at once.

Also, by taking moments about P, and denoting the co-ordinates of A by a and b, while, to avoid confusion, we denote the co-ordinates of any point on the curve between A and P by ξ and η, we have

$$L = \lambda + \alpha(y-b) - \beta(x-a) + \int\{X(y-\eta) - Y(x-\xi) + G\}\,ds. \tag{6}$$

Or the value of L may often be better obtained from (3), U having been determined from (4) and (5).

305.] **Particular Case of Plane Spring.** Suppose that there are no forces or couples *continuously* distributed along the rod, or spring, but merely a force, H, at one end, B, the other being either simply or tangentially fixed; and suppose that before strain the spring had the form of any plane curve, of which the radius of curvature at any point was r. Considering the equilibrium of any portion PB, we see that stress at P (force and couple) must reduce to a force equal to the force H and directly opposed to it in its line of action.

Hence at all points the internal force I is constant in magnitude and direction—equal to H.

Again, by moments about P, if we take the line of action of H as axis of x, we have

$$L = H.y. \tag{1}$$

Now the magnitude of L is assumed to be equal to the change in curvature at P produced by strain, multiplied by a certain constant, A, whose magnitude depends on the stiffness of the material of the spring; so that

$$L = A\left(\frac{1}{\rho} - \frac{1}{r}\right). \tag{2}$$

$y = \frac{A}{H}\left(\frac{1}{\rho} - \frac{1}{r}\right) = a^2\left(\frac{1}{\rho} - \frac{1}{r}\right) \therefore \frac{y}{a^2} = \frac{1}{\rho} - \frac{1}{r}$

$\frac{1}{\rho} = \frac{y}{a^2} + \frac{1}{r} = \frac{1}{a^2}\left(y + \frac{a^2}{r}\right)$

The constant A is called a *flexural rigidity*, and it is evidently of the nature of a force multiplied by the square of a line.

We may therefore put $\frac{A}{H} = a^2 =$ a constant, and then we have

$$\frac{1}{\rho} = \frac{1}{a^2}\left(y + \frac{a^2}{r}\right), \tag{3}$$

which is the equation determining the form of the curve.

If the spring when free from strain was straight, $\frac{1}{r}$ is zero, and the equation becomes

$$\frac{1}{\rho} = \frac{y}{a^2}. \tag{4}$$

If the rod was in the form of a circular arc when unstrained, (3) could be put into the form (4) by taking the axis of x parallel to the line of action of H at a distance $\frac{a^2}{r}$ from it.

The force, H, may be applied either directly to the end, B, of the rod itself or to a rigid arm attached to B.

The latter case is the same as if H were directly applied to B and accompanied by a couple whose moment is the moment of H about B—in fact, a rigid arm at the extremity of which H is applied may be regarded as a means of applying a force and a couple at the end, B, of the rod (see Art. 202).

306.] **Elastic Curves and Elliptic Functions.** Let ADB (Fig. 268) represent the rod, with two rigid arms, Aa, Bb,

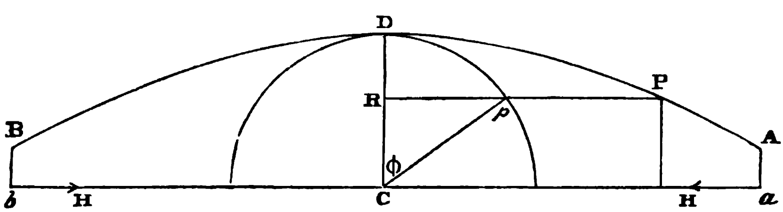

Fig. 268.

attached to its ends, two equal and directly opposed forces, H, being applied perpendicularly to these arms. We assume that the rod was straight when unstrained.

Taking the line ab as axis of x, the equation of the bent rod is

$$\rho y = a^2. \tag{1}$$

We shall express ρ in terms of the element of arc, ds, and the angle, $d\theta$, between the tangents at its extremities; and for definiteness we shall assume s to be measured from a point, D, at which the tangent is parallel to the line, ab, of action of the force H, and the angle θ made with the tangent at D by the tangent at any point, P, on the curve to be measured positively in the sense of clockwise rotation.

The curve of equilibrium may be concave at some points and convex at others to the line of action of the terminal forces; and if at any point it intersects this line, its curvature vanishes at the point. It may, again, never intersect this line at all.

If P and Q are any two very close points on the curve, the extremity of the curve, which we should reach by travelling from P to Q and then continuously along the curve, may be called *the extremity adjacent to* Q, while the other extremity may be called *the extremity adjacent to* P.

The terminal forces, H, may act along ab either towards each other, as represented in Fig. 268, or from each other, and the sense of the bending (or concavity) at any point will depend on the senses of the terminal forces. In every case, of course, the sense of the bending at any point P is such that the moment about P of the terminal force at either extremity is opposed by the stress couple exerted at P by the remaining portion of the rod; or, in other words, for all the figures which the curve can assume we have the following rule—*the sense in which the tangent revolves in passing from P to a consecutive point Q is the sense of the moment about P of the force at the end adjacent to Q.*

If in (1) we put $\rho = \dfrac{ds}{d\theta}$; and $\dfrac{dy}{ds} = -\sin\theta$, we have

$$a^2 \frac{d^2\theta}{ds^2} = -\sin\theta, \tag{2}$$

$$\therefore \quad a^2 \left(\frac{d\theta}{ds}\right)^2 = C + 2\cos\theta,$$

where C is a constant. Let D (Fig. 268) be a vertex, or point at which the tangent is parallel to ab, and let the ordinate $DC = h$; then from the last equation we have

$$y^2 = h^2 - 4a^2 \sin^2\frac{\theta}{2}. \tag{3}$$

Now different cases arise according as h is $< 2a$, $= 2a$, or $> 2a$.

CASE 1; $h < 2a$. Let $h = 2ak$, where k is a fraction. Then

$$y = 2a\sqrt{k^2 - \sin^2\frac{\theta}{2}}. \tag{4}$$

Let $\sin\frac{\theta}{2} = k\sin\phi$; then

$$y = h\cos\phi. \tag{5}$$

The angle ϕ can be easily exhibited: with C, the foot of the ordinate from D, as centre, describe a circle of radius h; from P, which is any point on the curve, draw PR parallel to ab, meeting the circle in p. Then the angle DCp is obviously ϕ.

To find the length of the arc DP, or s, we have $\frac{dy}{ds} = -\sin\theta$; but from (5), $\frac{dy}{ds} = -h\sin\phi\frac{d\phi}{ds}$; hence

$$\frac{ds}{d\phi} = \frac{h\sin\phi}{\sin\theta}$$

$$= \frac{a}{\sqrt{1-k^2\sin^2\phi}},$$

$$\therefore \quad s = a\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{6}$$

$$= a\,.\,F(k, \phi), \tag{7}$$

according to the notation for an elliptic integral of the first kind.

To express the abscissa, CM, of P. If $CM = x$, we have

$$\frac{dx}{ds} = \cos\theta, \quad \therefore \quad \frac{dx}{d\phi} = h\sin\phi\cot\theta = a\frac{1-2k^2\sin^2\phi}{\Delta\phi},$$

where $\Delta\phi \equiv \sqrt{1-k^2\sin^2\phi}$. Hence

$$\frac{dx}{d\phi} = 2a\,.\,\Delta\phi - \frac{a}{\Delta\phi}, \tag{8}$$

$$\therefore \quad x = 2a\,.\,E(k, \phi) - a\,.\,F(k, \phi), \tag{9}$$

$$\therefore \quad x + s = 2a\,.\,E(k, \phi), \tag{10}$$

where, as usual, E denotes the elliptic integral of the second kind.

EXAMPLES.

1. If the ends A and B are fixed by smooth pins at a given distance apart, and the rod is placed in the form of n bays, or spans, between the pins, find the pressures exerted by the pins.

The pressures exerted by the pins will be directly opposed in the line AB. Let $AB = c$, and let $l =$ whole length of the rod. Then the length of the arc in one span is obtained from (7) by putting $\phi = \frac{\pi}{2}$. Hence

$$l = 2naF(k, \tfrac{\pi}{2}). \tag{11}$$

Similarly, from (10), we have

$$c+l = 4naE(k, \tfrac{\pi}{2}); \tag{12}$$

so that k is determined from the equation

$$2l \,.\, E(k, \tfrac{\pi}{2}) = (c+l) \,.\, F(k, \tfrac{\pi}{2}), \tag{13}$$

which is independent of the number of bays.

Now there is only one value of k which satisfies this equation, as we can see graphically thus. The values of k range from 0 to 1. Draw two rectangular axes, Ox and Oy, and let abscissae (along Ox) represent values of k while ordinates represent the corresponding values of E.

When $k = 0$, $E = \frac{\pi}{2}$, and when $k = 1$, $E = 1$. Also, we easily find by differentiation that

$$\frac{dE}{dk} = \frac{E-F}{k}, \tag{14}$$

$$\frac{dF}{dk} = \frac{1}{k}\left(\frac{E}{k'^2} - F\right), \tag{15}$$

in which we use E and F, for shortness, instead of $E(k, \frac{\pi}{2})$ and $F(k, \frac{\pi}{2})$; and also k'^2 for $1-k^2$.

Measure off $OV = \frac{\pi}{2}$ along Oy, and $OT = 1$ along Ox, and at T draw an ordinate $TR = 1$. Then the curve representing the values of E passes through V and R, touching TR at R, and touching at V a line parallel to Ox. (The value of $\frac{dE}{dk}$ at V assumes the form $\frac{0}{0}$, but it is easy to find, by the help of (15), that it is equal to zero.) This curve is continuously concave towards the axis of x.

Again, the curve whose ordinates represent the values of F passes through V, touching the previous curve at this point, its ordinate being thenceforth always $> OV$, until when $k = 1$, the ordinate $= \infty$. This curve, therefore, approaches the line TR asymptotically. An inspection of the figure shows that if the ordinates of these curves have

a given ratio, there is only one value of k which will answer, since the second curve is continuously convex towards the axis of x, E being always $> k'^2 F$, except at the point V.

The value of k, then, must be found empirically from (13), and if the corresponding value of F is μ, equation (11) gives

$$H = \frac{4n^2\mu^2}{l^2} A \tag{16}$$

for the pressure exerted by each pin.

Hence the pressure is proportional to the square of the number of bays.

If a is the angle made with the line AB by the tangent at either extremity of the rod, since, in general, $\sin\frac{\theta}{2} = k\sin\phi$, we have

$$\sin\frac{a}{2} = k, \tag{17}$$

for any figure which crosses the line of force, since for all such curves $\phi = \frac{\pi}{2}$ for the point of crossing.

In the present case, therefore, whatever be the number of bays, their terminal tangents are all equally inclined to the line AB.

The particular case in which the ends A and B are brought together deserves to be noticed. One form of equilibrium is, of course, that of a single loop starting from A and coming round to A again, there being two distinct tangents to the curve at A.

For this case put $c = 0$ in (13), and the value of k is obtained from the equation

$$2E(k, \tfrac{\pi}{2}) = F(k, \tfrac{\pi}{2}), \tag{18}$$

and the inclination of each tangent is given by (17).

Another form of equilibrium in this case is that of a figure of 8, the two tangents at the double point making with the axis of the curve the same angles as those just found for the case of a single loop.

2. Show that if a rod is slightly bent between its extremities, its figure is that of the curve of sines, $y = h\sin\frac{x}{a}$.

Case 2; $h = 2a$. In general, the radius of curvature at the vertex D is equal to $\frac{a^2}{h}$, so that when $h > a$, the curve on leaving D comes inside the circle Dp (Fig. 268). Such happens, then, in the present case; and we easily find from (7) and (10)

$$s = a\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right), \tag{19}$$

$$x = 2a\sin\phi - a\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right). \tag{20}$$

Also $\phi = \frac{\theta}{2}$, and $y = 2a\cos\frac{\theta}{2}$. On leaving the vertex, D, the value of x begins by increasing; but the logarithmic term in (20) must soon destroy the term $2a\sin\phi$, so that $x = 0$, or the curve cuts the axis of y at some such point as J (Fig. 269). The course of the curve from D is $DPJB$; and there is obviously a similar and equal portion represented by $DQJA$; and the productions of the curve beyond the arms, at the extremities of which the forces H are applied, are asymptotic to the line of force.

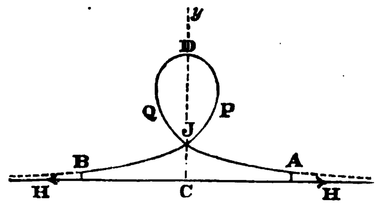

Fig. 269.

Case 3; $h > 2a$. In this case put $2a = k.h$, where k is, of course, < 1. Hence $y = h\sqrt{1-k^2\sin^2\frac{\theta}{2}}$; and if we put $k\sin\frac{\theta}{2} = \sin\phi$, we have $y = h\cos\phi$, so that we have the same geometrical representation of ϕ as before. But in this case the curve can never cross the line of force, since $\sin\frac{\theta}{2}$ cannot be equal to $\frac{1}{k}$, and, consequently, y cannot vanish.

The following results are easily found:

$$s = ka\,.\,F\left(k,\frac{\theta}{2}\right); \quad x = \frac{2a}{k}\cdot E\left(k,\frac{\theta}{2}\right) - \frac{2-k^2}{k}\,a\,.\,F\left(k,\frac{\theta}{2}\right).$$

The form of the curve is that represented in Fig. 270, in which ab is the line of action of the terminal forces. This figure represents also the curve of what is called the *Hydrostatic Arch* (omitting, of course, the looped portions).

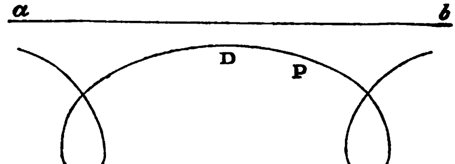

Fig. 270.

The idea in the construction of this arch is as follows: If a perfectly flexible string with fixed ends is in equilibrium under the action of continuously applied external force, which is everywhere

normal to the string, the tension, T, is constant, and at each point it $= N\rho$, where N is the normal force per unit length and ρ the radius of curvature at the point (Art. 183). In such a system there is no shearing force and no stress couple at any point. Now, if we imagine the sense of every force to be reversed, i.e. let the normal force, N, be converted into pressure towards the centre of curvature, and the tension, T, to be converted into thrust, while the string becomes a body capable of resisting tangential thrust (i.e. a body of the nature of a wire), *no shear and no bending couple would be called into play in the new body.* But these are the objects to be desired in an arch, DP, supporting water, since with no shear or bending couple, there will be no tendency in the joints to separate. The stress in the arch will then consist of direct tangential thrust between stone and stone.

Now if P is any point on the arch, and ab the level of the superincumbent water, the pressure per unit area at P is wy, where w is the weight of the water per unit volume, and y the depth of P below ab. If, then, the arch is merely a rigidified string devoid of shear and bending stress, $T =$ constant, and $\frac{T}{\rho} = wy$, therefore $\rho y =$ constant, where T is the thrust in the arch per unit of breadth (i.e. per unit distance perpendicular to the plane of the figure).

Practically the elastic curve $\rho y = a^2$ can be constructed as an assemblage of small circular arcs thus. Take any axis of abscissae, ab (Fig. 268), and, starting with a point D, describe a small circular arc, DD', whose centre is on the ordinate DC. Let this centre be O, and let y' be the ordinate of D'. Then on the line $D'O$ take a point O' such that $O'D' = OD \cdot \frac{y}{y'}$, where y is the ordinate, DC, of D, and with O' as centre draw the small circular arc $D'D''$; continue by a small circular arc from D'' to D''', and so on, and we get an approximate figure of an elastic curve.

A plane flexible string, every element of which is acted upon by a *normal* external force only, whose magnitude is proportional to the distance of the element from a fixed line in the plane of the string, assumes the form of one of the elastic curves, since by equation (2) of Art. 183, we have

$$\rho y = \text{constant}.$$

This would be the case of a flexible cylindrical sheet filled with water. A section of the surface perpendicular to the axis of the cylinder, would—at least in places not near the ends of the cylinder—be a curve satisfying the equation $\rho y =$ constant, since each element is acted upon by a normal force (the water pressure) whose magnitude is proportional to the depth, y, of the element below the free surface of the water—the lateral tensions proceeding from the elements of the sheet outside the plane of the section contributing no component tension in the plane of the section.

307.] **Kinetic Analogue of Plane Elastic Wire.** As we shall subsequently prove, there is an intimate connection between the statical problem of a bent and twisted wire and the kinetical problem of the motion of a rigid body round a fixed point. The kinetical analogue of the problem of the plane elastic rod is a very simple one, when the rod is acted upon only by terminal forces. For the equation of the curve assumed by the rod is $\rho y = a^2$; so that if θ is the angle made by the tangent at any point with the axis of x, we have (see p. 205)

$$a^2 \frac{d^2\theta}{ds^2} = -\sin\theta. \tag{1}$$

If a point travel along the curve with constant velocity, β, so that $\frac{ds}{dt} = \beta$, the equation which gives the position of the tangent to the curve, or direction of motion of the moving point, at any time is therefore

$$\frac{a^2}{\beta^2}\frac{d^2\theta}{dt^2} = -\sin\theta. \tag{2}$$

Now the equation of motion of a simple pendulum of length l is

$$\frac{l}{g}\frac{d^2\theta'}{dt^2} = -\sin\theta', \tag{3}$$

θ' being its inclination to the downward-drawn vertical through its fixed end. Hence the direction of the pendulum and the direction of the motion of the point which describes the elastic curve can be made the same at all times; for we have simply to place the axis of the elastic curve vertical, to make the tangent at the initial position of the moving point parallel to the initial position of l (the pendulum string), and to make $\beta = a\sqrt{\frac{g}{l}}$.

308.] **Tortuous Curve.** When a curve does not lie in one plane, it is generally called a 'curve of double curvature'; but the term *tortuous curve*, which is more expressive and appropriate, has been employed by Thomson and Tait. This is the term which we shall adopt. The measure of the tortuosity—or departure from planeness—at any point of such a curve is thus made. If P and P' are two points on the curve distant by the infinitesimal arc ds, and if $d\phi$ is the angle between the osculating planes at P and P', the *rate of change of direction of the osculating plane per unit length of arc* is obviously the limit of the ratio $\frac{d\phi}{ds}$ as P' approaches infinitely near to P.

This limiting ratio is called the *tortuosity* of the curve at P.

309.] **Resolution of Curvature.** *If through any point, P, of a tortuous curve any plane be drawn, and the given curve be orthogonally projected on this plane, it is required to find the curvature at P of the projection.*

Take the plane drawn through P, on which the given curve is projected, as plane of x, y. Let ρ be the radius of absolute curvature (i.e. radius of curvature in the osculating plane) of the given curve at P; let P' and P'' be points consecutive to P on the curve, the distances PP', $P'P''$ being each ds. Let the projections of P', P'' on the plane xy be Π', Π''. Then if, for simplicity, P is taken as origin, the co-ordinates of

$$\Pi' \text{ are } \frac{dx}{ds}ds; \quad \frac{dy}{ds}ds; \quad 0;$$

$$\Pi'' \text{ are } 2\frac{dx}{ds}ds + \frac{d^2x}{ds^2}ds^2; \quad 2\frac{dy}{ds}ds + \frac{d^2y}{ds^2}ds^2; \quad 0;$$

so that double the area of the triangle $P\Pi'\Pi''$ is

$$\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right)ds^3.$$

But the coefficient of ds^3 is well-known to be $\frac{\cos\nu}{\rho}$, where ν is the angle which the binormal (perpendicular to the osculating plane) makes with the axis of z (perpendicular to plane of projection). And if $\delta\chi$ is the acute angle between $P\Pi'$ and $\Pi'\Pi''$, double the area is also $P\Pi' \times \Pi'\Pi''.\delta\chi$. Also

$$P\Pi' = \sin\gamma.ds,$$

where γ is the angle made by the tangent PP' with the axis of z; so that we have

$$\sin^2\gamma \,.\, ds^2 \,.\, \delta\chi = \frac{\cos\nu}{\rho}\, ds^3.$$

Now if r is the radius of curvature of the projection $P\Pi'\Pi''\ldots$ at P, we have $r\delta\chi = P\Pi' = \sin\gamma \,.\, ds$. Hence

$$\frac{1}{r} = \frac{1}{\rho}\cdot\frac{\cos\nu}{\sin^3\gamma}. \qquad (a)$$

Cor. 1. If the plane of projection contains the tangent to the given curve, $\gamma = \frac{\pi}{2}$, and we have

$$\frac{1}{r} = \frac{1}{\rho}\cdot\cos\nu. \qquad (\beta)$$

Cor. 2. If the plane of projection contains the radius of absolute curvature, $\gamma + \nu = \frac{\pi}{2}$, and

$$\frac{1}{r} = \frac{1}{\rho}\cdot\sec^2\nu \qquad (\gamma)$$

(ν is, of course, the angle between the plane of projection and the osculating plane of the given curve).

It is obvious that (β) furnishes for the resolution of curvature exactly the same rule as that which holds for the resolution of a force into two components—viz. the curvature of the projection of a curve on any plane containing the tangent line at a point is equal to the curvature of the given curve multiplied by the cosine of the angle between the osculating plane and the plane of projection.

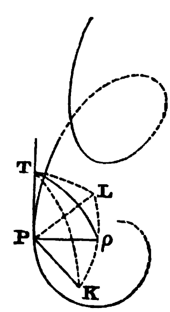

Fig. 271.

Thus (Fig. 271), let PT be the tangent to a tortuous curve at P; let $P\rho$ be the direction of the radius of absolute curvature; let PK and PL be any two lines perpendicular to PT and to each other; and imagine these lines and their planes projected on a sphere described with centre P.

Then, if ν is the angle $KP\rho$, the curvatures, $\frac{1}{r}$ and $\frac{1}{r'}$, in the planes TK and TL are $\frac{1}{\rho}\cos\nu$ and $\frac{1}{\rho}\sin\nu$, so that

$$\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r'^2}.$$

We may consider the element of the given curve itself at P (which, of course, lies in the plane $P\rho$) as 'resolved' into its two projections on the planes TK and TL; since this is only regarding the said element as given by the intersection of two cylinders.

And just as we can infer the magnitude of a force from one of its components and the angle between this component and the force, so we can infer the 'resultant' (or absolute) curvature of an element of a curve from the curvature of its projection on any plane containing the tangent, and the angle between this plane and the osculating plane.

EXAMPLE.

A spiral of inclination a is traced on a cylinder of radius r, what is the curvature of the spiral?

The angle a is the complement of that which the spiral makes with the generating lines of the cylinder. Projecting the spiral on a plane perpendicular to the axis of the cylinder, the plane of projection contains the radius of absolute curvature, so that by (γ) we have

$$\frac{1}{r} = \frac{1}{\rho}\sec^2 a, \quad \therefore \quad \rho = \frac{r}{\cos^2 a}.$$

310.] **Twist of a Wire.** Take a straight wire, AB, whose cross section at any point is a circle. Imagine a right line passing through the centres of all these circular sections, and call this the *axis* of the wire. Now on the surface of the wire mark a right line parallel to the axis; and imagine that from every point, P, on the axis a perpendicular, PK, is drawn to the marked line. The line PK at any point P is called a *transverse* of the wire at the point. At each point of the axis can, of course, be drawn an infinite number of transverses, all of them lying in the cross section of the wire at the point.

Suppose now, for definiteness, that the end B is held fixed, and that the end A is connected with a milled head—as in the case of suspension wires in some Electrometers and in Coulomb's Torsion Balance—the wire being kept vertical, and that the milled head is turned round through any angle. The result is that while the axis remains a straight line, all the transverses, PK, are rotated, those near B being least, and those near A most, disturbed. What is called 'the angle of torsion' of the wire in this case is the angle through which a transverse at the end A is turned from its original position; and if we divide this angle (in

circular measure) by the length of the wire AB, we obtain the *rate of twist* which the wire undergoes. If the end B is not absolutely fixed, but hampered in its movement, while the end A is twisted round—as is the case in Coulomb's Torsion Balance, in which the end B is attached to a repelled electrified ball—the angle of torsion will be the difference between the angles through which the transverses at A and B are turned, i.e. it will be the angle between the terminal transverses in their strained positions—or, as we prefer (for a reason to be presently given) to put it, *the angle made by the transverse at A with the plane containing the transverse at B and the axis of the wire.* It is only when the axis of the twisted wire remains a right line that the twist can be measured simply by the angle between transverses.

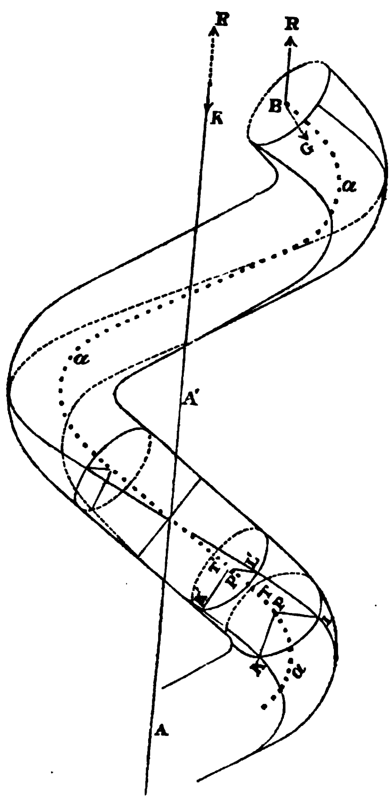

Fig. 272.

Let us now consider the case of a wire which when unstrained was straight, but which is bent and twisted so that its axis has the form of any tortuous curve—represented* by $a\,a\,a$ ... in Fig. 272. This curve we shall call the *elastic central line* of the wire.

It is obviously impossible in this case to estimate twist, as previously, by the rotation of transverses at the extremities. What we must do is to speak of a *rate of twist* at every point P of the wire, and to estimate this by treating a portion of the wire between two very close cross sections near P as straight.

* For the drawing of this figure I am indebted to Mr. Alfred Lodge.

Thus, suppose that LL'... is a line which was marked on the surface parallel to the axis when the wire was unstrained. It has assumed the spiral form represented.

Let P' be a neighbouring point on the axis, and let PL and $P'L'$ be two transverses at P and P'. Now, treating the portion PP' as straight, if $\delta\psi$ is *the angle made by $P'L'$ with the plane of PL and PP'* (or of PL and of PT, the tangent at P to the central line), the rate of twist at P is the limit of the ratio $\frac{d\psi}{PP'}$, when P' is taken infinitely close to P.

We shall denote the rate of twist at any point of the wire by τ.

Every strain, of whatever kind, being a *relative* displacement of adjacent parts, we may imagine the cross section, KL, at P to be held fixed in a vice, or by a couple, when the wire was unstrained, and the substance of the wire in its neighbourhood to be rotated round the tangent line, PT, to the axis.

Observe, then, that the amount of twist in any element, PP', of the wire is not the *angle between the transverses at* P and P'; for if the wire is simply bent in one plane, without any twist, the transverses at P and P' will be inclined to each other. It is the angle between the transverse at P' and the plane containing that at P and the tangent at P to the central line.

EXAMPLES.

1. Supposing a straight wire with a line marked on its surface parallel to its axis to be wound round a circular cylinder in such a way that the marked line is throughout kept in contact with the cylinder and forms a helix on its surface, what is the rate of twist at any point of the wire?

The transverse of any point, being perpendicular to the marked line and to the surface of the wire, will be normal to the surface of the cylinder, since the two surfaces touch all along.

Employ the method of spherical projection. Take any point O, and round it describe a sphere of unit radius. Draw OA parallel to the axis of the cylinder and meeting the sphere in A; draw OL parallel to the normal to the cylinder at a point Q of contact with the wire; draw OL' parallel to the normal at Q' which is infinitesimally distant from Q on the marked line; draw OT and OT' parallel to the tangents at Q and Q' to the helix of contact. Then the great-circle arcs AT and AT' are each $\frac{\pi}{2}-\alpha$, if α is the

inclination of the spiral. The great circles LT and $L'T'$ are parallel to the osculating planes of the helix at Q and Q', and they intersect, in the limit, in T.

Then the whole twist of the portion QQ' is the angle, $\delta\psi$, between OL' and the plane of LT. Now the great circle LL' is perpendicular to OA, so that the angle $TLL' = a$; hence $\delta\psi = LL' . \sin a$; but $LL' = \angle TAT' = \dfrac{TT'}{\cos a}$; $\therefore$ $\delta\psi = \tan a . TT'$, and the rate of twist is $\dfrac{\delta\psi}{\delta s}$, where $\delta s = QQ'$; hence the rate of twist $= \tan a . \dfrac{TT'}{\delta s}$; but $\dfrac{TT'}{\delta s} = \dfrac{1}{\rho} =$ the curvature at Q; therefore

$$\tau = \frac{1}{\rho} \cdot \tan a,$$

$$= \frac{\sin a \cos a}{r},$$

where r is the radius of the cylinder.

2. When the wire is laid on, as in last example, show that the rate of twist at any point is equal to the tortuosity of the spiral.

Produce the arc TA to N so that $TN = \dfrac{\pi}{2}$. Then ON is the binormal at Q. Similarly, produce $T'A$ to N' so that $T'N' = \dfrac{\pi}{2}$. Then the tortuosity $= \dfrac{NN'}{ds}$ in the limit. But $NN' = TT' . \tan a$; therefore the tortuosity $= \dfrac{1}{\rho} \tan a$, which is also the rate of twist.

3. If the marked line of the wire, instead of being wound on a cylinder, is laid (by bending) in contact with a sphere along any curve whatever traced on the surface, show that the rate of twist is zero at every point.

In this case the transverse at Q' lies in the plane containing that at Q and the element QQ'.

In general, if the marked line is kept in contact with *any* surface all along a line of curvature, the twist is zero at each point of the wire.

4. If the marked line is laid along a right cone so that it cuts the generators all at the same angle, i, prove that the rate of twist at any point is

$$\frac{\sin i \cos i \cos a}{r},$$

where $a =$ semivertical angle of cone, and $r =$ distance of the point from the axis of the cone.

311.] **Stress Couples of Bending and Twisting.** In the strained condition of equilibrium of the wire the equations connecting the (given) applied external forces and couples at any point with the stress forces and couples called into play, may be obtained by either of two methods—

(*a*) The ordinary method of considering the equilibrium of a portion of the wire of length ds, between two cross sections at P and P' (Fig. 272), and using equations of resolution and moments with respect to any three rectangular axes; that is, equating the component of the external forces in any direction to the component in the same direction of the forces produced on the element by the portions of the wire at the other sides of the sections through P and P'; and similarly for external couples and couples produced by strain; or

(*b*) Employing the principle of virtual work and, imagining any *further* (see Art. 268) small deformation of the element, equating the work done in producing it by external forces and couples to the work done against the forces and couples of strain —*assuming that none of the applied work passes into heat, or any other form of energy of motion.*

It is evident that before either method can be employed we must be able to express the forces and couples produced by elongations, bendings, and twistings of specified amounts in terms of the magnitudes of these strains—just as Hooke's Law expresses the magnitude of the stress produced (tension) in an elastic string in terms of the corresponding strain (extension).

Supposing that in the bent and twisted wire (Fig. 272), which we may take to be straight when unstrained*, PL and PK are any two transverses at P, and PT is the tangent to the elastic central line, and supposing the *resultant* curvature of this line to be resolved, by Art. 309, into two components, k and λ, in the planes TPL and TPK, respectively, these curvatures will be the rates of bending per unit length of arc in these planes—or round the lines PK and PL, respectively. Moreover, let τ be the rate of twist at P; then τ is a rotation of the substance round PT.

Now, if we imagine any further deformation from the configuration of equilibrium, so that k, λ, ds become $k+\delta k$, $\lambda+\delta\lambda$,

* If it is tortuous when unstrained, k and λ will be *changes* of component curvatures.

$ds + \delta ds$, the work done against the stresses must be of the form

$$T\delta ds + (K\delta k + L\delta\lambda + \Theta\delta\tau)ds, \tag{1}$$

where T is the longitudinal tension at P, and K, L, Θ are obviously stress couples, the directions of whose axes we do not at present know. Beyond bending, twisting, and elongation, the element cannot suffer any other deformation, so that the internal work is fully expressed by (1).

But we may assume that the work, $T\delta ds$, done against elongation is small compared with that done against bending and twisting, so that the internal work per unit length is

$$K\delta k + L\delta\lambda + \Theta\delta\tau. \tag{2}$$

Further, we shall assume the stresses to have a potential. This is another way of saying that to bend and twist the wire from its unstrained condition to the condition (k, λ, τ) always requires the same amount of work, no matter what may be the various intermediate conditions through which, in various experiments, we cause the wire to pass in reaching this condition (k, λ, τ); or, again, that the wire in unbending and untwisting would, by means of the stresses, give out exactly the same amount of work as that which was required to bend and twist it.

Hence the work done against strain, per unit length of the wire, must be of the form $f(k, \lambda, \tau)$, so that

$$K\delta k + L\delta\lambda + \Theta\delta\tau = \delta f(k, \lambda, \tau),$$

$$\therefore \quad K = \frac{df}{dk}, \quad L = \frac{df}{d\lambda}, \quad \Theta = \frac{df}{d\tau}. \tag{3}$$

The form of the function f cannot be determined unless some further supposition is made. The supposition is this—any *infinitely small* portion of the wire is so little deformed by strain that if the strain (k, λ, τ) is replaced by the strain $(nk, n\lambda, n\tau)$, where n is any finite number, the stress couples will become nK, nL, $n\Theta$—that is, the principle of superposition of strains and stresses holds for each infinitely small portion of the wire. [Observe that assuming the element between the cross sections at P and P' to differ infinitely little in shape from its unstrained figure, is by no means the same thing as assuming the figure of the whole wire to differ infinitely little from its unstrained figure.]

Hence K, L, and Θ must be linear functions of k, λ, τ, of the forms

$$\left.\begin{aligned} K &= Ak + c\lambda + b\tau, \\ L &= ck + B\lambda + a\tau, \\ \Theta &= bk + a\lambda + C\tau, \end{aligned}\right\} \tag{4}$$

the coefficients of λ and k in the values of K and L, respectively, being equal because, from the relations (3), $\frac{dK}{d\lambda} = \frac{dL}{dk}$, &c.

The constants A, B, C, a, b, c are *rigidities*—called by Thomson and Tait 'torsion-flexure rigidities'—of the wire at the point P considered.

Hence the work per unit length absorbed in bending and twisting the wire to the condition (k, λ, τ) is

$$\tfrac{1}{2}(Ak^2 + B\lambda^2 + C\tau^2 + 2a\lambda\tau + 2b\tau k + 2ck\lambda). \tag{5}$$

The values (4) enable us to represent the axes of the couples K, L, Θ. For, taking PK, PL, and PT as axes of x, y, z, construct the quadric

$$Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy = h, \tag{6}$$

where h is any constant. Let (k, λ, τ) be taken as the co-ordinates of a point; let this point be Q. Then the polar plane of Q with respect to the quadric is

$$Kx + Ly + \Theta z = h. \tag{7}$$

Let R be the point whose co-ordinates are (K, L, Θ). Then R is obviously on the perpendicular from P on the plane (7); and if p is the length of this perpendicular we have

$$PR = \frac{h}{p}; \tag{8}$$

thus R is found, and its three co-ordinates give the three axes of those stress couples at P which are called into play by bendings round PK and PL and twisting round PT.

Is there any line at P *such that a strain rotation of the substance round it calls into play a stress couple whose axis coincides with the line itself?* It is evident that there are *three* such lines—viz. the axes of the quadric (5); for if Q lies on any one of these axes, PQ will be perpendicular to the polar plane of Q, so that PR and PQ are coincident in direction; and if the axes of the quadric are taken as those of x, y, z and the strain rotation is $(k, 0, 0)$—i.e. simply a rotation round the first axis, the y and z co-ordinates of R will both be zero.

These are called the *principal torson-flexure axes* at P.

If ω_1, ω_2, ω_3 denote any strain rotations round these axes, and L_1, L_2, L_3 the corresponding stress couples called into play, we have, therefore,

$$L_1 = A_1\omega_1; \quad L_2 = A_2\omega_2; \quad L_3 = A_3\omega_3, \tag{9}$$

where A_1, A_2, A_3 are the principal torsion-flexure rigidities at P. Also the potential work of the stress, per unit length at any point, is

$$\tfrac{1}{2}(A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2). \tag{10}$$

There is, of course, no *essential* difference between the quantities k, λ, τ; each is simply a strain rotation of the substance round an axis. But when the axes of these rotations are two transverses and a tangent line to the wire (or rather to its elastic central line), it is usual to distinguish the last by the name *twist* and the others by the name *bending*. The *principal* axes may have any positions whatever at P, and it is impossible to use distinctive names for the rotations round them. That this is so will be at once obvious if we imagine a solid body with any fibrous or laminated structure, the fibres or laminae running in a definite direction. Now imagine *a wire cut in any way out of this body*, its tangent line at any point being oblique to the fibres or to the laminae. It is clear that at any point of this wire the tangent and two transverses have no special relations whatever to its rigidities, and therefore no special properties with regard to simplicity.

For a wire drawn in the ordinary manner, we may assume that, whatever may be the form of the cross section (circular, elliptical, or rectangular) the tangent to the central line is one principal axis of strain rotation, the other two being necessarily some two transverses.

If, in this case, we use A_3 for the rigidity round the tangent to the central line, and if the other two rigidities, A_1 and A_2, are equal, there will be equal flexibility round *all* transverses; for the quadric will become

$$A_1(x^2 + y^2) + A_3 z^2 = h, \tag{11}$$

and any two lines whatever in the cross section are principal axes of the quadric.

In all cases the axis of *resultant* bending at P is the binormal to the central line—the resultant bending taking place in the osculating plane; but this plane will not, in general, be a

principal plane of flexure; i.e. this bending will not, in general, call into play a stress couple whose axis is the binormal.

It is essential to observe that the values of the principal rigidities A_1 and A_2 round the two principal transverses (when the tangent to the central line is a principal axis)—i.e. the two principal bending rigidities—depend not merely on the structure of the material, but also on the *shape* of the cross section; and that for wires drawn in the ordinary manner they are evidently equal if this section is a circle or a square, since in each of these cases the quadric will be given by (11).

The determination of the rigidities A_1, A_2, A_3 in terms of the shape, area, &c., of the cross section must be postponed until we consider the subject of strain and stress more particularly.

312.] **Wire held in a given Spiral with given Twist.** Before proceeding to the general case in which the wire is acted upon by any forces and couples, we shall consider the case in which there are only a force and a couple applied to each end, or to one end while the other is held fixed.

Assuming the tangent to the central line to be a principal axis of strain, and also that its rigidities round all transverses are equal, it is required to find the force and couple which must be applied to each end in order to keep the wire in the form of a given helix, and with a given constant rate of twist.

[This problem is solved in Art. 602 of Thomson and Tait, vol. ii.]

Let R (Fig. 272) be the force and G the axis of the couple applied to one end. Then, reducing R and G to a wrench, the axis of this wrench must be the axis of the cylinder on which the given spiral lies.

Let (R, K) be the force and couple of the wrench, and consider the equilibrium of the portion between the cross section at P and the end B. We shall simply equate to zero the sum of the moments of the forces acting on this portion about the axis AA', and about a line through P perpendicular to AA' and to the radius of the cylinder drawn to P.

Take AA' as axis of z; the radius of the cylinder drawn to P as axis of y; and a perpendicular to both as axis of x. Let α be the inclination of the spiral, r the radius of the cylinder, τ the rate of twist.

Then the stress couples at P exerted on the portion PB by the

portion PA can be reduced to two, viz. one about the tangent PT, and one about the binormal to the central line. If $A_1 = A_2$, and ρ is the radius of curvature at P, the second couple is $\frac{A_1}{\rho}$, or (Art. 309) $A_1 \frac{\cos^2 a}{r}$.

Also the magnitude of the couple exerted on PB about PT (the twisting couple) is $A_3 \tau$, and its axis is in the sense PT if the twist is in the sense LK, since, if the portion PA were removed, we should have to apply to PB a couple in the sense KL.

Again, the direction-angles of PT are $\left(a, \frac{\pi}{2}, \frac{\pi}{2}-a\right)$ and those of the binormal are $\left(\frac{\pi}{2}+a, \frac{\pi}{2}, a\right)$, and the axis of the bending couple exerted on PB is to be drawn along the binormal from P towards the upper portion of the figure as we view it, by Art. 200.

Hence, by moments about AA',

$$K - A_3 \tau \sin a - A_1 \frac{\cos^2 a}{r} \cos a = 0; \tag{1}$$

and by moments about a line through P parallel to the axis of x,

$$R.r - A_3 \tau \cos a + A_1 \frac{\cos^2 a}{r} \sin a = 0. \tag{2}$$

It is scarcely necessary to observe that these two equations of moments are only a part of the condition of equilibrium of the portion of the wire considered; and that the stress at P does not consist wholly of the two couples $A_3 \tau$ (that of twisting) and $\frac{A_1}{\rho}$ (the 'resultant' bending couple, Art. 309); for there is also, manifestly, at P a stress which must be equal, parallel, and opposite, to R; but this last stress contributes nothing in the equations of moments about axes selected as above.

We can now consider particular cases of this problem.

EXAMPLES.

1. If the given rate of twist is that which would be produced by laying the wire along the cylinder in the manner described in example 1, p. 216, determine the necessary wrench.

Ans.
$$R = \frac{A_3 - A_1}{r^2} \cos^2 a \sin a,$$
$$K = (A_3 \sin^2 a + A_1 \cos^2 a) \frac{\cos a}{r}.$$

2. One end of a wire is held fixed, while a couple, G, is applied (without any force) to the other end, the axis of the couple making a given angle, θ, with the tangent at the end; determine the form of the spiral assumed by the wire, and its rate of twist.

Ans. The axis of the cylinder must be parallel to that of G, so that the inclination of the spiral is $\frac{\pi}{2}-\theta$. Also, since $R=0$, we have

$$A_3\tau - A_1\frac{\sin\theta\cos\theta}{r}=0,$$

and
$$A_3\tau\cos\theta + A_1\frac{\sin^2\theta}{r}=K(=G),$$

from which
$$r=\frac{A_1\sin\theta}{G};\quad \tau=\frac{G\cos\theta}{A_3};$$

so that we have the radius of the cylinder and the rate of twist of the wire. Thus everything relating to the spiral is found.

3. Show that, one end of a wire being held fixed, the wire may be kept by means of a terminal force alone in the form of a given spiral, and find the accompanying rate of twist.

Putting $K=0$, we get $\tau=-\frac{A_1}{A_3}\frac{\cos^3 a}{r\sin a}$ (r and a being given). Then $R=-\frac{A_1\cos^2 a}{r^2\sin a}$, which shows that the ends must be pressed towards each other.

The negative sign in the value of τ means that the twist is in a sense reverse to that of the winding of the spiral.

4. How can a wire be held in the form of a given helix so as to have no twist at any point?

Ans. If $a=$ the inclination of the spiral, $r=$ radius of cylinder, A_1 the flexural rigidity, apply at the free end a couple in the osculating plane, whose moment $=\frac{A_1\cos^2 a}{r}\left(=\frac{A_1}{\rho}\right)$, and apply also at this end a compressive force parallel to the axis of the cylinder, equal to $\frac{A_1\sin a\cos^2 a}{r}$. [The free end is supposed to terminate on the cylinder, and not to be carried in towards the axis.]

If the wire when unstrained has the form of a helix of radius b and inclination β, and if, in its strained form, it has radius r and inclination a, one end being fixed while the other is acted upon a force and a couple which are equivalent to the wrench (R, K), the *change* of curvature at any point is $\frac{\cos^2 a}{r}-\frac{\cos^2\beta}{b}$, and it is to this *change* that the resultant stress couple of bending

(that in the osculating plane) is now proportional. Moreover, if we mark on the wire the spiral line along which in its unstrained condition it touched the cylinder of radius b, and if the wrench is such as to keep this marked line in contact with the cylinder of radius r, the rate of twist will be

$$\frac{\sin a \cos a}{r} - \frac{\sin \beta \cos \beta}{b}.$$

Hence equations (1) and (2) become

$$K = A_3 \sin a \left(\frac{\sin a \cos a}{r} - \frac{\sin \beta \cos \beta}{b}\right) + A_1 \cos a \left(\frac{\cos^2 a}{r} - \frac{\cos^2 \beta}{b}\right), \tag{3}$$

$$R.r = A_3 \cos a \left(\frac{\sin a \cos a}{r} - \frac{\sin \beta \cos \beta}{b}\right) - A_1 \sin a \left(\frac{\cos^2 a}{r} - \frac{\cos^2 \beta}{b}\right). \tag{4}$$

Thomson and Tait (*Nat. Phil.* vol. ii. p. 141) take other variables than a and r to express the new spiral—viz. the axial distance, z, between the ends of the spiral, and the whole angle ϕ through which the spiral winds round its axis.

If $l =$ the whole length of the spiral, it is obvious that $z = l \sin a$, and $\phi = \frac{l}{r} \cos a$. The angle ϕ may be considered as that included between a plane through the axis and one end, A, and a plane through the axis and the other end, B, of the spiral.

Using these new variables, we have

$$K = A_3 \frac{z}{l^3} (z\phi - h\gamma) + A_1 \frac{\sqrt{l^2 - z^2}}{l^3} \left(\phi \sqrt{l^2 - z^2} - \gamma \sqrt{l^2 - h^2}\right), \tag{5}$$

$$R = A_3 \frac{\phi}{l^3} (z\phi - h\gamma) - A_1 \frac{z\phi}{l^3 \sqrt{l^2 - z^2}} \left(\phi \sqrt{l^2 - z^2} - \gamma \sqrt{l^2 - z^2}\right), \tag{6}$$

where h is the axial distance between the ends and γ the value of ϕ in the unstrained position.

We now proceed to a verification.

Since z is measured along the line of action of R, and ϕ is measured round the axis, it is clear that the work done by the wrench in a small further deformation is

$$Pdz + K\,d\phi,$$

and since this is the work done against the stresses, which we have assumed to have a potential (Art. 311), it follows that this

expression must be the differential of a single function; and we see at once on trial that it is the differential of the function

$$\frac{A_3}{l^3} z\left(\tfrac{1}{2} z\phi^2 - h\gamma\phi\right) + \frac{A_1}{l^3}\sqrt{l^2-z^2}\left(\tfrac{1}{2}\phi^2\sqrt{l^2-z^2} - \gamma\sqrt{l^2-h^2}.\phi\right),$$

or of the function

$$\tfrac{1}{2}\frac{A_3}{l^3}(z\phi - h\gamma)^2 + \tfrac{1}{2}\frac{A_1}{l^3}\left(\phi\sqrt{l^2-z^2} - \gamma\sqrt{l^2-h^2}\right)^2,$$

which differs by a constant from the previous function.

If ϖ is the curvature in the strained state, $\varpi = \dfrac{\phi\sqrt{l^2-z^2}}{l^2}$; and if ϖ_0 is the curvature in the unstrained state,

$$\varpi_0 = \frac{\gamma\sqrt{l^2-h^2}}{l^2}. \quad \text{Also } \tau = \frac{z\phi - h\gamma}{l^2}.$$

Hence the above function is

$$\frac{l}{2}\left[A_3\tau^2 + A_1(\varpi - \varpi_0)^2\right],$$

which is, as anticipated, the potential of the stress, i.e. the Static Energy, of the whole spring. [See Art. 311, expression (10).]

Equations (5) and (6) give the magnitudes of the couple and force, constituting a wrench about the axis of the spiral, which are required to produce given changes, $z-h$ and $\phi-\gamma$, in the wire; or, conversely, the changes which are produced in it by a given wrench.

It remains to notice the case in which the change of form from the unstrained to the strained state is very small.

We may then put $\phi = \gamma + \delta\phi$, and $z = h + \delta z$. Then

$$K.l^3 = (A_3 - A_1)\gamma h.\delta z + \left[A_3 h^2 + A_1(l^2-h^2)\right].\delta\phi, \qquad (7)$$

$$R.l^3 = \left(A_3 + A_1\frac{h^2}{l^2-h^2}\right)\gamma^2.\delta z + (A_3 - A_1)\gamma h.\delta\phi. \qquad (8)$$

Suppose the wire to form *a spiral of very small inclination*. Then, either from the two equations just written, or, more directly, from the fundamental equations (3) and (4), the values of the requisite couple and force may be found. In (3) and (4), if for $\sin a$ we put a, &c., we have

$$K = A_1\left(\frac{1}{r} - \frac{1}{b}\right); \quad R.r = A_3\left(\frac{a}{r} - \frac{\beta}{b}\right),$$

neglecting in the value of Rr the term $a\left(\dfrac{1}{r} - \dfrac{1}{b}\right)$. If for a we put $\dfrac{z}{l}$, or $\dfrac{h+\delta z}{l}$, we have

$$R = \frac{A_3}{b^2 l}\delta z, \qquad (9)$$

if we neglect the term $\frac{h}{l}\left(\frac{1}{r}-\frac{1}{b}\right)$, which is evidently an infinitesimal of the second order. The value of K

$$\left(\text{since } \gamma+\delta\phi=\frac{l}{r}, \quad \gamma=\frac{l}{b}\right)$$

can be written

$$K=\frac{A_1}{l}\delta\phi. \tag{10}$$

In this case, then, the couple produces simply rotation and no lengthening, while the force produces a lengthening and no rotation.

Prof. J. Thomson has given a simple rule for the elongation, δz, of the axial length of a spiral of small inclination, produced by a given axial force at one end, the other being held fixed. The rule is the interpretation of (9) above. Suppose the wire constituting the spiral to be made straight and coincident with the axis of the cylinder (of radius b) on which it was wound. Let one end, A, be held fixed; let the other, B, be rigidly connected with a cap or torsion-head which just fits the cylinder and can be turned about its axis.

If the given force R is applied tangentially to the circumference of this torsion-head, its point of application will move through a certain circular arc until it is stopped by the torsional resistance of the wire. The length of this arc is the axial motion (δz) which the force R produced when applied to draw out the spiral.

For, considering the equilibrium of the portion of this straight wire between any point P, and the extremity, B, we have by moments about its central line

$$A_3\omega_3=Rb; \quad \therefore \ \omega_3=\frac{Rb}{A_3}.$$

But ω_3 is the strain rotation per unit length, therefore $l\omega_3$ is the whole angular rotation of the end B, and the circular arc described by a point on the circumference of the torsion-head is $bl\omega_3$, which is $\frac{Rb^2 l}{A_3}$; and this is precisely δz in (9).

It is worthy of note, then, that in the case of a wire which forms a cylindrical spiral of small inclination—such, for instance, as the spiral wire employed in Siemens's Ammeter and Wattmeter—if the spiral is drawn out in the direction of its axis through a small distance (so that it is still a spiral of small inclination),

it is the torsional rigidity (A_3) *and not the flexural rigidity* (A_1) *that is called into play.* Moreover, if one end of such a spiral is rotated (instead of being drawn out), as happens in the Ammeter and Wattmeter, it is the *flexural* and not the torsional rigidity that is called into play; for $\delta\phi$ in (10) involves A_1 and not A_3.

And if the spiral is not one of very small inclination, these results are not true—i.e. *both* rigidities will be called into play by axial stretching, and both by terminal rotation.

313.] **Case of Continuously Distributed Force.** Assuming that the principal torsion-flexure axes at any point of the wire are the tangent, PT, to the central line and two lines, PK and PL (Fig. 272) in the normal section, suppose that instead of mere *terminal* force and couple we have external *bodily* force (like weight, for example,) acting on all the elements, as well as external bodily couple. There may, in addition, be terminally applied forces and couples.

Assume, for the present, that the wire when unstrained was straight.

Consider the equilibrium of the elementary portion of the wire between the cross sections at P and P'; then before strain the principal axes, $P'K'$, $P'L'$, $P'T'$, at P' were parallel to those at P. After strain they will occupy positions, relatively to those at P, which are obtained by holding the normal section at P fixed and rotating the system of axes at P' round PK, PL, PT through angles equal to $\omega_1 ds$, $\omega_2 ds$, $\omega_3 ds$, respectively.

Hence we easily find the following table for the direction-cosines of the strained positions of the axes at P' with reference to those at P:—

	PK	PL	PT
$P'K'$	1	$\omega_3 ds$	$-\omega_2 ds$
$P'L'$	$-\omega_3 ds$	1	$\omega_1 ds$
$P'T'$	$\omega_2 ds$	$-\omega_1 ds$	1

magnitudes of the order ds^2 being neglected.

Let L_1, L_2, L_3 be the principal stress couples at P, and S_1, S_2, S_3 the components along PK, PL, PT of the internal force exerted on the cross-section at P by the lower portion, PA, of

the wire in Fig. 272, the same letters with dashes defining the corresponding things at P'. Also let the components of external bodily couple applied to the portion between the two cross-sections be $G_1 ds$, $G_2 ds$, $G_3 ds$; and of bodily force $X_1 ds$, $X_2 ds$, $X_3 ds$.

Equating to zero the total component force parallel to PK, we have
$$S_1 - S_1' + S_2'.\omega_3 ds - S_3'.\omega_2 ds + X_1 ds = 0.$$
But $S_1' = S_1 + \frac{dS_1}{ds} ds$; therefore in the limit
$$\frac{dS_1}{ds} - \omega_3 S_2 + \omega_2 S_3 = X_1. \tag{1}$$
Similarly,
$$\frac{dS_2}{ds} - \omega_1 S_3 + \omega_3 S_1 = X_2, \tag{2}$$
$$\frac{dS_3}{ds} - \omega_2 S_1 + \omega_1 S_2 = X_3. \tag{3}$$

Again, by moments round PK we easily find
$$L_1 - L_1' + L_2'.\omega_3 ds - L_3'.\omega_2 ds + S_2' ds + G_1 ds = 0,$$
or
$$\frac{dL_1}{ds} - \omega_3 L_2 + \omega_2 L_3 - S_2 = G_1. \tag{4}$$
Similarly,
$$\frac{dL_2}{ds} - \omega_1 L_3 + \omega_3 L_1 + S_1 = G_2, \tag{5}$$
$$\frac{dL_3}{ds} - \omega_2 L_1 + \omega_1 L_2 \quad = G_3. \tag{6}$$

[With a view to *homogeneity* in our equations, it may be well to observe that S_1, S_2, S_3 are forces—not forces per unit area; X_1, X_2, X_3 are forces per unit length; G_1, G_2, G_3 are couples per unit length; ω_1, ω_2, ω_3 are curvatures, or angular rotations per unit length.]

Observing now that $L_1 = A_1\omega_1$; $L_2 = A_2\omega_2$; $L_3 = A_3\omega_3$, the last three equations become, if A_1, A_2, A_3 are constant throughout the wire
$$A_1 \frac{d\omega_1}{ds} - (A_2 - A_3)\,\omega_2\omega_3 = G_1 + S_2, \tag{7}$$
$$A_2 \frac{d\omega_2}{ds} - (A_3 - A_1)\,\omega_3\omega_1 = G_2 - S_1, \tag{8}$$
$$A_3 \frac{d\omega_3}{ds} - (A_1 - A_2)\,\omega_1\omega_2 = G_3; \tag{9}$$
which (see Routh's *Rigid Dynamics*, chap. v., or Williamson and Tarleton's *Dynamics*, chap. x.) are identical with the equations of motion of a rigid body with one point fixed, the arc s measured

along the wire corresponding to the time t in the motion of the body, and the component curvatures and twist, $\omega_1, \omega_2, \omega_3$, from point to point of the wire corresponding to the component angular velocities, from time to time, of the rotating body—this body being supposed to be at each instant acted upon by couples numerically equal to G_1+S_2, G_2-S_1, and G_3 about its principal axes, while its principal moments of inertia at the fixed point are numerically equal to A_1, A_2, A_3.

If the unstrained wire is not straight, let a_1 and a_2 be the component curvatures in the planes perpendicular to PK and PL, respectively, before strain. Then the direction-cosines of PK' with respect to PK, PL, PT before strain are $(1, 0, a_2 ds)$; and if $\theta_1, \theta_2, \theta_3$ are the bending and twisting rotations per unit length, the direction-cosines of PK' after strain are easily found to be $[1, \theta_3 ds, -(a_2+\theta_2) ds]$. Now $a_2+\theta_2$ is the new component curvature in the plane perpendicular to PL. If we denote this by ω_2 and use ω_3 for θ_3 to preserve the previous notation, we find that all the equations (1) ... (6) hold for this case, ω_1 and ω_2 being the new component curvatures (and not *changes* in them).

Equations (7), (8), (9) will not hold in this case; for we have $L_1 = A_1(\omega_1-a_1)$, and $L_2' = A_2(\omega_2-a_2)$, which must be substituted in (4), (5), (6); so that the equations become

$$A_1\frac{d(\omega_1-a_1)}{ds}-A_2\omega_3(\omega_2-a_2)+A_3\omega_2(\omega_3-a_3) = G_1+S_2, \quad (10)$$
$$\&c., \qquad\qquad \&c.$$

314.] **Kinetic Analogies.** In a very simple case (Art. 307) of a strained wire it has been shown that the curvatures at different points are numerically equal to the angular velocities, at different times, of a certain compound pendulum.

We propose to notice a less restricted case now.

Suppose that a wire (Fig. 272) is bent and twisted under the influence solely of *terminal* forces and couples, and suppose that the wire was originally straight. Then, from last Article, it appears that the equations for its component curvatures and its twist are

$$A_1\frac{d\omega_1}{ds}-(A_2-A_3)\,\omega_2\omega_3 = S_2, \quad (1)$$

$$A_2\frac{d\omega_2}{ds}-(A_3-A_1)\,\omega_3\omega_1 = -S_1, \quad (2)$$

$$A_3\frac{d\omega_3}{ds}-(A_1-A_2)\,\omega_1\omega_2 = 0, \quad (3)$$

where S_1, S_2 are the components of the stress (force) exerted on the normal section at P along the lines PK and PL.

Now, considering the equilibrium of the whole length PB of the wire between P and the extremity B, we see that the force and couple of strain at P must be equal and opposite to R and G at B; i.e. S_1, S_2, S_3 are simply equal to the components of R along PK, PL, PT.

Imagine a rigid body moveable round a fixed point O, at which point we draw three lines Ok, Ol, Ot parallel to the lines PK, PL, PT in the wire, and let the body have moments of inertia equal to A_1, A_2, A_3 about Ok, Ol, Ot, which are the principal axes of the body at O.

Moreover, let this body be acted upon by a force equal and parallel to R at a point p on the axis Ot such that Op is a unit length (the unit implied in the measurement of curvature in the wire). Then the components of R parallel to Ok and Ol will be obviously S_1 and S_2, and the equations for the angular velocities of the body round Ok, Ol, Ot will be precisely the same as (1), (2), (3) for the component curvatures and twist of the wire. [It is understood, of course, that A_1, A_2, A_3 are each constant all along the wire, which may have a section of any (but constant) shape.]

If, then, the body is started with angular velocities the same in magnitudes and senses as the component curvatures at any point, P, of the wire, its principal axes being at starting made parallel to the principal axes of strain at P, while it is acted upon by the constant force R as described, its principal axes of inertia will at any time be parallel to the principal axes of strain at the point (Q, suppose) reached at that instant by a point travelling originally from P with constant unit velocity along the wire; and the component angular velocities and curvatures will also correspond.

Hence *the Kinetic Analogue of a wire of constant section bent and twisted by terminal forces and couples is a rigid body moving round a fixed point while acted on by a force of constant magnitude, direction, and point of application;* e.g. a heavy rigid body moving round a fixed point.

The case in which the motion of this pendulous body is uniplanar is that discussed in Art. 307. This analogy is due to Kirchhoff, and the reader will find it discussed from a different

point of view in a paper by Professor Larmor (*Proceedings of the London Math. Soc.*, Nos. 225–228).

The case in which the wire when unstrained formed a helix does not correspond to the motion of a rigid body about a point, because its equations are of the type (10) of last Article; but Professor Larmor (*loc. cit.*) points out that such a case is analogous to that of a rigid body rotating about a fixed point, a revolving fly-wheel being attached to the body, the axis of the fly-wheel being fixed in the body.

Examples.

1. Deduce the equations of the plane elastic spring (Art. 304) from the general equations, p. 229.

2. In a plane elastic rod, without external bodily couple, the shearing stress vanishes at points of maximum or minimum curvature.

3. A heavy uniform inextensible, but imperfectly flexible, string is suspended from two fixed points; prove that the figure of the 'catenary' is given by the equation

$$A\frac{d^2\theta}{ds^2}+mgs\cos\theta-\tau\sin\theta=0,$$

where A is the flexural rigidity, τ the tension at the lowest point, and mg the weight per unit length of the string; θ being the inclination of the tangent to the horizon, and s the length of the arc measured from the lowest point.

(Consider the equilibrium of the portion of the curve between the lowest point and any point P. See p. 202; and observe the rule with regard to signs.)

4. Assuming the flexural rigidity to be small, prove that, approximately,

$$\tan\theta=\frac{s}{c}-\frac{2A}{\tau}\frac{s}{(c^2+s^2)^{\frac{3}{2}}},$$

in which $\tau=mgc$. Compare with the result for the flexible catenary.

5. If the string in Ex. 3 is replaced by a thin steel ribbon magnetised in the direction of its length, find the differential equation of the curve formed under the influence of gravity and the earth's magnetism.

Ans. If μ is its magnetic moment per unit length, $F=$ the resultant earth-force per unit pole, $i=$ the dip, the equation is

$$A\frac{d^2\theta}{ds^2}+mgs\cos\theta-\tau\sin\theta+\mu F\sin(\theta+i)=0,$$

the arc being, as previously, measured from the lowest point—which is not now, however, a point of zero shearing stress.

6. If a uniform elastic rod is freely pivoted at its extremities and made to assume the figure of n bays (as in example 1, p. 207), prove that its Static Energy is equal to

$$\frac{4n^2 A}{l} F(E - k'^2 F),$$

F and E denoting the complete elliptic integrals with modulus k, where k is (as shown in p. 207) a determinate constant.

If Π is the Potential Work, or Static Energy, we have, by Art. 311, $d\Pi = \frac{1}{2}\frac{A}{\rho^2}ds$, since $\omega_1 = \frac{1}{\rho}$; and $\rho y = a^2$, therefore $d\Pi = \frac{A}{2a^4}y^2 ds$; therefore substituting for y and ds their values (Art. 306),

$$d\Pi = \frac{A}{2a^3}\,.\,h^2\cos^2\phi\,\frac{d\phi}{\Delta\phi},$$

$$\therefore\quad \Pi = \frac{2nAk^2}{a}\int_0^{\frac{\pi}{2}}\frac{\cos^2\phi\, d\phi}{\Delta\phi},$$

which at once gives the above result, since (p. 207) $l = 2naF$.

7. If a uniform rod has both extremities fixed by smooth pins and is continuously acted upon by a constant normal force, show that its equation is

$$\frac{A}{\rho} = \beta y - \tfrac{1}{2}p\{(c-x)^2 + y^2\},$$

where β is a constant, p = normal force per unit length, $2c$ = distance between the pins, $\beta = \sqrt{f^2 - c^2p^2}$, and f is the pressure on each pin; the axes of co-ordinates being the line joining the pins and a perpendicular to it at its middle point. (The value of β must be found from the fact that the length of the curve is given; and if a is the angle made by f with the line of pins, $f\sin a = cp$.)

NOTE. Several of the results in Arts. 306, &c., were obtained by Professor Greenhill and published by him in the *Messenger of Mathematics*.

CHAPTER XVII.

THEORY OF ATTRACTION.

SECTION 1.—*Direct Calculation of Attraction.*

315.] **Newtonian Law of Attraction.** The motions of the planets and comets of the solar system can be explained completely on the hypothesis that each body of this system attracts every other body of the system with a force which in any position of the two bodies is directly proportional to the product of the masses of the bodies, and which in different positions is inversely proportional to the square of the distance between them. The fact that the positions which will be occupied by comets can be predicted with certainty, that the existence of Neptune was mathematically deduced from the assumption that certain disturbances in the motion of Uranus were due to the attraction of an unknown planet according to the above law, and several other facts of the same kind, all prove that the law holds with all the accuracy that human observation is capable of testing, so far as the action upon each other of large masses separated by distances which are great compared with their linear dimensions is concerned.

As to the cause, or mechanism, to which this attraction is due, nothing is known. Newton says in the Scholium at the end of Book III of *The Principia*, 'To us it is enough that gravity does really exist and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies and of our sea.' A little before this in the same Scholium he says, 'But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypothesis (*hypotheses non fingo*).'

Although Newton framed no hypothesis on the mode by which gravitation is propagated through space, he mentions certain speculations which were current in his time, and which have been brought into great prominence in our days. Thus, at the end of section xi. of book I. he says, 'I here use the word attraction in general for any endeavour, of whatever kind, made by bodies to approach each other; whether that endeavour arise from the action of the bodies themselves, as tending mutually to or agitating each other by spirits emitted; or whether it arises from the action of the æther or of the air, or of any medium whatsoever, whether corporeal or incorporeal, anyhow impelling bodies placed therein towards each other.'

By far the most promising step that has been taken towards a solution of this great difficulty is the discovery by Faraday that the attraction between two electrified bodies is influenced by the insulating medium in which they are placed, inasmuch as this discovery renders it highly improbable that any force produced by one body on another is a *direct action at a distance.* This discovery has been worked by Clerk Maxwell into a consistent mathematical theory of the mechanism by which magnetic and electromagnetic actions are propagated by a rare medium filling space.

Newton does not, however, confine the law of attraction according to the inverse square of distance to large masses like the planets; for he investigates the attraction of a solid on a particle, even when the particle is within the matter forming the body, on the supposition that this particle is attracted by *every* elementary particle of the body—however close to the attracted particle—with a force expressed by this law.

The assumption that *every indefinitely small particle of matter attracts every other particle with a force which acts in the right line joining the particles and whose magnitude is directly proportional to the product of the quantities of matter in the particles and inversely proportional to the square of the distance between them* is the formula of what is called the *Law of Universal Gravitation.*

The terms of the enunciation render it clear that *the linear dimensions of the particles must be infinitely small compared with the distance between them*—otherwise, indeed, the term 'distance between them' would be unmeaning. We shall soon prove,

however, that if the particles are homogeneous and spherical, this limitation may be removed, and the 'distance between them' is the distance between their centres.

But it is just at this point—i. e. when dealing with forces exerted on each other by indefinitely close molecules—that our ignorance of the cause or mechanism of this attraction introduces a most unsatisfactory dualism—or rather *multiplicity* of laws—into physical science.* For we are often presented with *repulsions* instead of attractions, and the phenomena of Elasticity and Capillarity have hitherto compelled physicists to assume other laws of force between molecules than the Newtonian law of the inverse square of distance, or the *law of nature*, as it is often called.

Electrical and magnetic attractions and repulsions are proved by experiment to obey this law, and therefore the theory of attraction is almost wholly a discussion of its consequences.

The quantitative expression of the Newtonian law is as follows. Suppose two very small particles whose masses are m grammes and m' grammes to be placed at a distance of r centimètres apart—this distance being, as before said, very great compared with the linear dimensions of either particle; then each will attract the other with a force equal to

$$\gamma \frac{m \,.\, m'}{r^2} \text{ dynes,} \tag{a}$$

in which expression γ is an absolute constant, i. e. one whose magnitude is independent of the magnitudes of the masses and their distance.

We shall subsequently calculate the value of γ, which is called the *absolute constant of gravitation.* With the units of mass and distance chosen as above, γ is evidently *the number of dynes in the force with which a mass of one gramme condensed into an infinitely small volume attracts an equal mass similarly condensed*

---

* For example, Clerk Maxwell, in his article on Capillary Action (*Encyclop. Brit.*) says: 'The forces which are concerned in these phenomena are those which act between neighbouring parts of the same substance, and which are called forces of cohesion, and those which act between portions of matter of different kinds, which are called forces of adhesion. These forces are quite insensible between two portions of matter separated by any distance which we can directly measure. It is only when the distance becomes exceedingly small that these forces become perceptible.'

Clearly science still needs a vigorous application of Occam's Razor.

at a distance of one centimètre; or, as we shall see, the force of attraction between two homogeneous spherical grammes with a distance of one centimètre between their centres.

316.] **Conical Angles.** Let $ABCDE$ (Fig. 273) be any closed curve, plane or tortuous, and O any point. If from O lines OA, OB, &c., are drawn to every point on the curve, we obtain a cone. If round O a sphere of 1 centimètre radius is described, the rays OA, OB, &c., constituting the cone will intersect the spherical surface in a curve $abcde$; and the number of square centimètres in the area of the spherical surface contained within this curve is called the *solid angle* subtended at O by the given curve $ABCDE$. Instead of this term (which is in no way expressive) we shall use the term *Conical Angle*. If the sphere described round O has a radius of 1 mile instead of 1 cm., the number of square miles of the spherical surface enclosed by $abcde$ will be the conical angle, and this number will be the same as that of square centimètres on a sphere of radius 1 cm. Generally, if a sphere of any radius, r, be described round O, and the curve $ABCDE$ conically projected, as above, on its surface, the ratio of the area of $abcde$ to the square of the radius r is the measure of the conical angle subtended at O by the given curve—just as the *plane* angle subtended at O by any two points, P, Q, is the ratio of the length of the arc of any circle, with O as centre in the plane POQ, intercepted by the rays OP and OQ, to the length of the radius.

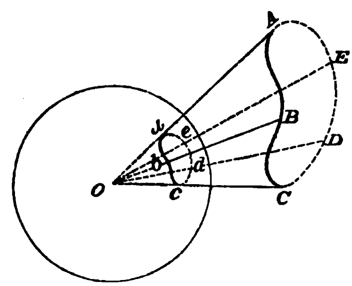

Fig. 273.

A conical angle is thus a mere *number*, like the circular measure of a plane angle.

The sum of all the conical angles round any point is 4π, because it is the whole area, in square centimètres, of a sphere of 1 cm. radius described round the point.

The conical angle subtended by any closed plane curve at any point which is in the plane of the curve and inside its area is 2π, since the rays OA, OB, &c., from O to the different points

on the curve intersect a spherical surface described round O as centre in a great circle of the sphere.

Let any closed *surface* be broken up into an indefinitely great number of small elements of area; then the sum of all the conical angles subtended by the contours of these elements at any point, O, inside the given closed surface is obviously 4π.

If O is *anywhere* on the surface itself, the sum of all the conical angles subtended at O by the elements of area of the surface is 2π, since the slender cones revolving round O lie all on one side of the tangent plane at O, and they will cut off only the area of half the sphere described round O.

If O is anywhere *outside* the given closed surface, the sum of all the conical angles subtended at O by the elements of area on the surface is *zero*. This case requires a little explanation.

Let any line drawn through O meet the given closed surface in points P_1, P_2, P_3, P_4 (Fig. 274), of which there must be an even number; and let a very slender cone of rays drawn through O intersect the surface in the small elements of area represented at these points. Then although *numerically* the conical angles subtended at O by these elements of area are all the same, distinctions of *sign* must be made between them.

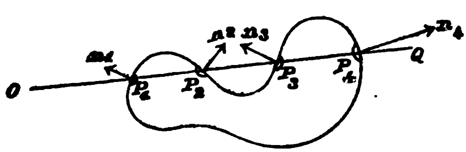

Fig. 274.

These distinctions can easily be made thus. At P_1 it is the *outside* of the surface that is turned towards O; at P_2 it is the *inside*; at P_3 the outside; and at P_4 the inside. Hence if $d\omega$ is the magnitude of the conical angle subtended at O by these elements, we may agree to make it *plus* for the inside aspects, P_2 and P_4, and *minus* for the outside aspects, P_1 and P_3; so that the sum of the conical angles subtended at O by these four elements of the given surface is zero.

For the purpose of projecting any element of area—as that at P_1—on any plane, we may adopt the convenient and consistent plan of drawing at the point the normal, $P_1 n_1$, *outwards* from the surface proportional in length to the element of area, marking its extremity with an arrowhead, thus treating it as

$* \quad \dfrac{dS_1 \times \cos n_1 P_1 Q}{(OP)^2}$

a *directed* magnitude, like a force, and taking its *component* along the normal to the plane as representing in magnitude and sign the projection of the element of area at P_1 along the plane in question.

Thus, the conical angle subtended at O by the element, dS_1, of area at P_1 is represented by the projection of Pn_1 along the line OQ which is the normal to the surface of the sphere of projection; this gives the conical angle $= dS_1 \times \cos n_1 P_1 Q$, $*$ which is negative. Similarly for the other points, P_2, P_3, P_4.

If dS is any element of area of a surface at a point P, and $d\omega$ is the conical angle subtended at any point O by this element, while ψ is the angle between OP and the outward-drawn normal at P, we have

$$dS = \frac{OP^2}{\cos\psi} \cdot d\omega. \qquad (a)$$

For, if a sphere is described through P having O for centre, the cone of rays which intercepts the area $d\omega$ square centimètres on the sphere of 1 cm. radius will intercept on this sphere an area of $OP^2 . d\omega$ square centimètres (if OP is measured in centimètres); and since this is the projection of dS on the surface of the sphere, we have the result (a).

The locus of the point O at which a given closed curve, or *circuit*, subtends a constant conical angle is a surface which contains the given curve as an edge—just as *in plano* the locus of a point O at which two fixed points, A, B, subtend a constant angle is a curve (circle) passing through A and B. The constant angle belonging to any one of a series of circles passing through A and B may be found by joining any point on the circle to A and B; but if the point selected on the circle is either A or B itself, an indeterminateness naturally arises, since the line joining B to itself is indeterminate. However, for any one circle if we take a point on the curve infinitely close to B, the direction of the line joining it to B is the tangent to the circle at B; so that the angle pertaining to that circle is the angle between AB and the tangent to the circle at B.

Similarly when the point O is taken on the given circuit, the conical angle subtended at it by the curve is naturally indeterminate; and to determine the angle pertaining to any one surface of the series of surfaces of constant conical angle having the given circuit for an edge, we must take a point, O', infinitely close to O *in the tangent plane to the particular surface.* The rays joining O' to the various points on the neighbouring part of the circuit form a semicircular fan of rays in the tangent plane, and they will intersect the sphere of unit radius described round O as centre in a semicircle; thus the

projection of the given circuit (which projection answers to the curve *abcde* in Fig. 273) on the unit sphere at O consists of a great semicircle and some irregular curve, U (suppose), completing this semicircle into a closed curve on the sphere; and the area of the sphere inside this closed curve is the conical angle belonging to the selected surface locus.

To find the angle at which two surfaces of constant conical angles, ω_1 and ω_2, cut each other at any point, O, on their common edge of intersection, describe the unit sphere round O as centre. Then we have just seen that the conical angle belonging to the surface ω_1 is the area of the sphere included by a closed curve on its surface consisting of a great semicircle S_1 and an irregular curve U; and the conical angle ω_2 belonging to the other surface is the area of the sphere included between a great semicircle S_2 (having the same diameter as S_1) and the same irregular curve U. Hence $\omega_1 \sim \omega_2$ is the area of the lune included between S_1 and S_2; but S_1 and S_2 lie in the tangent planes to the surfaces ω_1 and ω_2, respectively, so that the angle, θ, between them is the angle at which the two surfaces intersect; and the area of the lune $= 2\theta$ square centimètres, if the radius of the unit sphere is 1 cm.

Hence *two surface-loci of constant conical angles* ω_1, ω_2 *for a given circuit intersect at a constant angle at all points on this circuit, the angle between them being*

$$\tfrac{1}{2}(\omega_1 \sim \omega_2).$$

316, *a*. Line-Integrals and Surface-Integrals. The discussion of the Conical Angles subtended at various points in space by a given circuit depends on certain theorems of integration with reference to unclosed surfaces and their bounding edges, and as the whole subject is of much importance, particularly in the theory of Electromagnetism, it is considered advisable to devote special consideration to it here.

Let $\phi\,(x, y, z)$, which we shall briefly denote by ϕ, be any function of the co-ordinates of a point in space; then if any surface (closed or unclosed) be broken up into infinitesimal elements of area and the element, dS, of area at any point be multiplied by the value of ϕ which belongs to that point, the sum of all such products, viz.

$$\int \phi\, dS,$$

taken all over the surface, is called the *Surface-Integral* of ϕ over the given surface.

In the same way, if any curve (closed or unclosed) be taken in space, and if it is broken up into infinitesimal elements of length,

and the element of length, ds, at any point be multiplied by the value of ϕ which belongs to that point, the sum of all such products, viz.,

$$\int \phi\, ds,$$

taken all over the curve, is called the *Line-Integral* of ϕ over the given curve.

THEOREM 1. *If ϕ and ψ are any two functions of x, y, and if any closed plane curve be described in the plane xy, the double integral*

$$\iint \left(\frac{d\psi}{dx} - \frac{d\phi}{dy}\right) dx\, dy \tag{α}$$

taken over the area of the curve, is equal to the integral

$$\int (\phi\, dx + \psi\, dy) \tag{β}$$

taken along the perimeter of the curve in the sense in which the positive axis of x should rotate in order to coincide with the positive axis of y.

Let $rpq\ldots$, Fig. 275, represent the given curve. Take the term $\iint \frac{d\psi}{dx}\, dx\, dy$ first. Now to find this, we may first integrate with respect to x considering y constant.

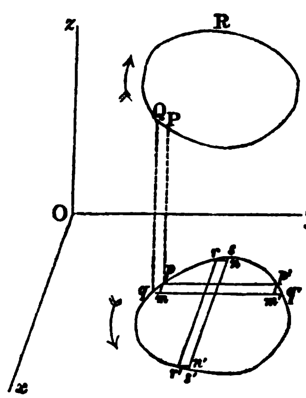

Fig. 275.

Let r be any point on the contour and rr' a line parallel to the axis of x; let s be a point on the curve infinitely near r, and ss' a line parallel to the axis of x; and let $r'n'$ and rn be perpendiculars on ss'.

Taking y, then, as constant, we are to sum $\frac{d\psi}{dx}\, dx\, dy$ over the narrow strip $srr's'$. This sum is

$$rn . \int \frac{d\psi}{dx}\, dy, \quad \text{or} \quad rn\,(\psi' - \psi), \tag{γ}$$

if ψ' and ψ are the values of ψ at r' and r, respectively.

Now observe that if we travel over the contour in the sense of the arrow, taking at each point the value of the product

$$\psi\, dy,$$

where, of course, dy is the algebraic increment of y in each

infinitesimal step, we should have in travelling from r' to s' the term

$$\psi' \times r'n',$$

and in travelling from s to r the term

$$\psi \times (-rn),$$

since the value of ψ at s may obviously be taken the same as at r. These two terms, therefore, give the sum (γ), so that the summation of ψdy over the contour will correctly give the result of the integration of the strip $rr's's$, and over all other similar strips.

In the same way, the term $-\iint \frac{d\phi}{dy} dx\,dy$ is to be found by integrating first with respect to y, considering x constant. Let, then, pp' and qq' be two indefinitely close parallels to the axis of y, enclosing a narrow strip. The summation is to be performed over this strip from p to p', so that if ϕ' and ϕ are the values of ϕ at p' and p, respectively, we have

$$dx\int\frac{d\phi}{dy}dy = pm\,(\phi' - \phi);$$

$$\therefore\quad -dx\int\frac{d\phi}{dy}dy = pm\,(\phi - \phi'); \qquad (\delta)$$

and in travelling over the contour in the sense of the arrow, while taking at each point the value of the product

$$\phi\, dx,$$

we should have at p the term $\phi \times pm$, and at q' the term $\phi' \times (-p'm')$, the sum of which is $pm\,(\phi - \phi')$, which is precisely (δ).

Hence, then, the area-integral (α) is equal to the contour-integral (β), which can, of course, be expressed in the form of the line-integral

$$\int\left(\phi\frac{dx}{ds} + \psi\frac{dy}{ds}\right)ds, \qquad (\epsilon)$$

where ds is the element of length at any point of the curve.

THEOREM 2. *If ϕ is any function of x, y, z, the co-ordinates of a point in space, and l, m, n the direction-cosines of the outward normal at any point of an unclosed surface, the integral*

$$\int\left(l\frac{d\phi}{dy} - m\frac{d\phi}{dx}\right)dS \qquad (\zeta)$$

taken over the surface, is equal to the integral

$$\int \phi\, dz \qquad (\eta)$$

taken along the bounding edge of the surface by a motion whose projection on the plane of xy is in the sense in which the positive axis of x should rotate in order to coincide with the positive axis of y.

It must be observed that $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$ are the partial differential coefficients of ϕ with respect to x and y, and that they take no account of any variation of z—belonging, as they are supposed to do, indifferently to all points in space, and not being restricted to the (related) points which lie on the given surface.

Suppose that the co-ordinates of points on the given surface are related by the equation

$$z = f(x, y),$$

or

$$dz = p\,dx + q\,dy, \tag{1}$$

as is usual, where p and q are functions of x and y only.

Then $l = \frac{-p}{\sqrt{1+p^2+q^2}}$, $m = \frac{-q}{\sqrt{1+p^2+q^2}}$: $n = \frac{1}{\sqrt{1+p^2+q^2}}$,

and $\quad dS = \sqrt{1+p^2+q^2}\,dx\,dy.$

Then the given integral (ζ) can be expressed in the form

$$\iint\left(q\frac{d\phi}{dx} - p\frac{d\phi}{dy}\right)dx\,dy, \tag{2}$$

which is a double integral over the area of the projection, $srpq\ldots$, of the given surface S on the plane xy.

Now, of course, a passage from point to point of the area of this projection will correspond to a motion from one point to another on the given surface S, and will necessarily involve a variation of z in both $\frac{d\phi}{dx}$ and $\frac{d\phi}{dy}$.

Denote by $\frac{\partial\phi}{\partial x}$ the total differential coefficient of ϕ with respect to x in the passage from one point on S to a neighbouring point when y remains constant but z is altered with x. Then

$$\frac{\partial\phi}{\partial x} = \frac{d\phi}{dx} + p\frac{d\phi}{dz}.$$

Similarly

$$\frac{\partial\phi}{\partial y} = \frac{d\phi}{dy} + q\frac{d\phi}{dz}.$$

Hence (2) becomes

$$\iint\left(q\frac{\partial\phi}{\partial x} - p\frac{\partial\phi}{\partial y}\right)dx\,dy. \tag{3}$$

Taking the terms of this double integral separately, we have first to integrate $q\dfrac{\partial\phi}{\partial x}$ with respect to x, considering y constant, i.e., to perform a summation along the strip rs'. Denote, as usual, $\dfrac{dq}{dx}$ by s.

Now $$\int q\frac{\partial\phi}{\partial x}dx = (q\phi)' - (q\phi) - \int s\phi dx,$$
where $(q\phi)'$ is the value of $q\phi$ at r', and $(q\phi)$ its value at r.

In a motion round the curve $srpq\ldots$ in the sense of the arrow, the term $[(q\phi)' - (q\phi)] \times r'n'$ is the same as the sum of the values of
$$q\phi . dy$$
at r' and r, as explained in Theorem 1. Hence
$$\iint q\frac{\partial\phi}{\partial x}dx\,dy = \int q\phi dy - \iint s\phi\,dx\,dy, \tag{4}$$
in which the single integral is one along the contour $srpq\ldots$.

Similarly $$-\iint p\frac{\partial\phi}{\partial y}dx\,dy = \int p\phi dx + \iint s\phi\,dx\,dy, \tag{5}$$
the single integral on the right side being taken round $srpq\ldots$ in the sense of the arrow. Hence (3) becomes simply
$$\int\phi(p\,dx + q\,dy). \tag{6}$$

But, if x, y are the co-ordinates of any point, p, on the curve $srpq\ldots$, the point P on the edge of S, of which p is the projection, will have the same x and y, and by (1) the increment of z in passing from P to Q (of which q is the projection) is the multiplier of ϕ in (6), so that (6) is the value of
$$\int\phi dz$$
in a motion round the bounding edge $PQR\ldots$, in the sense of the arrow, which was to be proved.

In the same way, of course, $\displaystyle\int\left(n\frac{d\phi}{dx} - l\frac{d\phi}{dz}\right)dS =$ the line-integral $\int\phi\,dy$ taken along the bounding edge.

We shall find it convenient to denote the operations
$$m\frac{d}{dz} - n\frac{d}{dy},\quad n\frac{d}{dx} - l\frac{d}{dz},\quad l\frac{d}{dy} - m\frac{d}{dx},$$
with regard to any surface the direction-cosines of whose normal are l, m, n, by the symbols
$$\partial_1, \partial_2, \partial_3.$$

THEOREM 3. *If u, v, w are any functions of x, y, z the co-ordinates of a point in space, and l, m, n the direction-cosines of the normal at any point on an unclosed surface, the integral*

$$\int\left\{l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dx}-\frac{du}{dy}\right)\right\}dS$$

taken over the surface, is equal to the integral

$$\int(u\,dx+v\,dy+w\,dz)$$

taken over the bounding edge of the surface by a motion which projects on the co-ordinate planes in the senses of positive rotation these planes.

This follows at once from the last Theorem. For, taking the term

$$\int\left(l\frac{dw}{dy}-m\frac{dw}{dx}\right)dS,$$

we have found that it is simply $\int w\,dz$ taken along the edge. Similarly

$$\int\left(m\frac{du}{dz}-n\frac{du}{dy}\right)dS=\int u\,dx,$$

taken along this edge; &c.

This is the result that *the line-integral of any vector taken along any circuit is equal to twice the surface-integral of the normal component of its 'rotation,' or 'curl,' taken over any surface having the given circuit for a bounding edge*, of which a different proof will be given in the Chapter on Strain and Stress.

Another discussion of this Theorem will be found in Clerk Maxwell's *Electricity and Magnetism*, vol. I., Art. 24.

Put into a quaternion form, this Theorem is thus expressed—

$$\int S\nu\nabla\rho\,.\,dS=\int S\tau\rho\,.\,ds, \tag{7}$$

where ν is a unit-vector in the direction of the outward-drawn normal at any point of an unclosed surface, τ is a unit-vector along the tangent at any point of the bounding edge of the surface, ρ is any vector, and ∇ is the Hamiltonian operator

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}.$$

The result in this Theorem gives the solution of the following inverse problem:—Given the components, U, V, W, of a vector, ρ, which satisfy at all points in space the equation

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=0,$$

to determine the components of another vector, σ, such that the surface-integral of the normal component of ρ over any unclosed surface shall be equal to the line-integral of the tangential component of σ taken along the bounding edge of the given surface.

For, in order to transform the given surface-integral into a line-integral along the edge, we must have

$$lU + mV + nW \equiv \partial_1 u + \partial_2 v + \partial_3 w,$$

that is
$$\frac{dw}{dy} - \frac{dv}{dz} = U; \; \frac{du}{dz} - \frac{dw}{dx} = V; \; \frac{dv}{dx} - \frac{du}{dy} = W. \qquad (8)$$

(Stokes's method of solving these will be found in Lamb's *Treatise on the Motion of Fluids*, Art. 129.)

316, *b*.] **Calculation of Conical Angles.** Let ω be the conical angle subtended by a given circuit, $PQR\ldots$, at any point A whose co-ordinates are α, β, γ. Then by (a), p. 239, if P is any point on any surface having the circuit for edge, and dS an element of area of this surface at P,

$$d\omega = \frac{1}{r^2}\cos\psi\, dS, \qquad (1)$$

where $r = AP$, $d\omega$ = conical angle subtended by dS at A, and ψ is the angle between AP and the normal to the surface at P.

Now if x, y, z are the co-ordinates of P, and l, m, n the direction-cosines of the normal,

$$\cos\psi = \frac{1}{r}\{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}. \qquad (2)$$

Hence
$$\omega = \int \frac{1}{r^3}\{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}\, dS. \qquad (3)$$

But since $r^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$, we have

$$\frac{x-\alpha}{r^3} = \frac{d}{d\alpha}\left(\frac{1}{r}\right),$$

with similar values of $\frac{y-\beta}{r^3}$ and $\frac{z-\gamma}{r^3}$. Hence (3) can be written

$$\omega = \int \left\{ l\frac{d}{d\alpha}\left(\frac{1}{r}\right) + m\frac{d}{d\beta}\left(\frac{1}{r}\right) + n\frac{d}{d\gamma}\left(\frac{1}{r}\right)\right\} dS. \qquad (4)$$

But since α, β, γ are completely independent of all co-ordinates on the surface S, and therefore have nothing to do with the

limits of integration, the symbols of differentiation with respect to them can be taken outside the integrals, and we have

$$\omega = \frac{d}{d\alpha}\int\frac{l}{r}dS + \frac{d}{d\beta}\int\frac{m}{r}dS + \frac{d}{d\gamma}\int\frac{n}{r}dS. \tag{5}$$

Differentiate both sides with respect to α, and observe that

$$\left(\frac{d^2}{d\alpha^2} + \frac{d^2}{d\beta^2} + \frac{d^2}{d\gamma^2}\right)\frac{1}{r} \equiv 0,$$

so that for $\frac{d^2}{d\alpha^2}$ we may write $-\left(\frac{d^2}{d\beta^2} + \frac{d^2}{d\gamma^2}\right)$. Then

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\left[\frac{d}{d\alpha}\int\frac{m}{r}dS - \frac{d}{d\beta}\int\frac{l}{r}dS\right] - \frac{d}{d\gamma}\left[\frac{d}{d\gamma}\int\frac{l}{r}dS - \frac{d}{d\alpha}\int\frac{n}{r}dS\right]. \tag{6}$$

Now obviously $\frac{d}{d\alpha}\left(\frac{1}{r}\right) = -\frac{d}{dx}\left(\frac{1}{r}\right)$; $\frac{d}{d\beta}\left(\frac{1}{r}\right) = -\frac{d}{dy}\left(\frac{1}{r}\right)$; &c.

Hence, first bringing the symbols of differentiation which are within the square brackets under the integral signs, (6) can be written

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\int\left(l\frac{d\frac{1}{r}}{dy} - m\frac{d\frac{1}{r}}{dx}\right)dS - \frac{d}{d\gamma}\int\left(n\frac{d\frac{1}{r}}{dx} - l\frac{d\frac{1}{r}}{dz}\right)dS. \tag{7}$$

Now, by Theorem 2 of last Article, the surface-integral on which $\frac{d}{d\beta}$ operates is $\int\frac{1}{r}dz$ taken along the given circuit, while that on which $\frac{d}{d\gamma}$ operates is $\int\frac{1}{r}dy$ taken along the circuit; so that

$$\frac{d\omega}{d\alpha} = \frac{d}{d\beta}\int\frac{dz}{r} - \frac{d}{d\gamma}\int\frac{dy}{r}, \tag{8}$$

similar values holding for $\frac{d\omega}{d\beta}$ and $\frac{d\omega}{d\gamma}$.

Denoting the line-integrals $\int\frac{dx}{r}$, $\int\frac{dy}{r}$, $\int\frac{dz}{r}$ along the given circuit by F, G, H, respectively, as in example 42, p. 64, we have for the differential coefficients of the conical angle subtended by the given circuit at any point (α, β, γ) the equations

$$\left.\begin{aligned} \frac{d\omega}{d\alpha} &= \frac{dH}{d\beta} - \frac{dG}{d\gamma}, \\ \frac{d\omega}{d\beta} &= \frac{dF}{d\gamma} - \frac{dH}{d\alpha}, \\ \frac{d\omega}{d\gamma} &= \frac{dG}{d\alpha} - \frac{dF}{d\beta}. \end{aligned}\right\} \tag{9}$$

It is evident that the conical angle subtended by a given circuit at any point (α, β, γ) satisfies the differential equation

$$\left(\frac{d^2}{d\alpha^2} + \frac{d^2}{d\beta^2} + \frac{d^2}{d\gamma^2}\right)\omega = 0, \text{ or } \nabla^2\omega = 0. \qquad (10)$$

Again, the quantities F, G, H which have reference to a given circuit and any point (α, β, γ) satisfy the equations

$$\frac{dF}{d\alpha} + \frac{dG}{d\beta} + \frac{dH}{d\gamma} = 0, \qquad (11)$$

$$\nabla^2 F = 0, \quad \nabla^2 G = 0, \quad \nabla^2 H = 0. \qquad (12)$$

For, the left-hand side of (11) is

$$-\int\left(\frac{d\frac{1}{r}}{dx}dx + \frac{d\frac{1}{r}}{dy}dy + \frac{d\frac{1}{r}}{dz}dz\right),$$

which, being taken along a closed curve, is zero. Hence if space were imagined to be filled with a fluid in motion, or a substance in a state of strain, its velocity components, or components of strain, at each point, A, being F, G, H, the cubical compression at every point would be zero, and the axis of resultant vortical spin at the point would be the direction in which the conical angle subtended by the circuit increases most rapidly—as will be understood after a study of the Chapter on Strain and Stress.

Another method of calculating the conical angle subtended at a point by a circuit is the following. Let $ABCDE$ (Fig. 273) be the circuit, and O the point at which the conical angle is subtended. Then if a is the radius of the sphere described round O, the conical angle is the area of the spherical curve *abcde* divided by a^2. Through O draw any line, Oz, meeting the surface of the sphere in z (not represented in the figure). For definiteness, suppose z to be within the part of the spherical surface which we regard as the area of *abcde*. Then the position of any point, p, within *abcde* may be expressed by its angular distance, θ', from z, and the angle, ϕ, which the plane pzO makes with any fixed plane through Oz. These angles are the co-latitude and the longitude of p. An element of spherical area at p is $a^2 \sin\theta'\, d\theta'\, d\phi$, so that the strip of area of *abcde* contained between two longitude planes including an angle $d\phi$ is

$$a^2\, d\phi \int_0^\theta \sin\theta'\, d\theta',$$

where θ is the colatitude of the point in which the arc zp intersects the contour of *abcde*.

Hence the conical angle is given by the equation

$$\omega = \int_0^{2\pi} (1-\cos\theta)\,d\phi \tag{13}$$

since ϕ runs from 0 to 2π round z.

It is, of course, quite indifferent which portion of the spherical surface (the upper or the lower) we regard as being the area of any curve traced on the sphere. If Oz is drawn so that z is in that portion of the surface which is regarded as outside the area, the longitude, ϕ, of a point within the area will not run from 0 to 2π, but from its initial value it will, after increasing and diminishing, return to this initial value, so that $\int d\phi = 0$. With an axis Oz so chosen, we should have

$$\omega = \int \cos\theta\, d\phi, \tag{14}$$

the upper and lower limits of ϕ being identical.

In the case of any *plane* circuit we obtain another expression for the conical angle subtended at any point in space. Taking the plane of the circuit as that of x, y, let (α, β, γ) be the co-ordinates of the point, A, at which the conical angle is required. At any point, P, in the area of the curve let the element of area be dS, and let $AP = r$. Then in (a), p. 239, we have

$$\cos\psi = \frac{\gamma}{r}, \quad \therefore\ d\omega = \frac{\gamma}{r^3}dS. \quad \text{But } \frac{\gamma}{r^3} = -\frac{d}{d\gamma}\left(\frac{1}{r}\right),$$

therefore

$$\omega = -\frac{d}{d\gamma}\int\frac{dS}{r}. \tag{15}$$

The method of calculating ω from this equation will be understood when we come to the treatment of Potential; and it will then be seen that (15) expresses the fact that the conical angle subtended at any point by a plane curve is the same (to a factor *près*) as the component of the attraction-intensity normal to the plane of the curve exerted at the point by a uniform plane lamina bounded by the curve.

Thus for a circular curve of radius a, if R is the distance of A from the centre,

$$\omega = -\frac{d}{d\gamma}\int_0^{2\pi}\int_0^a \frac{r\,d\phi\,dr}{\sqrt{R^2 - 2\gamma r\cos\phi + r^2}}, \tag{16}$$

which reduces to Elliptic Integrals.

EXAMPLES.

1. Find the conical angle subtended at any point on a sphere by a given circle lying on the sphere.

Ans. Let r be the angular radius of the given circle, a = angular distance of the point considered from the pole of the circle,

$$r+a=2\sigma,\ r-a=2\delta,\ k^2=\frac{\sin a\sin r}{\sin^2\sigma},\ \text{ and }\ n=\frac{\sin a\sin r}{\cos^2\sigma};$$

then
$$\omega=2\pi-\frac{2}{\sin\sigma}\left\{-\cos r\,.\,F(k)+\frac{\cos\delta}{\cos\sigma}\Pi(n,k)\right\},$$

where $F(k)$ is the complete elliptic integral of the first kind with modulus k, and $\Pi(n, k)$ the complete integral of the third kind with modulus k and parameter n.

The *complete* integral of the third kind is expressible in terms of integrals of the first and second kinds (Hymers's *Integral Calculus*, p. 290); thus

$$\frac{\cos\delta}{\sin\sigma\cos\sigma}\Pi(n,k)$$
$$=\frac{\pi}{2}+\left\{\frac{\cot\sigma}{\cos\delta}-E(k',\beta)\right\}F(k)-\{E(k)-F(k)\}F(k',\beta),$$

where
$$k'=\frac{\sin\delta}{\sin\sigma},\ \text{ and }\ \sin\beta=\frac{\cos\sigma}{\cos\delta}.$$

2. Show that the conical angle subtended at any point, A, by a circuit is the line-integral along the circuit of the tangential component of a vector whose magnitude at each point, P, of the circuit is

$$\frac{1}{r}\frac{\cos\theta\cos\lambda}{\sin^2\theta},$$

the vector being perpendicular to AP in the plane of AP and the tangent at P, $r = AP$, θ is the angle made by AP with a fixed line, and λ the angle made with this line by the normal to the plane of AP and the tangent at P.

3. Find the conical angle subtended at any point, P, in space by two intersecting right lines OA, OB, their extremities A and B being both at infinity.

Ans. If ϕ and ϕ' are the angles between the plane AOB and the planes containing P and the lines OA, OB, and $a = \angle AOB$

$$\omega=\pi-\phi-\phi'+\cos^{-1}(\sin\phi\sin\phi'\cos a-\cos\phi\cos\phi'). \qquad (1)$$

When $a = 0$, the plane from which ϕ and ϕ' are reckoned is indeterminate, but in this case $\phi+\phi'$ is π, so that ω is constant whatever be the position of P. When $a=\pi$, $\phi=\phi'$, and $\omega=2\pi-2\phi$, which is indeterminate and may be taken as 2ϕ simply, where ϕ is the longitude of P with reference to any fixed plane through the infinite line AOB.

If $t=\tan\phi$, $t'=\tan\phi'$, the equation of the surface locus of constant conical angle is

$$(t+t')\sin\omega+tt'(\cos\omega-\cos a)+2\sin^2\frac{\omega}{2}=0. \qquad (2)$$

To all points on the same right line OP, through O belongs the same value of ω; moreover, this equation shows that the planes determining any given angle ω can be represented in pairs by the points of an involution system.

The surfaces of constant conical angles are cones of the second degree whose equations are easily found from (2). For, if OA is taken as axis of x, and the plane AOB as that of xy, we find for the locus

$$2y(x\sin a - y\cos a)\sin^2\frac{\omega}{2} + z\left(x\sin a + 2y\sin^2\frac{a}{2}\right)\sin\omega + z^2(\cos\omega - \cos a) = 0; \quad (3)$$

or, taking the internal and external bisectors of the angle AOB as axes of x and y,

$$\sin^2\frac{\omega}{2}\left(x^2\sin^2\frac{a}{2} - y^2\cos^2\frac{a}{2}\right) + z^2\left(\sin^2\frac{a}{2} - \sin^2\frac{\omega}{2}\right) + xz\sin\frac{a}{2}\sin\omega = 0. \quad (4)$$

The conical angle is a measure of the Magnetic Potential at any point due to a current in the given circuit; hence the case $a = 0$ corresponds to a current returning on itself, which, of course, produces no effect at any point; while $a = \pi$ corresponds to a straight current of (practically) infinite length.

4. In the case of any plane triangular circuit whose angles are A, B, C, prove the following construction for points on the surface-locus of constant conical angle, ω:—

From any point, O, on a sphere draw arcs, OL, OM, ON, of three great circles including between them angles equal to $\pi - C$, $\pi - A$, $\pi - B$; then describe any spherical triangle, L, M, N, whose vertices lie on these arcs, such that the sum of its sides $= 2\pi - \omega$; the angles OL, OM, ON will be the inclinations to the plane of the triangle ABC of planes drawn through its sides BC, CA, AB intersecting in a point, P, at which the conical angle is ω.

Thus, then, to find the point on any given line, AP, drawn through A at which the triangle subtends ω, we take two given points M, N on two of the arcs and find the point, L, on the third such that $LM + LN$ is a given quantity. There are two solutions, since, given base and sum of sides of a spherical triangle, the locus of the vertex is a sphero-conic.

5. Show that the *complete* solution of equations (8), p. 246, from u, v, w will necessarily be indeterminate.

(To any values found for u, v, w may be added terms $\dfrac{dP}{dx}$, $\dfrac{dP}{dy}$, $\dfrac{dP}{dz}$, where P is any function of x, y, z which has a single (unambiguous) value for given values of x, y, z.)

6. For the conical angle subtended by a given plane circle at any point in space, show that the angles a, r, σ, δ in example 1 can be exhibited by a plane construction.

317.] **Attraction of a Thin Uniform Straight Bar.** Let the line AB (Fig. 276), represent a straight bar the area of whose transverse section is k square centimètres, this area being very small; let the mass of the bar be ρ grammes per cubic centimètre of substance; let P be the position of a mass of 1 gramme supposed to be condensed into an infinitely small volume. It is required to find the magnitude and direction of the attraction of the bar on the particle at P.

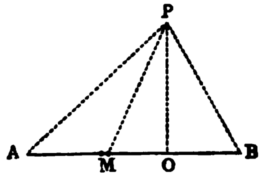

Fig. 276.

Draw PO perpendicular to AB; take any point, M, on AB; let $OM = s$, and consider the attraction on P of the elementary length ds at M. The mass at P being 1 gramme, and the mass of ds being $\rho \,.\, kds$, if γ is the constant of gravitation (Art. 315), the attraction of ds on P is, in dynes,

$$\gamma \frac{k\rho\, ds}{PM^2}. \tag{1}$$

This force acts in the line PM. Resolve it into a component along PO and a component perpendicular to PO. Let

$$\phi = \angle OPM, \quad \text{let} \quad PM = r, \quad \text{let} \quad PO = p,$$

and let these components be dY and dX, respectively.

Then
$$dX = \frac{\gamma k \rho}{r^2} \sin\phi \,.\, ds,$$

$$dY = \frac{\gamma k \rho}{r^2} \cos\phi \,.\, ds.$$

But $s = p\tan\phi$, $\therefore\ ds = p\sec^2\phi\, d\phi$, and $r = p\sec\phi$. Hence

$$dX = \frac{\gamma k \rho}{p} \sin\phi\, d\phi;$$

$$dY = \frac{\gamma k \rho}{p} \cos\phi\, d\phi.$$

Then, obviously, if $\angle OPA = \alpha$, and $\angle OPB = \beta$, and if X and Y are the total component attractions parallel and perpendicular to BA produced by all the elements of the bar, we have

$$X = \frac{\gamma k \rho}{p}\int_{-\beta}^{\alpha} \sin\phi\, d\phi = \frac{\gamma k \rho}{p}(\cos\beta - \cos\alpha), \tag{2}$$

$$Y = \frac{\gamma k \rho}{p}\int_{-\beta}^{\alpha} \cos\phi\, d\phi = \frac{\gamma k \rho}{p}(\sin\beta + \sin\alpha). \tag{3}$$

If the resultant attraction on P makes the angle ψ with PO, we have $\tan\psi = \frac{X}{Y} = \tan\frac{\alpha-\beta}{2}$, $\therefore \psi = \frac{\alpha-\beta}{2}$, which shows that *the resultant*, R, *bisects the vertical angle* APB. Also

$$R = \frac{2\gamma k\rho}{p}\sin\frac{APB}{2}\text{ (dynes).} \tag{4}$$

We may also notice the simple fact that *the attraction of the bar AB on P is the same in all respects as the attraction of a circular arc having P as centre with radius PO, this arc being terminated by the lines PA and PB*, the density and area of transverse section of this arc being the same as those of the given bar. For, if N is the point on AB distant ds from M, and if the lines PM and PN meet the circular arc in m and n, the attractions of MN and mn on P are exactly the same, because if from M a perpendicular MQ is drawn to PN, we have

$$MN = \frac{MQ}{\sin PMO} = \frac{MQ\,.\,PM}{PO} = \frac{mn\,.\,PM^2}{Pm\,.\,PO} = \frac{mn\,.\,PM^2}{Pm^2};$$

therefore $\frac{MN}{PM^2} = \frac{mn}{Pm^2}$, and the attractions of corresponding elements of the bar AB and the circular arc are the same.

The attraction of the particle P on the bar is R exactly reversed.

For an *infinitely long bar*, or one so long that the distances of P from its extremities are each very great compared with the distance, PO, of P from the bar, the attraction is

$$\frac{2\gamma k\rho}{p}, \tag{5}$$

since the angle APB is in this case π.

If the law of attraction be not that of the inverse square of distance, let the attraction of the element $k\rho ds$ at M on the unit mass at P be expressed by the law

$$k\rho ds \times \lambda f'(r),$$

where λ is a constant and $f'(r)$ any function of the distance PM.

Then, if $PA = r_2$ and $PB = r_1$, we easily find

$$X = \lambda k\rho[f(r_2) - f(r_1)], \tag{6}$$

$$Y = \lambda k\rho\int_{r_1}^{r_2}\frac{f'(r)\,dr}{\sqrt{r^2-p^2}}. \tag{7}$$

The expression (5) brings us back to the observation made at p. 235 with regard to the application of the law of inverse square to particles separated by an infinitely small distance; for it would appear from this expression that if $p = 0$, or the attracted particle is on the surface of the bar, the attraction is ∞: a result which is obviously absurd. The whole of our investigation depends on the assumption that every point in the element ds of length at M is at the same distance, r, from P. Now if P is in contact with the surface, the particles of the bar in the normal section at P are at all distances ranging from zero to the diameter of the bar from P, so that we cannot expect our result to hold for this case. In fact, k, the area of the normal section, ought in this case to be *infinitely* small, and then the expression (5) is indeterminate. To find what is really the intensity of attraction at a point on the surface of the bar, we must consider this latter as a cylinder of finite radius, a, and break it up into slender filaments in such a way that a filament to which P is infinitely close is one of infinitely small section. Such a mode of breaking up the bar is obtained by a polar resolution. Thus: draw the normal section through P; take any point Q in the area of this section, let $PQ = r$, and take the usual polar element, $r\,dr\,d\theta$, of area at Q. Consider now the attraction at P due to the filament of the bar, parallel to its axis, which stands on this element of area. It is clear that the filaments are now taken in such a way that when the distance of P from one of them vanishes, so does the transverse section of the filament.

For greater generality, let P be assumed outside the bar at a distance c from its centre, O; let the transverse section be circular and the length of the bar *practically* infinite, i.e. P is so close to the surface, that for each filament the angle APB may be taken as π.

The attraction of the filament at Q on a unit mass condensed at P is $\dfrac{2\gamma\rho . r\,d\theta\,dr}{r}$, or $2\gamma\rho\,d\theta\,dr$; and since PO is the direction of the resultant, we resolve this along PO; thus we have $2\gamma\rho\cos\theta\,d\theta\,dr$, where $\theta = \angle QPO$. Integrating this first with respect to r between the points at which the line PQ enters and leaves the circular section, we have

$$4\gamma\rho\sqrt{a^2 - c^2\sin^2\theta}\cos\theta\,d\theta,$$

as the contribution of the wedge of bars corresponding to the angle θ. The extreme values of θ are $\pm\sin^{-1}\frac{a}{c}$, so that a further integration gives

$$\frac{2\pi\gamma\rho a^2}{c}$$

for the attraction of a cylindrical bar at a point near its surface, the length of the bar being very great compared with its diameter. Now if the position of the attracted point is on the surface, $c = a$, and the attraction is

$$2\pi\gamma\rho a.$$

318.] **Uniform Thin Circular Plate.** Consider a circular plate of uniform density (ρ grammes per cubic centimètre of substance) and very small uniform thickness (τ centimètres); and let 1 gramme mass be condensed into a point P situated on the axis of the plate, i.e. a line drawn through O, the centre of the plate, perpendicular to the plane of the plate. It is required to find the attraction of the plate on the particle at P. Let a (centimètres) be the radius of the plate, and let $OP = z$ (centimètres). Take any point, Q, in the plane of the plate and describe a circle with centre O and radius OQ $(= r)$. Concentric with this describe another circle of radius $r+dr$, so that a narrow strip of area is included between these circles. We may in this way break up the plate into an infinitely great number of circular strips; the mass of the typical strip is $2\pi\rho\tau r\,dr$ grammes, and all points in the strip are at the same distance, PQ, or $\sqrt{z^2+r^2}$, from P. Also, if ϕ is the angle OPQ, since the resultant force exerted on P by the strip is obviously along PO, this resultant is

$$\gamma\cdot\frac{2\pi\rho\tau r\,dr}{z^2+r^2}\cos\phi, \quad\text{or}\quad 2\pi\gamma\rho\tau\sin\phi\, d\phi,$$

since $r = z\tan\phi$, γ being the constant of gravitation.

If a is the semiangle of the cone whose vertex is P and base the rim of the plate, ϕ ranges from 0 to a, so that

$$R = 2\pi\gamma\rho\tau\,(1-\cos a), \tag{1}$$

in dynes. This can be written $2\pi\gamma\rho\tau\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)$.

From this expression is deduced a result of great importance in Electrostatics. Suppose the attracted particle P to be very

close to the plate, *at the same time that the latter is infinitely thin compared with the distance of P*—this supposition being obviously necessary if we are to assume that all the particles in the body of the plate and included in a circular strip are equidistant from P. Then lines drawn from P to the rim of the plate are practically at right angles to OP, so that $a = \frac{\pi}{2}$, and

$$R = 2\pi\gamma\rho\tau \text{ (dynes)}, \tag{2}$$

and the result is independent of the radius of the plate. Thus, if P is at a distance of 1 millimètre from the centre of such a plate, the attraction on P is practically the same whether the radius of the plate is infinitely great or only 1 decimètre.

Again, the expression (1) shows that any two uniform plates of the same substance and of the same small thickness will exert equal forces on P if they are cut from the cone having P for vertex (their planes being parallel). Hence any two frustums of equal thickness, h, however great, cut from this cone will equally attract the particle P at its vertex, the magnitude of the force being

$$2\pi\gamma\rho h\,(1-\cos a).$$

The result holds also in the case of an oblique cone standing on any plane base whatever, the attracted particle P being at its vertex. To prove this geometrically we have merely to show that if two plates of the same thickness, each parallel to the base, be taken anywhere in the cone, they exert equal attractions on a particle at the vertex. Through the vertex P draw an infinite number of rays forming a very slender cone which intercepts on the two plates two small similar areas, dS and dS', at the points M and M', suppose. Then the attraction of dS on P is $\frac{\gamma\rho\tau dS}{PM^2}$, and that of dS' is $\frac{\gamma\rho\tau dS'}{PM'^2}$, these forces being coincident in the line PMM'. But since the contours of dS and dS' are similar curves, $\frac{dS}{dS'} = \frac{PM^2}{PM'^2}$; therefore these attractions are equal. Similarly for all other pairs of corresponding elements of the plates.

This put into the usual form of analysis would be as follows: Let (r, θ, ϕ) be the radius vector from P to M, the colatitude and longitude (Art. 175) of M. Then the element of volume at M may be taken as $r^2 \sin\theta\, dr\, d\theta\, d\phi$, and the attraction on P of the element of mass included is $\gamma\rho \sin\theta\, dr\, d\theta\, d\phi$, and this

depends only on dr and not on r (so long as θ and ϕ are constant); hence the attractions of the elements at M and M' above are equal, since these points have obviously the same θ and ϕ.

319.] **Uniform Spherical Shell.** Suppose a thin shell of attracting matter of uniform density, ρ grammes per cubic centimètre, to be contained between two very close concentric spheres. Then *this shell exercises no resultant attraction on any internal particle.* For, let P' be the position of any internal particle, and through P' draw a pencil of rays, $QP'Q'$, $RP'R'$, &c., forming a very slender cone; then if

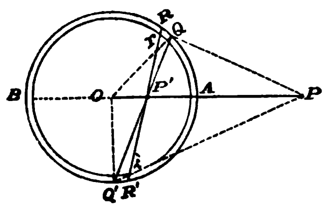

Fig. 277.

a ray RR' meet in the inner sphere in r and r', the lengths Rr and $R'r'$ are equal; hence the spherical surfaces at Q cut off a frustum whose thickness is equal to that of the frustum cut off at Q', and by Art. 318 the attractions of these frustums on the particle P' at their common vertex are equal and opposite. Hence the attractions of these elements of the shell destroy each other at P', and similarly all the vertically opposite elements of the shell cut off in the same way annul each others' effects at P'.

The resultant force on the particle is therefore zero.

Precisely the same result holds for an ellipsoidal shell bounded by two very close concentric and *similar* ellipsoids, since the intercepts Rr, $R'r'$ made by the shell on any line cutting its bounding surfaces are equal. This proof is given by Newton, Cor. 3, Prop. 91, Book I. These results we shall find useful in Electrostatics—in which occurs the general problem: Given the outer bounding surface of a shell, find the law of its thickness (or, in other words, find its inner bounding surface) so that its resultant attraction on every internal particle shall be zero. The simple result in the case of surfaces of the second degree, that the inner surface is one concentric with and similar to the outer, is due to the fact that they have diametral planes which bisect all parallel chords.

If the law of attraction is not that of the inverse square, let it be expressed by $\lambda \dfrac{f(r)}{r^2}$, and consider the narrow belt of the shell

which is generated by the revolution of the element of arc QR about OP'. Let $OP' = c$, $P'Q = r$, a = radius of shell; then the area of this strip $= 2\pi\frac{a}{c}r\,dr$; for in the usual notation it $= 2\pi y\,ds$, or $2\pi a^2 \sin\theta\,d\theta$, where $\theta = \angle QOA$, and

$$r^2 = a^2 - 2ac\cos\theta + c^2, \quad \therefore\ r\,dr = ac\sin\theta\,d\theta,$$

so that if dS = area of belt,

$$dS = 2\pi\frac{a}{c}r\,dr, \tag{A}$$

and the mass of this belt $= 2\pi\rho\tau\frac{a}{c}r\,dr$, where τ = thickness of shell.

Every particle of this strip is at the distance r from P', and its resultant attraction on P' (which is in the direction OP) is $2\pi\lambda\rho\tau\frac{a}{c}r\,dr.\frac{f(r)}{r^2}.\cos QP'P$, which is $\frac{\pi\lambda a\rho\tau}{c^2}.\frac{a^2-c^2-r^2}{r^2}f(r)\,dr$.

Hence, if R is the resultant attraction at P',

$$R = \frac{\pi\lambda a\rho\tau}{c^2}\int_{a-c}^{a+c}\frac{a^2-c^2-r^2}{r^2}f(r)\,dr. \tag{1}$$

When the law of attraction is that of the inverse square, $f(r)$ is constant, and the value of the integral is zero.

From this expression can be deduced a result which is fundamental in Electrostatics—viz. *the law of the inverse square is the only law of attraction for which a spherical shell of uniform thickness and density will produce no resultant attraction on any internal particle.*

For, whatever $a+c$ and $a-c$ may be, i.e. wherever P' is situated inside, the definite integral must vanish identically. Denote $a+c$ by m and $a-c$ by n. Then for all values of m and n,

$$\int_n^m \frac{mn-r^2}{r^2}f(r)\,dr = 0.$$

Differentiating this with regard to m and n, successively,

$$\frac{n-m}{m}f(m) + n\int_n^m \frac{f(r)}{r^2}\,dr = 0,$$

$$\frac{n-m}{n}f(n) + m\int_n^m \frac{f(r)}{r^2}\,dr = 0.$$

Hence $f(m) = f(n)$, whatever m and n may be; i.e. $f(r)$ must be absolutely constant, so that the law of attraction is that of the inverse square.

For a particle placed at any external point, P, the attraction (for the law of the inverse square) is the same as if the shell were condensed into a particle at its centre.

This may be shown in several ways. Thus, take the inverse of P with respect to the spherical surface; let this point be P', that is, $OP \times OP' = OQ^2 = a^2$. From this equation it follows that the triangles POQ and QOP' are similar, and therefore

$$\frac{QP}{QP'} = \frac{D}{a}, \tag{α}$$

where $D = OP$ and $a = OQ$; that is, *the ratio of the distances of all points on the sphere from P and P' is constant.* Again, from the similarity of these triangles $\angle QPO = \angle P'QO$; similarly, $\angle Q'PO = \angle P'Q'O$; *therefore the angle QPQ' is bisected by PO.*

Denote QP by r and QP' by r'. Let $d\omega =$ the conical angle subtended at P' by the elements of surface of the sphere cut off at Q and at Q' by a thin cone of rays $QP'Q'$, $RP'R'$,.... Then (Art. 316) the area of the element of surface at Q is $r'^2 \sec OQP'.d\omega$, and the attraction of the mass of this element on a unit (gramme) mass at P is $\gamma\rho\tau\frac{r'^2}{r^2}\sec OQP'd\omega$ acting in PQ (γ being the constant of gravitation). This, by (α), is $\gamma\rho\tau\frac{a^2}{D^2}\sec OQP'.d\omega$. The attraction on P produced by the element at Q' is similarly $\gamma\rho\tau\frac{a^2}{D^2}\sec OQ'P'.d\omega$, and these two expressions are identical, i.e. P is equally attracted by the corresponding elements at Q and Q'. The resultant of these forces acts in PO and is equal to

$$2\gamma\rho\tau\frac{a^2}{D^2}d\omega.$$

The sum of all such forces is obtained by summing $d\omega$ from 0 to 2π. Hence the resultant attraction

$$R = \gamma.\frac{4\pi\rho\tau a^2}{D^2} \tag{β}$$

$$= \gamma.\frac{\text{mass of shell}}{D^2},$$

which is exactly the same as if the shell were condensed into an infinitely small particle at its centre.

To deduce the result analytically, break up the shell, as before, into strips formed by the revolution of elements of length, $QR, \ldots$ about OP. The area of such an element $= 2\pi \dfrac{a}{D} r\,dr$, where $r = QP$; and the attraction of the element of mass contained within this ring on the unit (gramme) mass at P is

$$2\pi\gamma\rho\tau \frac{a}{D}\frac{dr}{r} \cdot \cos QPO, \text{ i.e. } \frac{\pi\gamma\rho\tau a}{D^2} \cdot \frac{r^2 + D^2 - a^2}{r^2} dr; \text{ therefore}$$

$$R = \frac{\pi\gamma\rho\tau a}{D^2} \int_{D-a}^{D+a} \frac{r^2 + D^2 - a^2}{r^2} dr$$
$$= \gamma \cdot \frac{4\pi\rho\tau a^2}{D^2}.$$

If the law is not that of the inverse square, but expressed by $\lambda \dfrac{f(r)}{r^2}$, we have

$$R = \frac{\pi\lambda\rho\tau a}{D^2} \int_{D-a}^{D+a} \frac{r^2 + D^2 - a^2}{r^2} f(r)\,dr, \tag{2}$$

the limiting values of r in these integrals being PA and PB, i.e. $D-a$ and $D+a$.

To determine the laws of attraction for which the attraction of a uniform spherical shell on any external particle is the same as if the shell were condensed into an infinitely small particle at its centre. We know from Art. 23 (vol. i.) that this is the case for any material body if the law of attraction be that of the direct distance; and we have just proved that for a uniform spherical shell the result holds for the Newtonian law. We shall now prove that these two are the only laws.

Denoting $D+a$ by m, and $D-a$ by n, and observing that if the shell may be condensed into a particle of mass $4\pi\rho\tau a^2$ at its centre, the value of R must be $4\pi\lambda\rho\tau a^2 \dfrac{f(D)}{D^2}$, we have from (2)

$$\int_n^m \frac{r^2 + mn}{r^2} f(r)\,dr = 2(m-n) f\left(\frac{m+n}{2}\right). \tag{3}$$

Dividing out by $m-n$ and differentiating with respect to m and n successively, we have

$$\frac{m+n}{m(m-n)} f(m) - \frac{1}{(m-n)^2} \int_n^m \frac{n^2 + r^2}{r^2} f(r)\,dr = f'\left(\frac{m+n}{2}\right),$$

$$-\frac{m+n}{n(m-n)} f(n) + \frac{1}{(m-n)^2} \int_n^m \frac{m^2 + r^2}{r^2} f(r)\,dr = f'\left(\frac{m+n}{2}\right);$$

therefore by subtraction,

$$(m^2-n^2)\left[\frac{f(m)}{m}+\frac{f(n)}{n}\right]=\int_n^m \frac{m^2+n^2+2r^2}{r^2}f(r)\,dr.$$

Differentiating again successively, and eliminating $\int_n^m \frac{f(r)}{r^2}dr$ from the two equations, we have simply

$$\frac{f'(m)}{m^2}=\frac{f'(n)}{n^2},$$

which must hold whatever m and n may be. Hence

$$f'(r)=Cr^2,$$

where C is a constant. If $C=0, f(r)$ is constant, and we have the law of inverse square, as before. If C is not zero,

$$f(r)=\tfrac{1}{3}Cr^3, \quad \therefore \frac{f(r)}{r^2}\propto r,$$

and we have the law of the direct distance. These, therefore, are the only laws for which the result holds.

320.] **Solid Homogeneous Sphere.** Instead of a spherical shell, let Fig. 277 represent a solid sphere, and consider its attraction on a unit mass condensed at P. Imagine the sphere broken up into an infinite number of infinitely thin concentric spherical shells. Then each of these attracts P as if it were condensed into a particle at O. Hence the whole sphere may be considered as condensed into a particle of mass $\frac{4}{3}\pi\rho a^3$ at O, and if $R=$ the resultant force on the unit mass at P,

$$R=\gamma\cdot\frac{4\pi\rho a^3}{3D^2}\cdot \qquad (\alpha)$$

If the attracted particle is inside the sphere, at P', all those shells which lie outside the sphere described with centre O and radius OP' may be rejected, since none of them produce any *resultant** effect on P'; so that the whole force

$$=\gamma\,\frac{\text{mass of sphere with radius } OP'}{OP'^2},$$

or

$$R=\gamma\cdot\tfrac{4}{3}\pi\rho D', \qquad (\beta)$$

where $D'=OP'$, i.e. *inside a homogeneous solid sphere the attraction varies as the distance of the attracted particle from the centre.*

* To be carefully distinguished from the *pressure* effect which is produced at all internal points, and which is very great at great depths. The whole surface of a particle may be subject to great intensity of pressure, while the *resultant* force on the particle may be zero.

Also any two solid homogeneous spheres attract each other as if each were condensed into a single particle at its centre. If, then, m and m' are their masses, and if c is the distance between their centres, the magnitude of their mutual attraction is

$$\gamma \cdot \frac{mm'}{c^2}. \qquad (\gamma)$$

321.] **Value of the Constant of Gravitation.** We are now in a position to find γ, the absolute constant of gravitation. Let the two attracting spheres be the earth (assumed homogeneous and spherical) and a small particle whose mass is 1 gramme. The following data* relating to the magnitude and density of the earth may be assumed as approximately correct: the radius of the earth is 637×10^6 centimètres; the mass of the earth is 614×10^{25} grammes (its mean density, ρ, being 5·67 grammes per cubic centimètre); the weight of 1 gramme mass at the surface of the earth is 981 dynes. Hence, putting $R = 981$, $\rho = 5{\cdot}67$, $D' = 637 \times 10^6$ in (β), or using the expression (γ) and making $m = 614 \times 10^{25}$, $m' = 1$, $c = 637 \times 10^6$, we find

$$\gamma = \frac{1 \text{ dyne}}{1543 \times 10^4}.$$

Now a dyne being, roughly, the weight of a milligramme, we see how extremely small a magnitude is the constant of gravitation, at least, in our region of Space; for it is conceivable that in enormously distant portions of the Universe the values of γ may be different.

322.] **Sudden Change of Attraction through a Shell.** Let P and Q (Fig. 278) be two points on the normal to any thin shell at opposite sides of the surface. Suppose a unit (gramme) mass condensed at P, and regard the attraction of the shell on this particle as produced by a small circular plate just below P, and the remainder of the surface. Consider, similarly, the attraction of the shell on a unit mass at Q. Now the attractions at P and Q produced by the portion of the shell obtained by omitting the small circular plate above-mentioned are sensibly the same in mag-

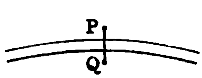

Fig. 278.

* See Everett's *Units and Physical Constants*, chap. vi.

nitude and line of action. Each of these attractions may be represented by $\bar{f}$, regarded as a vector.

But it has been shown (Art. 318) that the attraction of the small plate on the unit mass at P is

$$2\pi\gamma\rho\tau \text{ dynes,}$$

acting in the normal from P to Q; while the attraction of this plate on Q is this force exactly reversed in direction.

Denote this force on Q by the vector $\bar{n}$. Then the forces on P and Q are, vectorially,

$$\bar{f}-\bar{n} \text{ and } \bar{f}+\bar{n},$$

respectively.

If the shell is such that it exercises no resultant attraction at Q, $\bar{f}+\bar{n} = 0$, and the resultant attraction on P is normal to the surface and equal to $-2\bar{n}$, i.e. to

$$-4\pi\gamma\rho\tau,$$

considered as acting in the sense of the *outward-drawn* normal, QP. Numerically, of course, the force on P is $+4\pi\gamma\rho\tau$, acting in the sense, PQ, of the inward-drawn normal.

323.] **Force-intensity.** To save circumlocution, we shall define the *force-intensity* exerted by any attracting mass at any point P as *the force exerted by the given mass on a gramme mass condensed into a point at P.*

If instead of having 1 gramme mass at P, we have a particle whose mass is dm grammes, and if the given mass attracts it with a force of dF dynes, the force-intensity at P is

$$\frac{dF}{dm}.$$

Thus the force-intensity at P of the small circular plate in last Article is $2\pi\gamma\rho\tau$ (inwards), which will be in dynes if ρ is the density of the shell at P in grammes per cubic centimètre, τ is the thickness of the shell in cms., and γ is the constant of gravitation as defined in Art. 321.

324.] **Surface-integral of Normal Force-intensity.** If round any particle, dm, of matter attracting according to the law of the inverse square any closed surface whatever be described, the surface-integral of the normal force-intensity produced by the particle (the integration being taken over this surface) is equal to $4\pi\gamma . dm$; and if the surface is described so that the particle is outside it, the surface-integral is zero.

Begin with the latter case. Let O (Fig. 273) be the position of the attracting particle of mass dm grammes, and let the surface represented be any one whatever. Then the force-intensity at P_1 is $\gamma \frac{dm}{OP_1^2}$; the component of this along the outward normal is $\gamma \frac{dm}{OP_1^2} \cos OP_1 n_1$; and if dS_1 is the element of area of the surface at P_1, we have $\gamma \frac{dm}{OP_1^2} \cos OP_1 n_1 dS_1$ for the element of the surface-integral in question. But this is simply $\gamma dm \, . \, d\omega$, where $d\omega$ is the conical angle subtended at O by dS_1. Hence, if at any point on the given surface N is the magnitude of the normal component of the force-intensity and dS is the element of area, we have

$$\int NdS = \gamma dm \int d\omega$$
$$= 0, \qquad (1)$$

since, as explained in Art. 316, the total conical angle subtended at any external point by the elements of any closed surface is zero.

If O is internal to the surface, the whole of the investigation remains unaltered, but $\int d\omega$ is now 4π, so that for any internal particle, dm,

$$\int NdS = -4\pi\gamma dm. \qquad (2)$$

If instead of a single particle we have any number of them, all external to the given closed surface, and forming a mass which we may denote by M_e, we shall have (1) still holding, N being the normal component of the force-intensity due to the attraction of the whole mass M_e; and if inside the surface there is any mass M_i distributed in any way whatever, we have

$$\int NdS = -4\pi\gamma M_i, \qquad (3)$$

γ being the constant of gravitation, having in the C. G. S. system the value given in Art. 321.

If attracting matter can be spread as an infinitely thin layer *on* the surface, and the total mass thus spread be M_0, we should have

$$\int NdS = -2\pi\gamma M_0, \qquad (4)$$

N being the normal force-intensity at any point due to the action of M_0. This is obvious by Art. 316, since for any point on the surface $\int d\omega = 2\pi$.

The expression $\int NdS$ is sometimes described as the *normal flux of force through the surface outwards.* It is to be carefully

noted that N has been measured at all points on the surface along the outward-drawn normal. If at any point it really acts inwards, it is to be considered as *negative* at this point.

Many results in Electrostatics depend on the theorems expressed by (1), (3), (4). These theorems are due to Gauss, and are given in his famous paper on forces varying inversely as the square of the distance (Taylor's *Scientific Memoirs*, Vol. III, Part X).

325.] **General Components of Attraction.** Let there be any attracting body the matter of which attracts according to any law of the distance—suppose $\phi(r)$—and consider its attraction on a unit particle condensed into an infinitely small volume at any point, P, which may be either inside the attracting mass or in void space.

Let the co-ordinates of P referred to any fixed rectangular axes be (x, y, z); let P' be any point in the attracting mass, its co-ordinates being (x', y', z'); let ρ be the density of the matter at P', so that in a small parallelopiped cut out in the usual manner at P' the mass is $\rho\, dx'dy'dz'$; let r be the distance PP'. (We may, for definiteness, suppose these quantities to be measured in the units of the C. G. S. system.) Then the attraction of the element at P' on the condensed gramme at P is $\rho\phi(r)\, dx'dy'dz'$, acting in the sense $\overline{PP'}$, and the component of this parallel to the axis of x, in the positive sense of this axis, is

$$-\rho\phi(r) \cdot \frac{x-x'}{r}\, dx'dy'dz'.$$

Hence, if X, Y, Z denote the total components of the 'attraction-intensity' (see Art. 323) at P parallel to the axes, in their positive senses,

$$X = -\iiint \rho\phi(r)\frac{x-x'}{r}\, dx'\, dy'\, dz', \tag{1}$$

or simply,
$$= -\int \phi(r)\frac{x-x'}{r}\, dm, \tag{2}$$

with exactly similar values of Y and Z, the integrations being extended to all points, P', of the attracting mass, of which in the more succinct form (2) dm is the element of mass.

When P is within the attracting mass, the term $\frac{x-x'}{r}$ assumes the form $\frac{0}{0}$ for all the points P' in the immediate vicinity of P,

and though the distances of P from some *points* in the mass are zero, we must not conclude that the attraction is infinite, because, as we have pointed out at the very beginning (Art. 315), a law of attraction according to a function of the *distance* between two particles cannot be logically enunciated, or even conceived, except on the supposition that the dimensions of such particles are infinitely small compared with the (mean) distance between them.

As a matter of fact—and it is one of considerable importance to understand—the law of the inverse square leads to no such result as an infinite attraction when the position of the attracted particle is within the attracting mass; but the Cartesian expressions (1), (2) do not immediately show this. We shall show it by taking the elements, dm, of mass in polar co-ordinates.

Taking the position of the attracted particle P as pole, let (r, θ, ϕ) be the usual polar co-ordinates of P'. Then the element of mass at P' is $\rho r^2 \sin\theta\, dr\, d\theta\, d\phi$ (Art. 175), so that the attraction along $\overline{PP'}$ is $\rho r^2 \phi(r) \sin\theta\, dr\, d\theta\, d\phi$; hence

$$X = \iiint \rho r^2 \phi(r) \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \tag{3}$$

$$Y = \iiint \rho r^2 \phi(r) \sin^2\theta \sin\phi\, dr\, d\theta\, d\phi, \tag{4}$$

$$Z = \iiint \rho r^2 \phi(r) \sin\theta \cos\theta\, dr\, d\theta\, d\phi. \tag{5}$$

Now, for the law of Newton, $\phi(r) = \frac{\gamma}{r^2}$, so that

$$X = \gamma \iiint \rho \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi, \tag{6}$$

ith similar values of Y and Z; and even when $r = 0$, X contains no infinite term.

If, however, the attraction between two particles increased according to a law more rapid than the inverse square, the attraction-intensity at any internal point would be infinite.

For, if $\phi(r) = \frac{\mu}{r^3}$, we shall have the term $\frac{\rho}{r} \sin^2\theta \cos\phi\, dr\, d\theta\, d\phi$ in the value of X, and this becomes ∞ for the particles P' immediately in contact with P. This supposes the mass of P fixed and finite—1 gramme, suppose. But if the particle at P is itself of infinitely small mass, the infinite value of the attraction (no longer attraction-intensity) disappears.

As explained in the chapter on Centres of Mass, it is not necessary to take in all cases infinitesimal elements of the third order in breaking up the attracting mass. According to the

shape and law of density of the attracting body, we may take as elements, circular plates, thin bars, rings, &c., as will be illustrated in the following examples.

Examples.

1. Whatever may be the law of attraction, the force-intensity exerted by the smaller of two concentric solid homogeneous spheres at any point on the surface of the larger is to the force-intensity exerted by the larger at any point on the surface of the smaller in the ratio (radius of smaller)^2 : (radius of larger)^2.

Draw any radius OP meeting the surface of the larger in P and that of the smaller in p, O being the common centre. Draw a chord, ab, of the smaller parallel to OP; at a and b take equal and similar very small elements of area, each ds; draw lines from the various points of ds at a to the corresponding points of ds at b; we thus have a uniform bar of the substance of the smaller sphere lying along ab. Draw lines from O to all the points on the contour of ds at A; we thus get a slender cone; produce this cone outwards to intersect the surface of the outer sphere—at A, suppose—and let dS be the element of surface of the outer intercepted by this cone; draw similarly a cone with vertex O having ds at b for base, and let this intercept at B on the outer an element of area dS. Joining the points on the contour of dS at A to the corresponding points of dS at B, we have a bar, AB, of the substance of the larger sphere, also parallel to OP.

Now, if r and R are the radii of the smaller and larger spheres, it is obvious that $\frac{ds}{dS} = \frac{r^2}{R^2}$.

Consider the force-intensity at P due to the smaller, and at p due to the larger, sphere. Each acts in the line PO; hence to find the resultant force at P we may consider only the component attraction parallel to PO due to the bar ab and to all the other parallel bars into which the smaller sphere can be broken up. If the law of attraction is expressed by $\lambda f'(r)$, as in Art. 317, and if dX' is the intensity of attraction of the bar ab at P, we have by equation (6), Art. 317,

$$dX' = \lambda k \rho \,[f(Pa) - f(Pb)].$$

Similarly, if dX is the intensity of attraction at p due to the bar AB,

$$dX = \lambda K \rho \,[f(pA) - f(pB)],$$

k and K being the areas of the normal sections of the bars.

Now $Pa = pA$; $Pb = pB$; and $\frac{k}{K} = \frac{ds}{dS} = \frac{r^2}{R^2}$; therefore

$$\frac{dX'}{dX} = \frac{r^2}{R^2};$$

$$\therefore\quad X' = \frac{r^2}{R^2} X,$$

where X' and X are the resultant intensities of attraction at P and p due, respectively, to the smaller and larger spheres.

2. From the last result deduce a proof of the theorem that the only law of attraction for which a uniform spherical shell will exercise no resultant force at any internal point is the law of the inverse square. [This application is due to Duhamel.]

If a shell produces no attraction inside it, all the portion of the larger sphere between the two spheres may be neglected in finding the attraction of the larger at p. Hence, however great R may be, X is constant at p, so that $X' \propto \frac{1}{R^2}$, however small the inner sphere may be.

3. Calculate the intensity of attraction of a uniform thin rectangular plate at a point on the perpendicular to its plane drawn at its centre.

Let $2a$ and $2b$ be the lengths of its sides; h the height of the attracted particle, P, above O, the centre of the plate; ρ and τ the density and thickness of the plate. Break up the plate into bars parallel to the side $2a$; let y be the distance of one of these bars from O. Then the area of the normal section of this bar is τdy, and if the extremities of the bar are A and B and its middle point Q, we have for its attraction-intensity at P the expression (Art. 317)

$$2\frac{\gamma\rho\tau}{PQ}\sin APQ\,.\,dy.$$

Let $\theta = \angle QPO$; then $y = h\tan\theta$, $PQ = h\sec\theta$, and this expression becomes $2\gamma\rho\tau a\dfrac{\sec\theta d\theta}{\sqrt{a^2+h^2\sec^2\theta}}$; and since the resultant attraction is along PO, we multiply this expression by $\cos\theta$. Thus we have

$$R = 4\gamma\rho\tau a\int_0^{\alpha}\frac{\cos\theta d\theta}{\sqrt{h^2+a^2\cos^2\theta}},$$

where α is the extreme value of θ, i.e. $\tan^{-1}\frac{b}{h}$. Thus

$$R = 4\gamma\rho\tau\sin^{-1}\frac{ab}{\sqrt{(h^2+a^2)(h^2+b^2)}}.$$

If the plate is of infinite length ($a = \infty$),

$$R = 4\gamma\rho\tau\alpha.$$

4. Given the whole mass of a solid, find its shape so that its attraction, in any direction, on a particle placed at a given point may be a maximum. (*Solid of maximum attraction.*)

It is clear that the surface of the solid must pass through the given point, O. Let OA be the given direction, and let P and Q be any two points on the bounding surface of the solid. Consider an element of mass, dm at P, and an equal element at Q. Then, whatever be the law of attraction, the element dm at P and the element

dm at Q must give the same component attractions on O along OA; for if that of Q were the greater, advantage would be gained by transferring the element dm from P to Q.

Hence, if the law of attraction is expressed by $\phi(r)$, and if $\theta = \angle POA$, made with OA by the radius vector from O to any point on the bounding surface, we must have

$$\phi(r) . \cos\theta = \text{const.} \tag{1}$$

for all points on this surface. Hence the surface is one of revolution obtained by causing the curve (1) to revolve round OA. If $\phi(r) = \frac{\gamma}{r^2}$, the revolving curve is

$$\frac{\cos\theta}{r^2} = \frac{1}{a^2} = \text{const.} \tag{2}$$

Hence, if R is the resultant intensity of attraction along OA,

$$R = \gamma\rho \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^r \sin\theta \cos\theta \, dr \, d\phi \, d\theta$$

$$= 2\pi a \gamma\rho \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}}\theta \sin\theta \, d\theta$$

$$= \frac{4}{5} \pi a \gamma\rho.$$

The value of a must be found from the given mass of the solid, M; and we easily find $M = \frac{4}{15} \pi\rho a^3$;

$$\therefore \quad R = \left[\frac{48\pi^2\rho^2 M}{25}\right]^{\frac{1}{3}} . \gamma.$$

The attraction-intensity of a sphere of mass M at a point on its surface would be $\left[\frac{16\pi^2\rho^2 M}{9}\right]^{\frac{1}{3}} . \gamma$; so that the former exceeds the latter in the ratio $(27)^{\frac{1}{3}} : (25)^{\frac{1}{3}}$.

The curve (2) which generates the solid by revolution round OA may be thus drawn. Describe a circle with O as centre and OA as radius; describe another circle with OA as diameter; draw any line, OMN, meeting the second circle in M and the first in N; then take OP, a mean proportional between OM and ON, and we have a point P on the required curve.

5. To find the attraction-intensity of an infinite homogeneous elliptic cylinder at any external point situated on the major axis of a transverse section.

Let C be the centre of the ellipse which is the transverse section of the cylinder through the point O at which the intensity of attraction is to be found, O lying on the major axis of the ellipse at a distance ξ from C. Let P be any point on the circumference of the ellipse; with O as centre and OP $(= r)$ as radius describe a circular arc cutting the ellipse again in P'; take a point Q on

the ellipse indefinitely close to P, and with O as centre and OQ $(=r+dr)$ as radius describe another circular arc cutting the ellipse again in Q'. From all points on PP' and QQ' draw lines of infinite length perpendicular to the plane of the figure, and we shall have a thin cyliudrical plate of infinite length cut off from the given cylinder.

It is very easy to prove that the attraction of this plate on a unit mass at O, in the direction OC, is

$$4\gamma\rho \sin\theta \, dr,$$

where $\theta = \angle POC$, $\gamma =$ gravitation constant, $\rho =$ density of cylinder. (Consider this plate as formed of a number of bars.) Hence the attraction-intensity at O due to the whole cylinder is

$$4\gamma\rho\int \sin\theta \, dr.$$

But $\int \sin\theta \, dr = -\int r\cos\theta d\theta$, the other portion vanishing at both limits, since $\sin\theta = 0$ both at the beginning and end of the integration. Now if a and b are the semiaxes of the ellipse,

$$b^2(r\cos\theta - \xi)^2 + a^2r^2\sin^2\theta = a^2b^2;$$

$$\therefore \quad r = b\,\frac{b\xi\cos\theta \pm a\sqrt{b^2\cos^2\theta - (\xi^2 - a^2)\sin^2\theta}}{b^2\cos^2\theta + a^2\sin^2\theta}.$$

If we denote the values of r by r_2 and r_1, the integration will obviously contain the terms $-r_1\cos\theta d\theta$ and $r_2\cos\theta d\theta$, since after the radius vector OP passes the position of the tangent from O, the element $d\theta$ changes sign. Hence, if $-X$ is the intensity of attraction towards C,

$$X = -8\gamma\rho ab\int\frac{\sqrt{b^2\cos^2\theta - (\xi^2 - a^2)\sin^2\theta}}{b^2\cos^2\theta + a^2\sin^2\theta}\cos\theta d\theta,$$

the limits of θ being 0 and the value for which $r_1 = r_2$, i.e. $\tan^{-1}\dfrac{b}{\sqrt{\xi^2 - a^2}}$. Putting $\sqrt{\xi^2 - c^2}\sin\theta = b\sin\phi$, we have

$$X = -8\gamma\rho ab\sqrt{\xi^2 - c^2}\int_0^{\frac{\pi}{2}}\frac{\cos^2\phi \, d\phi}{\xi^2 - c^2\cos^2\phi}$$

$$= -4\pi\gamma\rho\,\frac{ab}{c^2}(\xi - \sqrt{\xi^2 - c^2}). \qquad (a)$$

When the cylinder is circular, the value of this expression is easily found to be $-2\pi\gamma\rho\,\dfrac{a^2}{\xi}$.

6. Draw a diagram representing the weight of a particle in its different positions as it is brought from the centre of the earth out through its surface and to infinity.

7. What should be the masses of two small equal homogeneous spheres so that when placed with a distance of 1 centimètre between their centres their mutual attraction shall be 1 dyne?

Ans. The mass of each must be $100\sqrt{1543}$, or 3928, grammes.

8. Prove that if there be two homogeneous solids of equal density bounded by similar surfaces, their attraction-intensities, for the law of inverse square, at two points similarly situated with respect to them are in the ratio of the corresponding linear dimensions of the solids. (Newton, Prop. 72, Cor. 3.)

Hence the attraction at any point on a given diameter inside a solid homogeneous ellipsoid varies as the distance of the point from the centre.

9. If the intensity of attraction of any body at a point is vastly greater when the point is very close to the surface of the body than when it is distant from this surface by a small interval, the attraction takes place according to a law more rapid than that of the inverse square. (Newton, Prop. 72.)

10. Find the intensity of attraction, for the law of inverse square, of any portion of a thin uniform spherical shell, cut off by a plane, at any point on its axis.

Ans. Let O be the centre of the sphere; OA the axis of the given segment, A being on the surface; AB the circular arc whose revolution round OA generates the given segment; P the position of the attracted particle on AO; $a=$ radius of sphere, $PO=c$, and β the angle PBO. Then the attraction is

$$\frac{2\pi a^2\rho\gamma\tau}{c^2}(1-\cos\beta).$$

If AB is a semicircle and P internal, $\beta=0$; if P is external, $\beta=\pi$.

11. If P coincides with O, find the attraction.

Ans. $\pi\rho\gamma\tau\sin^2 a$, where $a=\angle BOA$.

12. Find the intensity of attraction of a uniform right cone at the middle point of its base.

Ans. $2\pi\gamma\rho h\sin a\left[\sin a+\cos a-\sin a\cos a\left\{1+\log_e\cot\frac{a}{2}\cot\left(\frac{\pi}{4}-\frac{a}{2}\right)\right\}\right]$,

where h and a are the height and semivertical angle of the cone.

13. A platinum wire of uniform diameter 1 mm. and 1 mètre long attracts a gramme mass condensed into a point distant 1 cm. from the bar on a perpendicular to the bar at its middle point; find the magnitude of the force of attraction (specific gravity of platinum $=22{\cdot}06$).

Ans. $\dfrac{1}{89075\times 10^3}$ dynes, nearly.

14. If the law of attraction is expressed by any function, $\phi'(r)$, of the distance, prove that the intensity of attraction of any homogeneous solid, estimated in a given direction, at any point P is expressed by the surface-integral

$$\int\phi(r)\cos\lambda\, dS,$$

where r is the distance from P of any point on the surface bounding

the solid, dS is the element of surface area, and λ the angle made by the normal at this point with the given direction.

Take P as origin and the given direction as axis of x; at any point (x, y, z) in the mass let the element of volume $dxdydz$ be taken, and let the attraction of this element be $\phi'(r)\,dxdydz$. The component of this parallel to the axis of x is

$$\phi'(r)\frac{x}{r}\,dxdydz, \quad \text{or} \quad \phi'(r)\frac{dr}{dx}.dxdydz.$$

Integrating this, considering y and z constant, i.e. along a thin bar parallel to the axis of x, we have

$$[\phi(r_2)-\phi(r_1)]\,dydz,$$

where r_1 and r_2 are the distances from P of the points in which this bar cuts the bounding surface. Now

$$dydz = dS_2\,.\cos\lambda_2 = -dS_1\,.\cos\lambda_1,$$

the normal being at each point drawn outward; therefore, &c.

15. Calculate the attraction-intensity of a uniform elliptic plate at any point on the axis through its centre perpendicular to its plane.

Ans. If a, b are the semi-axes of the plate, $c=\sqrt{a^2-b^2}$, $z=$ distance of attracted particle from centre, $\tau=$ thickness of plate, $k^2=\dfrac{c^2}{a^2+z^2}$, $n=\dfrac{a^2}{a^2+z^2}$, the attraction-intensity is

$$\frac{4\gamma\rho\tau b\,.\,z}{a\sqrt{a^2+z^2}}\{\Pi(-n,k)-F(k)\}, \tag{1}$$

where $\Pi(-n, k)$ and $F(k)$ are the complete elliptic functions of the third and first kinds for the modulus k and parameter $-n$.

Again, this can be expressed entirely in terms of functions of the first and second kind, since the *complete* function of the third kind can be so expressed. Thus in general (Hymers's *Integral Calculus*, p. 290),

$$\Pi(-n,k)-F(k)=\frac{\Delta(k',\beta)}{k'^2\sin\beta\cos\beta}\{\frac{\pi}{2}-E(k',\beta)\cdot F(k) \\ -[E(k)-F(k)]F(k',\beta)\},$$

where $k'=\sqrt{1-k^2}$, and $\sin\beta=\dfrac{\sqrt{1-n}}{k'}$. Hence (1) becomes

$$4\gamma\rho\tau\{\frac{\pi}{2}-E(k',\beta).F(k)-[E(k)-F(k)]F(k',\beta)\}, \tag{2}$$

where $\sin\beta=\dfrac{z}{\sqrt{z^2+b^2}}$.

This obviously verifies for a circular plate.

SECTION II.—*Theory of Potential.*

326.] **Potential due to any Attracting Mass.** Consider an element, dm, of mass occupying any point, M, and let a unit mass condensed into an infinitely small volume be brought by any agent along any path whatever, plane or tortuous, from a position P_0 to a position P; it is required to calculate the amount of work done in this passage of the unit mass by the force exerted on it by the fixed particle dm. Suppose the law of attraction to be that of the inverse square, and at any point of the path of P let r be its distance from M. In this position let the force be $\frac{\gamma\, dm}{r^2}$. Then for any small displacement of P—say from P to P'—along its path the work done by the attracting force is $-\frac{\gamma\, dm}{r^2}dr$, where dr is $MP'-MP$. Hence the work done by the attraction from P_0 to P is $-\gamma\, dm\int_{r_0}^{r}\frac{dr}{r^2}$ (where $MP_0 = r_0$), i.e.

$$\left(\frac{1}{r}-\frac{1}{r_0}\right)\gamma\, dm. \tag{1}$$

If r and r_0 are measured in centimètres, dm in grammes, and if γ is the constant of gravitation (Art. 321), this expression for the work done is in *ergs*.

Now if the field of attraction is produced by several particles $dm, dm', dm'', \ldots$ at $M, M', M'', \ldots$ the sum of the works done by the attractions of all these on the unit mass in the passage of the latter from any initial position P_0 to any final one, P, is

$$\left(\frac{1}{r}-\frac{1}{r_0}\right)\gamma\, dm+\left(\frac{1}{r'}-\frac{1}{r_0'}\right)\gamma\, dm'+\left(\frac{1}{r''}-\frac{1}{r_0''}\right)\gamma\, dm''+\ldots, \tag{2}$$

$$\text{or }\ \gamma\left(\frac{dm}{r}+\frac{dm'}{r'}+\frac{dm''}{r''}+\ldots\right)-\gamma\left(\frac{dm}{r_0}+\frac{dm'}{r_0'}+\frac{dm''}{r_0''}+\ldots\right), \tag{3}$$

where $r, r', r'', \ldots$ are the distances of the final position P from the several particles, and $r_0, r_0', r_0'', \ldots$ the distances of the initial position from them.

If the initial position is infinitely distant from every attracting particle, $\frac{1}{r_0} = \frac{1}{r_0'} = \ldots = 0$, so that the work becomes

$$\gamma\left(\frac{dm}{r} + \frac{dm'}{r'} + \frac{dm''}{r''} + \ldots\right). \tag{4}$$

The amount of work done in bringing a particle of unit mass and infinitely small volume from any position in which the attractions exerted by the particles of any given system are zero (or insensible) to any point P in their field of attraction is called the **Potential** *of the field at that point.*

It will be seen that since the work done involves merely distances of P from the several particles, *it is wholly independent of the shape and length of the path along which P has been brought*; in other words, the attractions exerted by the several particles in the field are a system of conservative forces (Art. 272).

In the above formal definition of the Potential at any point produced by a given mass system, instead of saying that the unit particle is brought from *infinity* up to the final position P, we have said that it is to be brought from a position in which the attractive forces of the mass system are zero, although, *in general*, a position at infinity would satisfy this description. It will be shown soon, however, that there are cases in which the estimation of the work done on the unit particle *from infinity* up to the finite position P leads to infinite constants in the integration. If we define the Potential at P as the amount of work done in bringing the particle from infinity to this point, we must add the proviso that *when the particle is at infinity it is also infinitely distant from every attracting particle of the mass system*—i.e. that none of the attracting mass is contemplated as at infinity.

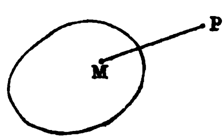

Fig. 279.

Throughout the sequel we shall speak of the position in which the forces of the field are insensible as the *zero position.*

Suppose now that the attracting particles form a continuous body of any shape represented in Fig. 279. Then the number of terms in (4) becomes infinitely great, and if we denote by V the Potential at P, we have

$$V = \gamma\int\frac{dm}{r}, \tag{a}$$

where dm is the element of mass at any point, M, and r is its distance from P. The integration is, of course, to be extended throughout the whole body, the position of P being fixed.

Thus to each position of P belongs a value, V, of the Potential. If P' is any other point at which the Potential is V', the work done by the attractions in transferring the unit particle *along any path whatever* from P' to P is

$$V - V',$$

since the particle might be brought from the zero position to P by passing through P' on the way.

It is to be remembered, then, that the expression (a) does not represent the work done in bringing a unit mass from infinity to P if any of the attracting matter is contemplated as being at infinity.

We might take $\gamma = 1$ by departing, to some extent, from the C. G. S. system, i.e. by taking the unit mass to be that which, condensed into a small sphere, attracts an equal spherical mass with a force of 1 dyne when the distance between the centres of the spheres is 1 centimètre; and this mass would be, by Ex. 7, p. 270, about 3928 grammes. We prefer, however, to adopt the C. G. S. system pure and simple and to retain γ, its value being that given in Art. 321.

It is to be observed that Potential is an *undirected* or *scalar* magnitude—unlike force, which has direction and is a *vector*. The Potential at P has magnitude but no direction.

Again, Potential is arithmetically additive; i.e. if V is the Potential at P due to any one mass system, and U the Potential at P due to any other mass system, the Potential at P due to their combined action is simply $V + U$.

327.] **Equipotential Surfaces.** The Potential produced at a point P by the attraction of any fixed masses may evidently be expressed as a function of the position of P, i.e. as a function of its co-ordinates, x, y, z, with reference to any fixed axes. If, then, $V = \phi(x, y, z)$, there must be a surface locus of points at each of which V has a given constant value, C; for the equation

$$\phi(x, y, z) = C$$

denotes a surface.

Let APB (Fig. 280) represent the surface at every point of which the Potential has the same value as that at P. [In the figure this surface is represented as closed; but, except for very

simple arrangements of attracting matter, the equipotential surfaces are very complicated, each consisting, perhaps, of several detached portions closed or unclosed.] Then no work, on the whole, is done in transferring a particle from any point P on this surface to any other point, A, on the same surface; the attractive forces of the field do as much positive work throughout a portion of any path connecting P with A as negative throughout the remainder.

If the particle is transferred from P to A *along any path lying on the equipotential surface*, then at no instant during the passage are the forces doing any work whatever; for no work is done in the passage from any point to the next consecutive.

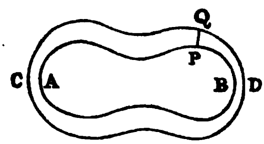

Fig. 280.

Hence the resultant attraction at any point on the surface acts along the normal to the surface at the point; for, every direction of displacement for which no work is done must be at right angles to the direction of the resultant force, and no work is done by the resultant attraction at P for any displacement of a particle at P in the tangent plane to the equipotential surface at this point.

An equipotential surface is often called a *level surface* (*surface de niveau*) from its analogy with a horizontal plane which is an equipotential surface for the case of gravity. (In reality, the equipotential surfaces for the earth's attraction are approximately spheres concentric with the earth, but a limited portion of one of them at any place may be considered a horizontal plane.) The horizontal plane is such that the work done by the weight of a particle in the descent of the particle, along any path, to the ground is the same from whatever point on the plane the particle falls; and, moreover, the particle, if placed on a smooth hard substance coinciding with this plane, would not move along it. All points on this plane have, therefore, the same Potential with reference to the earth's attraction, and are said to be at the same level. Hence the use of the term *level surface* in general, in any field of attraction, gravitational, electrostatic, or magnetic.

328.] **Relation between Force and Potential.** At any point, P (Fig. 280), construct the equipotential surface PAB; let PQ be an infinitesimal length measured on the normal at P;

and through Q describe another equipotential surface, QCD. Let V be the value of the Potential at P, and $V + \Delta V$ its value at Q. Now the resultant force at P acts along PQ, either inwards or outwards. Let it be R, and consider the work done in transferring a unit mass from P to Q. By definition this work $= \Delta V$, and if R acts from P to Q, it must also be $R \times PQ$, assuming that we may consider R as constant at all points between P and Q. Hence

$$R = \frac{\Delta V}{PQ}, \quad \text{or} = \frac{\Delta V}{\Delta n},$$

so that if ΔV is a positive increase of Potential, the sense of R is from P to Q. Similarly at B the magnitude of the force $= \dfrac{\Delta V}{BD}$, where BD is the normal distance between the two surfaces at B. Hence at different points on the same level surface the magnitude of the resultant force is inversely proportional to the normal distance between that surface and another level surface whose Potential exceeds that of the given one by an infinitesimal amount. An inspection of the figure (Fig. 280) shows the points at which the resultant force is most intense, and also those at which it is least; it is most intense where the two surfaces are closest together, and least where they are farthest apart. The value of R without approximation is to be found by diminishing PQ, or Δn, and therefore ΔV, indefinitely; i.e.

$$R = \frac{dV}{dn}, \tag{a}$$

which asserts that *at any point, P, the resultant force is the rate of increase of Potential along the normal to the level surface through the point, and it acts in the sense in which the Potential increases.*

Again, the component of force in any direction at any point, P, is the rate of variation of the Potential in that direction at P. For at P draw PP' in the given direction, meeting in P' the indefinitely close equipotential surface on which the Potential is $V + \Delta V$. Then if F is the component force along PP', and R the resultant force at P,

$$\begin{aligned} F &= R \cos QPP' \\ &= \frac{\Delta V}{\Delta n} \cos QPP' = \frac{\Delta V}{PP'}. \end{aligned}$$

Hence if $PP' = \Delta s$, and its length is diminished indefinitely,

$$F = \frac{dV}{ds}. \tag{β}$$

If ds lies anywhere in the tangent plane, the component force is zero; and the resultant force acts in the direction in which the Potential increases most rapidly.

Cor. The components of force at P parallel to three fixed rectangular axes are

$$\frac{dV}{dx}, \quad \frac{dV}{dy}, \quad \frac{dV}{dz}, \tag{γ}$$

(x, y, z) being the co-ordinates of P, and V being expressed in the form $V = \phi(x, y, z)$.

If V is expressed as a function of the polar co-ordinates (r, θ, ϕ) of P, with reference to any origin, O, and axes, the component force along the radius vector OP is

$$\frac{dV}{dr}; \tag{δ}$$

and the component along the tangent to the parallel of latitude at P is

$$\frac{1}{r \sin\theta} \cdot \frac{dV}{d\phi}, \tag{ϵ}$$

since PP' for this direction $= r \sin\theta . \Delta\phi$; while the component along the tangent to the meridian at P is

$$\frac{1}{r} \frac{dV}{d\theta}. \tag{ζ}$$

In general, V may be expressed in terms of any three independent variables which serve as co-ordinates to define the position of a point.

Starting with the notion of work, we have deduced the force-component in any direction from the Potential. In particular, we have proved that $X = \dfrac{dV}{dx}$. But we might have adopted the reverse process and shown that X is the differential coefficient with respect to x of a certain function of x, y, z.

Thus (Art. 325), if $\phi(r) = \dfrac{\gamma}{r^2}$, we have

$$X = -\gamma \int \frac{x - x'}{r^3} \, dm,$$

in which the integration has reference to x', y', z'; so that we can write this in the form

$$X = \gamma \int \frac{d\left(\frac{1}{r}\right)}{dx} . dm$$

$$= \frac{d}{dx}\left[\gamma \int \frac{dm}{r}\right] = \frac{dV}{dx},$$

if we denote $\gamma \int \frac{dm}{r}$ by V.

For any law of attraction, $\phi'(r)$, between elements of mass, the value of X is (Art. 325) equal to $-\int \phi'(r) \frac{x-x'}{r} dm$, or $-\int \frac{d\phi(r)}{dx} dm$, or $\frac{dV}{dx}$ if we denote $-\int \phi(r)\, dm$ by V.

Now $-\int \phi(r)\, dm$ is precisely the work done by the attraction on a unit mass from a zero position to the point P considered. For, the attraction exerted by dm at any distance being $\phi'(r)\, dm$, the element of work done by this for a small displacement of P is $-\phi'(r) dm . dr$, and the whole amount done from the zero position is $-dm \int \phi'(r) dr$, or $-\phi(r)\, dm$. Summing the works done by all the other elements of attracting mass, we have

$$V = -\int \phi(r) dm. \qquad (\eta)$$

The process, however, of deducing the idea and properties of Potential from the components of force is less in accordance with the methods of modern Physics than the reverse process, which we have here adopted.

It will be useful to the student to imagine the whole field of attraction, due to any arrangement of mass, as mapped out by a series of equipotential surfaces, the value of the Potential increasing from one surface to the next by a small constant amount.

329.] **Differential Equations of Potential.** At any point P describe the usual small rectangular parallelopiped whose edges are parallel to the axes of x, y, z. If in Fig. 228, p. 1 of this volume, we put P in place of O, and take the edges infinitely small, equal to dx, dy, dz, the parallelopiped there represented is such as we contemplate. Now take the surface-integral of normal force-intensity over this parallelopiped. The

outward normal force-intensity on the face $PBFC$ is $-X$ or $-\frac{dV}{dx}$; so that the contribution of this face is $-\frac{dV}{dx}dydz$; while the contribution of the opposite face is

$$\frac{dV}{dx}dydz+\frac{d}{dx}\left(\frac{dV}{dx}dydz\right).dx;$$

hence the sum contributed by these two faces is $\frac{d^2V}{dx^2}dxdydz$. Similarly the sum contributed by the two faces perpendicular to the axis of y is $\frac{d^2V}{dy^2}dxdydz$, and that contributed by the remaining faces is $\frac{d^2V}{dz^2}dxdydz$. The whole surface-integral for the elementary volume considered is therefore

$$\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dxdydz,$$

or $\nabla^2 V.dxdydz$, using the symbol

$$\nabla^2 \text{ for } \frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}.$$

Now if there is none of the attracting matter within the element of volume at P, this quantity must be zero, by Art. 324. Hence at every point in space at which none of the attracting matter exists

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0, \text{ or } \nabla^2V=0. \qquad (\alpha)$$

If, on the contrary, P is a point inside the attracting matter, and if ρ is the density, or mass per unit volume (cubic centimètre) at P, the mass contained in the parallelopiped is $\rho\,dxdydz$; so that by Art. 324,

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=-4\pi\gamma\rho, \text{ or } \nabla^2V=-4\pi\gamma\rho.... \qquad (\beta)$$

Equation (α) is known as *Laplace's Equation*, while (β) is *Poisson's Equation*.

We now proceed to find the equivalent equations in polar co-ordinates. To do this, we take the surface-integral of normal force-intensity over the polar element of volume $msqt$ (Fig. 219, p. 299, Vol. I). Let s in this figure represent the point, P, in

any field of attraction, and let the co-ordinates of s be (r, θ, ϕ), let the normal force-intensity on the face msq, measured in the sense Os, be R, while the area of this face $= s_1$. Then this face will contribute the term $-Rs_1$ to the surface-integral, while the opposite face will contribute $Rs_1 + \frac{d(Rs_1)}{dr}\,dr$; therefore these faces give conjointly $\frac{d(Rs_1)}{dr}\,dr$. Let the normal force-intensities on the faces mst and tsq be T and S, and the areas of these faces s_2 and s_3; then the first and its opposite face will conjointly give $\frac{d(Ts_2)}{d\theta}\,d\theta$; and the second with its opposite will give $\frac{d(Ss_3)}{d\phi}\,d\phi$. Hence

$$\frac{d(Rs_1)}{dr}dr + \frac{d(Ts_2)}{d\theta}d\theta + \frac{d(Ss_3)}{d\phi}d\phi = 0, \text{ or}$$
$$= -4\pi\gamma\rho r^2 \sin\theta\, dr\, d\theta\, d\phi,$$

according as there is not, or is, mass inside the element of volume.

Now $\quad R = \frac{dV}{dr},\quad T = \frac{1}{r}\frac{dV}{d\theta},\quad S = \frac{1}{r\sin\theta}\frac{dV}{d\phi};$

$$s_1 = r^2 \sin\theta\, d\theta\, d\phi,\quad s_2 = r\sin\theta\, dr\, d\phi,\quad s_3 = r\, d\theta\, dr,$$

so that the equations are

$$\frac{1}{r^2}\Big[\frac{d}{dr}\Big(r^2\frac{dV}{dr}\Big) + \frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{dV}{d\theta}\Big) + \frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2}\Big] = 0,$$
$$\text{or } -4\pi\gamma\rho; \quad (\gamma)$$

and it will be useful to note the identity (putting μ for $\sin\theta$)

$$\nabla^2 V \equiv \frac{1}{r^2}\Big[\frac{d}{dr}\Big(r^2\frac{dV}{dr}\Big) + \frac{d}{d\mu}\Big\{(1-\mu^2)\frac{dV}{d\mu}\Big\} + \frac{1}{1-\mu^2}\frac{d^2V}{d\phi^2}\Big]. \quad (\delta)$$

A result of importance may here be noted—namely, if the equation $\nabla^2 V = 0$ is satisfied by the value $V = r^n Y$, where Y is a function of θ and ϕ only, it will also be satisfied by the value $V = \frac{Y}{r^{n+1}}$; for, each of these values when substituted in (γ) gives the equation

$$\frac{d}{d\mu}\Big\{(1-\mu^2)\frac{dY}{d\mu}\Big\} + \frac{1}{1-\mu^2}\frac{d^2Y}{d\phi^2} + n(n+1)\,Y = 0.$$

Equation for V in Cylindrical Co-ordinates. The position of

any point, P, in space may be defined in the following manner by what are called *cylindrical co-ordinates*. Take any fixed rectangular co-ordinate axes, Ox, Oy, Oz; from P draw PM perpendicular to the plane of xy, meeting this plane in M. Then the cylindrical co-ordinates of P are the lengths PM and OM, and the angle MOx. Denote these, respectively, by (z, ζ, ϕ); then V at P must be expressible as a function of these. The corresponding small element of volume at P is obtained by drawing a cylinder passing through P having Oz for axis, and another cylinder very close to it (having for radius $\zeta + d\zeta$); a plane through P parallel to the plane xy, and another plane parallel to this at a distance dz from it; an 'azimuth plane,' PMO, containing P and Oz, and finally a close azimuth plane through Oz making the angle $d\phi$ with the previous azimuth plane. The volume of this element is $\zeta dz d\zeta d\phi$, and the areas s_1, s_2, s_3 of its faces through P are $s_1 = \zeta d\zeta d\phi$, $s_2 = \zeta dz\, d\phi$, $s_3 = dz\, d\zeta$; and the force-intensity perpendicular to the first and measured *outwards* from the surface of the element of volume is $-\frac{dV}{dz}$, so that this face gives $-s_1 \frac{dV}{dz}$, and its opposite gives

$$s_1 \frac{dV}{dz} + \frac{d}{dz}\left(s_1 \frac{dV}{dz}\right) \cdot dz$$

to the surface-integral. The sum of these is

$$\frac{d}{dz}\left(s_1 \frac{dV}{dz}\right) \cdot dz.$$

Similarly the other pairs of opposite faces contribute

$$\frac{d}{d\zeta}\left(s_2 \frac{dV}{d\zeta}\right) \cdot d\zeta \text{ and } \frac{d}{d\phi}\left(\frac{s_3}{\zeta} \frac{dV}{d\phi}\right) \cdot d\phi,$$

so that the whole surface-integral over this element of volume is

$$\left[\zeta \frac{d^2 V}{dz^2} + \frac{d}{d\zeta}\left(\zeta \frac{dV}{d\zeta}\right) + \frac{1}{\zeta} \frac{d^2 V}{d\phi^2}\right] dz\, d\zeta\, d\phi.$$

Hence the equations for V are

$$\frac{d^2 V}{dz^2} + \frac{1}{\zeta} \frac{d}{d\zeta}\left(\zeta \frac{dV}{d\zeta}\right) + \frac{1}{\zeta^2} \frac{d^2 V}{d\phi^2} = 0, \text{ or } = -4\pi\gamma\rho; \qquad (\epsilon)$$

and we have the identity

$$\nabla^2 V \equiv \frac{d^2 V}{dz^2} + \frac{d^2 V}{d\zeta^2} + \frac{1}{\zeta} \frac{dV}{d\zeta} + \frac{1}{\zeta^2} \frac{d^2 V}{d\phi^2}. \qquad (\zeta)$$

If the attracting matter is symmetrical, as to shape and density, about an axis (that of z, suppose), equations (ϵ) and (γ) become

$$\frac{d^2 V}{dz^2} + \frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right) = 0, \text{ or } = -4\pi\gamma\rho,$$

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\} = 0, \text{ or } = -4\pi\gamma\rho r^2,$$

and these are necessarily the same, and can be transformed one into the other by the relations $r = \sqrt{z^2+\zeta^2}$, $\theta = \tan^{-1}\frac{\zeta}{z}$, which give

$$\frac{d}{dz} = \cos\theta\frac{d}{dr} - \frac{\sin\theta}{r}\frac{d}{d\theta},$$

$$\frac{d}{d\zeta} = \sin\theta\frac{d}{dr} + \frac{\cos\theta}{r}\frac{d}{d\theta}.$$

330.] **Infinite Elliptic Cylinder.** In general, *to find the Potential at any point due to an infinite homogeneous cylinder whose transverse section is any plane curve symmetrical with respect to an axis, it is sufficient to know the value of the Potential at all points on this axis.* (Laplace, *Mécanique Céleste*, Vol. I, Book III, Chap. 6.)

For, if the axis of z is taken parallel to the axis of the cylinder, V will be a function of x and y only, and the equation for V will be

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} = 0.$$

The solution of this partial differential equation is

$$V = F(x+y\sqrt{-1}) + f(x-y\sqrt{-1}),$$

where F and f are two arbitrary functions.

Let the axis of x be taken coincident with the axis of symmetry of the transverse section; then the above value of V must be unaltered if for x and y we put x and $-y$, since V is obviously the same at the point $(x, -y)$ as at the point (x, y).

$$\therefore\ V = f(x+y\sqrt{-1}) + F(x-y\sqrt{-1}).$$

Hence

$$2V = (F+f)(x+y\sqrt{-1}) + (F+f)(x-y\sqrt{-1}),$$

$$= \phi(x+y\sqrt{-1}) + \phi(x-y\sqrt{-1}), \qquad (a)$$

so that at every point on the axis, if U is the Potential,

$$U = \phi(x),$$

and, by hypothesis, this is known, i.e. the form of the function ϕ is known. Then if in ϕ we put $x+y\sqrt{-1}$ and $x-y\sqrt{-1}$ for x successively and add the results, we get $2V$, by (α).

Similarly for the attraction-intensity. Its value at any point on the axis of symmetry of the transverse section is $\phi'(x)$, while if X and Y are its components at any point,

$$2X = \phi'(x+y\sqrt{-1}) + \phi'(x-y\sqrt{-1}), \qquad (\beta)$$

$$2Y = \sqrt{-1}\,[\phi'(x+y\sqrt{-1}) - \phi'(x-y\sqrt{-1})], \qquad (\gamma)$$

which are both known when ϕ' is known.

To apply this to the case of an infinite elliptic cylinder, the form of ϕ' has been already found (example 5, p. 269). Hence we have for the attraction-intensity at any point (x, y),

$$-2X = 4\pi\gamma\rho\frac{ab}{c^2}\left[x+y\sqrt{-1} - \sqrt{(x+y\sqrt{-1})^2 - c^2}\right]$$
$$+ 4\pi\gamma\rho\frac{ab}{c^2}\left[x-y\sqrt{-1} - \sqrt{(x-y\sqrt{-1})^2 - c^2}\right],$$

$$-2Y = 4\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\left[x+y\sqrt{-1} - \sqrt{(x+y\sqrt{-1})^2 - c^2}\right]$$
$$- 4\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\left[x-y\sqrt{-1} - \sqrt{(x-y\sqrt{-1})^2 - c^2}\right];$$

or

$$-X = 2\pi\gamma\rho\frac{ab}{c^2}\left[2x - \sqrt{x^2-y^2-c^2+2xy\sqrt{-1}}\right.$$
$$\left. - \sqrt{x^2-y^2-c^2-2xy\sqrt{-1}}\right],$$

$$-Y = 2\pi\gamma\rho\frac{ab}{c^2}\sqrt{-1}\left[2y\sqrt{-1} - \sqrt{x^2-y^2-c^2+2xy\sqrt{-1}}\right.$$
$$\left. + \sqrt{x^2-y^2-c^2-2xy\sqrt{-1}}\right].$$

These may be put into real forms by observing that if

$$\sqrt{A+B\sqrt{-1}} + \sqrt{A-B\sqrt{-1}} = u,$$

we have
$$u = \sqrt{2}\sqrt{A+\sqrt{A^2+B^2}}.$$

Hence

$$-X = 2\pi\gamma\rho\frac{ab}{c^2}\left[2x - \sqrt{2}\sqrt{x^2-y^2-c^2+\sqrt{(x^2-y^2-c^2)^2+4x^2y^2}}\right], \qquad (\delta)$$

$$-Y = 2\pi\gamma\rho\frac{ab}{c^2}\left[-2y + \sqrt{2}\sqrt{\sqrt{(x^2-y^2-c^2)^2+4x^2y^2} - (x^2-y^2-c^2)}\right], \qquad (\epsilon)$$

If the point (x, y) is on the surface of the cylinder, $x = a\cos\phi$, $y = b\sin\phi$, and

$$X = -4\pi\gamma\rho\frac{ab}{a+b}\cos\phi, \tag{ζ}$$

$$Y = -4\pi\gamma\rho\frac{ab}{a+b}\sin\phi, \tag{η}$$

so that the resultant is constant in magnitude, and it acts in a line parallel to the radius of the auxiliary circle of the ellipse.

331.] **Potential Work, or Static Energy, of a Self-Attracting System.** In a system in which forces of attraction are exerted between particle and particle, these forces will do an amount of (positive or negative) work if the form of the system is altered. We propose to find the amount of work thus done in a material system self-attracting according to the Newtonian law.

Consider a system of particles of masses $m_1, m_2, m_3, \ldots$ with distances $r_{12}, r_{13}, \ldots, r_{23}, \ldots$ between them in any given configuration, and with distances $r'_{12}, r'_{13}, \ldots, r'_{23}, \ldots$ between them in any final configuration.

First, let m_1 alone be brought into the second configuration, all the others being fixed. Then the amount of work done by the forces of attraction acting on it is

$$\gamma m_1\Big[\Big(\frac{1}{\rho_{12}} - \frac{1}{r_{12}}\Big)m_2 + \Big(\frac{1}{\rho_{13}} - \frac{1}{r_{13}}\Big)m_3 + \ldots\Big],$$

where $\rho_{12}, \rho_{13}, \ldots$ are the distances between m_1 and $m_2, m_3, \ldots$ after this change. Now let m_2 be brought into the final position, $m_3, m_4, \ldots$ being kept fixed. The amount of work thus done is

$$\gamma m_2\Big[\Big(\frac{1}{r'_{12}} - \frac{1}{\rho_{12}}\Big)m_1 + \Big(\frac{1}{\rho_{23}} - \frac{1}{r_{23}}\Big)m_3 + \ldots\Big].$$

Bringing m_3 now into the final position, $m_4, \ldots$ being fixed, the work is

$$\gamma m_3\Big[\Big(\frac{1}{r'_{13}} - \frac{1}{\rho_{13}}\Big)m_1 + \Big(\frac{1}{r'_{23}} - \frac{1}{\rho_{23}}\Big)m_2 + \ldots\Big].$$

Repeating this process for all the rest, and adding the works done, we have the whole work $\left(\text{multiplied by } \dfrac{1}{\gamma}\right)$,

$$= m_1 m_2\Big(\frac{1}{r'_{12}} - \frac{1}{r_{12}}\Big) + m_1 m_3\Big(\frac{1}{r'_{13}} - \frac{1}{r_{13}}\Big) + m_1 m_4\Big(\frac{1}{r'_{14}} - \frac{1}{r_{14}}\Big) + \ldots$$

$$+m_2 m_3\left(\frac{1}{r'_{23}}-\frac{1}{r_{23}}\right)+m_2 m_4\left(\frac{1}{r'_{24}}-\frac{1}{r_{24}}\right)+\ldots$$

$$+m_3 m_4\left(\frac{1}{r'_{34}}-\frac{1}{r_{34}}\right)+\ldots.$$

Now rearrange this by taking one-half of the first, second, third, ... terms in the first row and putting them, respectively, into the succeeding rows, and similarly treating the terms of the other rows. We thus find that the expression is the same as

$$\tfrac{1}{2} m_1\left[\frac{m_2}{r'_{12}}+\frac{m_3}{r'_{13}}+\ldots-\frac{m_2}{r_{12}}-\frac{m_3}{r_{13}}-\ldots\right]$$

$$+\tfrac{1}{2} m_2\left[\frac{m_1}{r'_{12}}+\frac{m_3}{r'_{23}}+\ldots-\frac{m_1}{r_{12}}-\frac{m_3}{r_{23}}-\ldots\right]+\&\text{c.},$$

or $$\tfrac{1}{2}(V_1'-V_1)m_1+\tfrac{1}{2}(V_2'-V_2)m_2+\tfrac{1}{2}(V_3'-V_3)m_3+\ldots, \qquad (\alpha)$$

all divided by γ, where V_1' is the value of the potential in the final position of m_1 and V_1 its value in the first position of m_1, with similar meanings of V_2', V_2, &c.

Or we may write the work in the form

$$\tfrac{1}{2}(\Sigma Vm)'-\tfrac{1}{2}(\Sigma Vm), \qquad (\beta)$$

where $(\Sigma Vm)'$ means the sum obtained by multiplying the mass of each particle of the system by the value of the potential at its position in the final configuration, and ΣVm the corresponding quantity in the first configuration.

If the particles are infinitely numerous and form a continuous mass, the work of the forces of attraction in changing the configuration is $$\tfrac{1}{2}\left(\int V dm\right)'-\tfrac{1}{2}\left(\int V dm\right). \qquad (\gamma)$$

Hence to scatter the particles of a given self-attracting system to (practically) infinite distances from each other requires an amount of work equal to $$\tfrac{1}{2}\int V dm, \qquad (\delta)$$

in which expression the integral is taken throughout the system in its given configuration. This expression (δ) may, therefore, be regarded as the Potential Work of the forces of the system, or its Static Energy, in this configuration.

Again, if V_1, V_2, ... are the potentials at the positions of a number of particles m_1', m_1', ... produced by a given system of particles m_1, m_2, ..., and if the system m_1', m_2',... (which we shall denote by M') either is not self-attractive or is absolutely

rigid, the work of removing the system M' completely out of the field of attraction of the other system (which we denote by M) is obviously

$$(m_1'V_1 + m'_2V_2 + m'_3V_3 + \ldots), \text{ or } \Sigma m'V;$$

or, again, $\int V dm'$, if the system M' forms a continuous mass.

But the work of removing the system M' out of the field of influence of M must be exactly the same as the work of removing M out of the field of influence of M'—since each is the work of separating the two attracting systems, each of which is considered as either rigid or not self-attractive.

But if V_1', V_2', ... be the values of the potential produced by the system M' at the positions of m_1, m_2, ... the expression for the work of removing M is

$$(m_1V_1' + m_2V_2' + \ldots) \text{ or } \Sigma mV',$$

or $\int V'dm$.

Hence we have a useful theorem due to Gauss, viz.

$$\int V dm' = \int V' dm. \qquad (\epsilon)$$

But this is also evidently true if the elements dm, dm', are multiplied by any function of the distance between them, as well as when this function is $\frac{1}{r}$; and, moreover, instead of two mass systems, M and M', we may have two volumes of empty space, so that if dm and dm' are elements of volume, equation (ϵ) still holds. The theorem in this case is of course not physical but merely analytical.

We shall find useful applications of this theorem of Gauss hereafter.

332.] **Magnetic Shell.** In the study of Magnetism we have to deal with a *magnetic shell*, which behaves like a material shell consisting of two layers indefinitely close together, each element of one of the layers—the outer, suppose—acting on a given material particle, placed anywhere, with a *repulsive* force following the Newtonian law, while each element of the other layer *attracts* the same particle according to the same law. Let Fig. 278, p. 262, represent such a shell, and suppose the points P and Q to be on the outer and inner layers, respectively. The outer layer we may imagine to be composed of *positive matter*, the amount of which per unit area is m at any point P; while at Q, the point directly opposite to P, on the inner shell we may imagine a quantity of *negative matter*, equal to $-m$ per unit area.

The inner shell is, then, wholly composed of *negative matter*, and the amounts of + and − matter, per unit area, are equal at the extremities of the (small) normal distance between the shells at all points. The terms 'positive' and 'negative' matter are, of course, only provisional; they stand merely for *causes of repulsion and attraction*. Again, the quantity m may vary from point to point on either shell. The product of m and the normal distance, Δn, between the shells at any point is called the *strength* of the shell at that point. Denote this product by ϕ; so that

$$\phi = m\Delta n.$$

We shall assume the shell to be of constant strength at all points; so that if the surface-density, m, of matter varies along either layer, the normal distance between the layers will also vary—but in such a way that ϕ remains constant.

For ordinary gravitating matter, whose constant of gravitation has the numerical value of γ previously given, such a combination of indefinitely close layers of repulsive and attractive matter would be almost absolutely nugatory—unproductive of anything but an infinitesimal force effect at any point—since, Δn being at all points infinitesimal, the product $m\Delta n$ would be infinitely small; but if a *very large* quantity of repulsive 'matter' could be concentrated on a small surface, the product $m\Delta n$ might not be infinitesimal, and the whole action of such a shell on a unit mass might amount to a very considerable force.

The discussion of the following properties of such a shell as we now imagine will not only serve to illustrate the subject of the present Chapter but prove a useful study for the student of the theory of Magnetism.

(*a*) *The potential produced by a magnetic shell at any point in space is proportional to the conical angle subtended at the point by the bounding edge of the shell.*

Let A be the point at which the value of the potential is to be found; let Q be any point on the inner surface, and P the opposite point on the outer surface, of the shell; let $AQ = r$, $AP = r + \Delta r$. Also let the constant of gravitation for the kind of matter now supposed be k—i.e., the number of dynes in the force of repulsion between two positive unit masses at a distance of 1 cm.—; suppose a unit mass placed at A; take any small element of area, dS, of the inner layer at Q, and on the contour of this erect a cylinder which will cut off an equal element of

area, dS, on the outer at P. The quantities of matter on these elements being, respectively, $-mdS$ and mdS, the sum of their potentials at A is

$$k\left(\frac{1}{r} - \frac{1}{r+\Delta r}\right) mdS, \text{ or } k\frac{m\Delta r}{r^2}dS. \tag{1}$$

Now if ψ is the angle made by AP with the normal to the shell at P, we have $\Delta r = \Delta n . \cos\psi$, so that this element of potential becomes

$$k\phi\frac{\cos\psi}{r^2}dS, \text{ or } k\phi . d\omega, \tag{2}$$

by Art. 316, where $d\omega$ is the conical angle subtended at A by the element dS of the surface of the shell. It is usual to assume the constant k equal to unity—which amounts to taking the unit mass as indicated near the end of Art. 326. On this understanding, then, if V is the potential of the shell at A, we have

$$V = \phi . \omega, \tag{3}$$

where ω is the conical angle subtended by the whole shell at A, i.e. the conical angle subtended by its bounding edge.

Hence if the bounding edge disappears—in other words, if the shell is a closed surface—it produces a zero potential, and therefore a null force effect, at all points outside it, and also a uniform potential, $4\pi\phi$, and null force effect, at all points inside it.

Hence also all magnetic shells of the same strength which have the same bounding edge produce the same effects at all points in space.

(*b*) *The potential produced by a magnetic shell at any point in space is proportional to the normal flux of force through the surface of the shell produced by a unit particle at the point.*

This follows at once from Art. 324.

(*c*) *If a magnetic shell is placed in any field of force which has a potential satisfying Laplace's equation, the whole action of the field on the shell can be produced by a distribution of force along its bounding edge only, according to a simple law.*

Let X, Y, Z be the components of the force-intensity of the field (forces exerted on the magnetic unit) at any point. Then we assume

$$X = \frac{dU}{dx}, \quad Y = \frac{dU}{dy}, \quad Z = \frac{dU}{dz},$$

where U is the potential of the field at the point. Hence

$$\frac{dX}{dy} = \frac{dY}{dx}, \text{ \&c.}$$

Calculate now the whole x-component of force exerted on the shell. On the quantity $-mdS$ at any point, Q, on the inner layer, the force is $-mXdS$. If l, m, n are the direction-cosines of the normal at Q, ν the thickness of the shell at Q, and x, y, z the co-ordinates of Q, the co-ordinates of P are $x + l\nu$, $y + m\nu$, $z + n\nu$; so that the value of X at P is

$$X + \nu\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)X.$$

Hence the resultant x-component on the corresponding elements at P and Q is

$$\phi\int\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)X.dS;$$

and the whole x-force on the shell is

$$\phi\int\left(l\frac{dX}{dx} + m\frac{dX}{dy} + n\frac{dX}{dz}\right)dS. \tag{4}$$

Now since $\nabla^2 U = 0$, we have

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0.$$

Substituting from this the value of $\frac{dX}{dx}$, and also putting $\frac{dY}{dx}$ for $\frac{dX}{dy}$ and $\frac{dZ}{dx}$ for $\frac{dX}{dz}$ in (4), we have (4) equal to

$$\phi\int\left\{\left(n\frac{d}{dx} - l\frac{d}{dz}\right)Z - \left(l\frac{d}{dy} - m\frac{d}{dx}\right)Y\right\}dS,$$

or $$\phi\int(\delta_2 Z - \delta_3 Y)\,dS. \tag{5}$$

But by Theorem 2, Art. 316, a, the first term in this integral is equal to the integral $\phi\int Z\frac{dy}{ds}\cdot ds$ taken along the bounding edge of the shell, while the second term is equal to the integral $-\phi\int Y\frac{dz}{ds}ds$ taken along this edge. Hence the whole x-component, F_x, of force on the shell is given by the equation

Similarly
$$\left.\begin{aligned} F_x &= \phi\int\left(Z\frac{dy}{ds} - Y\frac{dz}{ds}\right)ds. \\ F_y &= \phi\int\left(X\frac{dz}{ds} - Z\frac{dx}{ds}\right)ds, \\ F_z &= \phi\int\left(Y\frac{dx}{ds} - X\frac{dy}{ds}\right)ds, \end{aligned}\right\} \tag{6}$$

where F_y, F_z, are the components of force, parallel to the other axes, exerted by the field on the shell.

Now if R is the resultant force-intensity of the field at any point, P, of the bounding edge, and θ the angle between R and the tangent to the edge at P, the multipliers of ds in equations (6) are simply the x, y and z components of a force

$$R \sin \theta \tag{7}$$

acting along the line which is at once perpendicular to R and to the tangent to the edge at P. This force, R', may be graphically represented thus: at any point, P, on the edge of the shell draw a line representing in magnitude and direction the resultant force-intensity, R, of the field of force; draw also at P a unit length in the direction of the tangent to the edge at P, and complete the parallelogram determined by these two lines; then at P draw a perpendicular to the plane of this parallelogram proportional to its area; this perpendicular will represent the magnitude and direction of the force R' to be applied to the edge at P, per unit length. As to the sense in which the perpendicular to the plane is to be drawn, a watch-hand rule similar to that in Art. 200 may be adopted; or we may express the result by a quaternion notation thus: let a unit vector, τ, be drawn along the tangent at P to the edge in the sense in which a man walking on the positive side of the shell along the edge must travel so as to keep the shell at his left hand, and let R be the vector representing the resultant force-intensity at P; then

$$R' = V\tau R. \tag{8}$$

We have now to show that the system R' will produce the same moment about any axis as the force system (X, Y, Z).

To calculate the moment of the latter about the axis of x, let Q be a point on the inner layer and P on the outer, as before. Then the moment of force exerted on the element, $-mdS$, at Q is

$$-(Zy - Yz)\, mdS,$$

and therefore the resultant moment given by the masses $-mdS$ and mdS at Q and P is

$$\phi\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)(Zy - Yz)\,.\,dS.$$

But this is easily seen to be the same as

$$\phi\left\{\left(m\frac{d}{dz}-n\frac{d}{dy}\right)(yY+zZ)-\left(n\frac{d}{dx}-l\frac{d}{dz}\right)yX-\left(l\frac{d}{dy}-m\frac{d}{dx}\right)zX\right\}dS;$$

or $$\phi\left\{\delta_1(yY+zZ)-\delta_2(yX)-\delta_3(zX)\right\}dS,$$

with the notation of Theorem 2, Art. 316, *a*; and by this Theorem the result is the line-integral

$$\phi\int\left\{(yY+zZ)\frac{dx}{ds}-yX\frac{dy}{ds}-zX\frac{dz}{ds}\right\}ds,$$

taken along the edge of the shell. Hence if L denotes this moment,

$$L=\phi\int\left\{y\left(Y\frac{dx}{ds}-X\frac{dy}{ds}\right)-z\left(X\frac{dz}{ds}-Z\frac{dx}{ds}\right)\right\}ds. \qquad (9)$$

Now the coefficient of ds is exactly the moment of the force R' about the axis; therefore the system of edge-forces, R', is completely equivalent to the given forces acting on all the elements of the shell.

(*d*) To express the Static Energy of two magnetic shells occupying given positions.

Let their strengths be ϕ and ϕ'.

Take any point, Q', on the inner (supposed negative) surface of the second shell. The potential at this point due to the first is $\phi\omega$ by (3); and if P' is the point on the outer (positive) surface at the extremity of the normal at Q', the potential at P' is

$$\phi\omega+\phi\nu'\left(l'\frac{d}{dx'}+m'\frac{d}{dy'}+n'\frac{d}{dz'}\right)\omega,$$

where ν' is the thickness of the shell, l', m', n' are the direction-cosines of the normal, and $(x', y'\ z')$ the co-ordinates of Q'. Hence the potential work of the force of the first shell on the masses $-mdS'$ and mdS' at Q' and P' is

$$\phi\phi'\left(l'\frac{d}{dx'}+m'\frac{d}{dy'}+n'\frac{d}{dz'}\right)\omega\,.dS'.$$

Now by Art. 316, *b*, this is the same as

$$\phi\phi'\left\{l'\left(\frac{dH}{dy'}-\frac{dG}{dz'}\right)+m'\left(\frac{dF}{dz'}-\frac{dH}{dx'}\right)+n'\left(\frac{dG}{dx'}-\frac{dF}{dy'}\right)\right\}dS';$$

and the integral of this over the surface of the shell is by Theorem 3, p. 245, the line-integral

$$\phi\phi'\int\left(F\frac{dx'}{ds'}+G\frac{dy'}{ds'}+H\frac{dz'}{ds'}\right)ds',$$

taken along the edge of the shell. Substituting for F its value, $\int \frac{dx}{r}$, and similar values of G, H, the potential work of the forces of the first shell acting on the second is

$$\phi\phi' \iint \frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right) ds\, ds',$$

this double integral being taken over the edges of the two shells, (x, y, z) being the co-ordinates of any point, P, on the edge of the first, (x', y', z') those of any point, P', on the edge of the second, r being the distance PP', and ds, ds' elements of length of the edges at P and P'. If ϵ is the angle between the directions of ds and ds', and W stands for the Static Energy,

$$W = \phi\phi' \iint \frac{\cos\epsilon}{r}\, ds\, ds', \tag{10}$$

which is known as *Neumann's Formula.*

The Static Energy here expressed is merely the work which must be done against their mutual forces in withdrawing either shell, considered as a rigid body, to an infinite distance from the other. The result depends, then, merely on the shapes and positions of the *edges* and not at all on those of the *surfaces* of the shells.

(*e*) Static Energy of a Magnetic Shell and any Field of Force. Supposing the field of force to have at each point a potential, the static energy, in any position of the shell, is equal to the normal flux of force of the field through the shell, multiplied by the strength of the shell.

For, taking, as before, any points Q and P, at the extremities of the small normal thickness, on the negative and positive faces of the shell, if V is the potential of the field at Q, the potential work of the forces on the element $-mdS$ at Q is $-mVdS$, while for the element mdS at P it is

$$mVdS + m\nu dS\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)V,$$

where ν is the thickness of the shell at P. Hence if W is the whole potential work

$$W = \phi\int\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)dS, \tag{11}$$

which, since $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$ are the components of the force-inten-

sity of the field at P (or Q), is the normal flux of force-intensity of the field through the shell. When $\nabla^2 V = 0$, this can be expressed as a line-integral of the vector (u, v, w) along the edge of the shell by determining u, v, w as at the end of Art. 316, *a*.

Examples.

[Throughout these examples it may be assumed that length and mass are measured in centimètres and grammes, so that the constant of gravitation, $\gamma, = \dfrac{1 \text{ dyne}}{1543 \times 10^4}$; and V is in *ergs per gramme*.]

1. If the field of attraction is produced by two particles of masses m_1 and m_2 at two points N and S (Fig. 36, p. 46, vol. I.), and if r_1 and r_2 are the distances of any point P from them,

$$V = \gamma\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right),$$

γ being the gravitation constant (Art. 321).

Now m_1 and m_2 being both essentially positive, very large values of V will correspond to points P very near either N or S, while small values will correspond to points very distant from both, and zero values to points at infinity. The equipotential surfaces are evidently all surfaces of revolution round the line NS. If V is a very large constant, the equipotential surface will consist approximately of a sphere with centre N and radius $= \dfrac{\gamma m_1}{C}$ together with a sphere with centre S and radius $\dfrac{\gamma m_2}{C}$. As the values of V decrease, the equipotential surfaces are each formed by two oval shaped surfaces surrounding the points N and S; for a certain value of V these ovals join each other at a point between N and S, forming a surface generated by a kind of lemniscate revolving round NS; and for less values of V each surface becomes continuous, and is nearly a sphere for very distant points.

For Newtonian gravitation, however, if the masses m_1 and m_2 have moderate values—say a few grammes each—large values of V exist only at points infinitesimally distant from N or S. Thus if $m_1 = 1$ gramme and $m_2 = 2$ grammes, and if V is only 1 erg, the radius of the sphere round N is γ centimètres, i.e. $\dfrac{1}{1543 \times 10^4}$ cms., which is practically zero.

Unless the masses condensed at N and S are comparable with 1543×10^4 grammes, no sensible values of V (i.e. of the work of bringing 1 gramme mass from infinity into the neighbourhood of NS) will exist, except at infinitesimal distances from the points N and S.

This inconvenience does not exist in Electrostatics and Magnetism, because in these domains the analogues of the unit (gramme) mass

in Newtonian gravitation act upon each other at small distances with forces incomparably greater than that exerted by two condensed grammes at a distance of 1 cm.

If we suppose m_1 to exercise a repulsive force at P, while m_2 exerts an attractive force, we shall have

$$V=\gamma\left(-\frac{m_1}{r_1}+\frac{m_2}{r_2}\right),$$

and the surface of zero potential, instead of being wholly at infinity, is a sphere, with regard to which N and S are inverse points (p. 259).

The field produced by both particles together may be studied by superposing the fields produced by them separately. Thus the equipotential surfaces due to each are spheres. Describe round N the spheres for which the potential due to m_1 are C, $C+k$, $C+2k$... where k is any small potential magnitude; and round S the spheres for which V is C', $C'-k$, $C'-2k$,...; then the curves of intersection of these trace out the surface on which the potential is $C+C'$.

2. To calculate V at any point for a thin uniform bar (see Fig. 276, p. 252).

With the same notation as before,

$$V=\gamma k\rho\int\frac{ds}{PM}=-\gamma k\rho\int_{\pi-B}^{A}\frac{d\theta}{\sin\theta}=\gamma k\rho\log\left(\cot\frac{A}{2}\cot\frac{B}{2}\right). \quad (\alpha)$$

This may be put into another form. If $PA=r$, $PB=r'$, $AB=2c$,

$$V=\gamma k\rho\log\frac{r+r'+2c}{r+r'-2c},$$

$$\text{or}\quad V=\gamma k\rho\log\frac{a+c}{a-c}, \quad (\beta)$$

where $a=$ semi-axis major of the ellipse described through P with A and B for foci.

The equipotential surfaces are surfaces for which a is constant; they are therefore ellipsoids of revolution having the extremities A and B for foci.

If we assign to V a series of values, the corresponding values of a may be graphically represented. Equation (β) gives

$$\frac{a}{c}=\frac{e^{\frac{V}{2\gamma k\rho}}+e^{-\frac{V}{2\gamma k\rho}}}{e^{\frac{V}{2\gamma k\rho}}-e^{-\frac{V}{2\gamma k\rho}}}.$$

Draw a line Ox and represent a series of values of V by successive lengths measured along it from O. Construct a catenary whose equation is

$$y=\frac{\gamma k\rho}{2}\left(e^{\frac{V}{2\gamma k\rho}}+e^{-\frac{V}{2\gamma k\rho}}\right),$$

O being the origin and Ox the horizontal axis of this catenary.

Along the other axis draw a line parallel to Ox at a distance $c\,(=\frac{1}{2}$ length of bar); then the lengths intercepted on the successive tangents to the catenary between these two parallel lines are the semi-axes of the corresponding ellipses which generate the equipotential surfaces by revolving round AB.

From the value of V given in (β) we can deduce the value of the force-intensity at P. For, the resultant acts in the bisector of the angle APB, and if ds is an element of length of this line at P, $\dfrac{dV}{ds} = 2\gamma k\rho\dfrac{c}{a^2-c^2}\dfrac{da}{ds}$. Now if $\angle APB = 2\phi$, $AP = r$, we have $\dfrac{dr}{ds} = \cos\phi$. Also $r+r' = 2a$, and since at a point near P on the bisector of APB (tangent to a hyperbola confocal with the ellipse) $r-r'$ is constant, we have $dr = dr'$, therefore $\dfrac{da}{ds} = \cos\phi$, and

$$\frac{dV}{ds} = \frac{2\gamma k\rho c}{a^2-c^2}\cos\phi.$$

Again, $\cos\phi = \sqrt{\dfrac{a^2-c^2}{rr'}}$, by elementary trigonometry, and if p is the perpendicular from P on AB, we have $2cp = rr'\sin 2\phi$; therefore

$$\frac{dV}{ds} = \frac{2\gamma k\rho\sin\phi}{p},$$

which is the value already found (Art. 317).

A particular case must now be noted. If the bar is infinitely long, the expression (a) gives $V=\infty$, i.e. the sum $\int\dfrac{dm}{r}$ is really infinite in any given position of P. On the other hand, we can see that the work which would be done by the attraction of the bar in bringing the condensed unit mass from infinity up to the finite position P is *not* ∞. For if we imagine the bar to be a circle of immense diameter OO', the point O being near us and O' remote, and also that the unit mass is brought from O' up to P, it is quite clear that while P is moving from O' up to the centre of the circle, the attraction of the circle is doing *negative* work, the resultant force being all through this motion directed towards O'; and that when P leaves the centre and moves towards O, the attraction does *positive* work; so that the total amount done in the motion from O' to the final position P is numerically equal to that which would be done in bringing the unit simply from O to P—which obviously is far from being ∞.

But observe that, with some of the attracting mass contemplated as existing at infinity, we are no longer to regard the integral $\int\dfrac{dm}{r}$, in a given finite position P, extended over the body, as the work done by the attraction in bringing the unit mass from infinity to P.

That, in the case of an infinitely long bar, the amount of work done by the attraction in bringing the unit from a perpendicular distance q to a perpendicular distance p is simply

$$2\gamma k\rho \log_e \frac{q}{p}, \qquad (\gamma)$$

may be seen by taking the resultant force, R, at any distance, x, viz. $\frac{2\gamma k\rho}{x}$, and taking $-\int R dx$.

We must bear in mind that (γ) will not hold for positions of P very close to the surface of the bar, i.e. for very small values of p; because for such points the linear dimensions of the transverse section become comparable with the distances of P from the various points in the section—as has been already pointed out in Art. 317.

3. Without any consideration of force or of work done, show that the difference,

$$\left(\int\frac{dm}{r}\right)_P - \left(\int\frac{dm}{r}\right)_Q,$$

of the summations over an infinite bar with reference to any two finite positions P and Q is finite and, when multiplied by the gravitation constant, equal to the expression (γ).

Instead of finding each integral separately, perform the summation in a different order. Thus, M being any point on the bar, and $OM = s$ (Fig. 276), take at once the difference of effects at P and Q produced by the particle at M. This gives

$$\gamma k\rho\left(\frac{1}{\sqrt{p^2+s^2}} - \frac{1}{\sqrt{q^2+s^2}}\right)ds.$$

Integrating this from $s = -l$ to $s = +l$, we get

$$\gamma k\rho \log_e\left(\frac{\sqrt{p^2+l^2}+l}{\sqrt{q^2+l^2}+l}\cdot\frac{\sqrt{q^2+l^2}-l}{\sqrt{p^2+l^2}-l}\right),$$

which assumes an indeterminate form when $l = \infty$; but a simple binomial development of $\left(1+\frac{q^2}{l^2}\right)^{\frac{1}{2}}$ shows at once the true value to be (γ).

4. To find the potential at any point on the axis of a thin uniform circular plate.

With the notation of Art. 318, the potential at P due to the ring of radius r is $\gamma . \frac{2\pi\rho\tau r dr}{z\sec\phi}$, or $\gamma\frac{2\pi\rho\tau z\sin\phi}{\cos^2\phi}d\phi$; therefore the potential produced by the whole plate is

$$2\pi\gamma\rho\tau z(\sec a - 1),$$

or

$$2\pi\gamma\rho\tau(\sqrt{z^2+a^2}-z),$$

where a = radius of plate.

5. To find V for a uniform spherical shell.

Firstly, at an internal point, P' (Fig. 277, p. 257). Breaking up

the shell into elements QR, $Q'R'$ which are thin conical frustums, as in p. 259, if $d\omega$ is the conical angle subtended by either at P', the volume of the frustum is $\rho\tau \,.\, P'Q^2 \sec P'PO \,.\, d\omega$, or $\frac{2a\rho\tau \,.\, P'Q^2}{QQ'}\,d\omega$. The potential due to this at P' is $2\gamma a\rho\tau \frac{P'Q}{QQ'}d\omega$. Similarly the potential due to the frustum $Q'R'$ is $2\gamma a\rho\tau\frac{P'Q'}{QQ'}d\omega$; and the sum of these $= 2\gamma a\rho\tau d\omega$. Hence

$$V = 4\pi\gamma\rho\tau \,.\, a; \qquad (1)$$

which shows that V is constant wherever P' may be inside—a result for which we are already prepared, since everywhere inside the resultant attraction $= 0$, and this requires that V is constant.

Secondly, for an external point, P. This may be deduced from the value of V at the inverse point, P'. For, the element contributed by the frustum QR is $\gamma\frac{dm}{PQ}$, where $dm =$ mass of frustum. But $PQ = \frac{D}{a} \cdot P'Q$, therefore the element of potential $= \frac{\gamma a}{D} \cdot \frac{dm}{P'Q}$, which bears the constant ratio, $\frac{a}{D}$, to the potential of the element at P'. Hence the potential of the whole shell at P is $\frac{a}{D}$ times the potential at P', or

$$V = \frac{4\pi\gamma\rho\tau a^2}{D},$$

which is the same as if the shell were condensed into a particle at its centre.

The resultant attraction-intensity at $P = -\frac{dV}{dD}$ (measured towards O), which gives the same result as before.

These results can also be easily deduced analytically by breaking up the shell into zones, as has been done (p. 258) in calculating the force-intensity at P and P'. Thus, the mass of a zone being, as in p. 258, $2\pi\rho\tau\frac{a}{c}\,rdr$ where $r = P'Q$ or PQ, the potential produced by the zone is $2\pi\gamma\rho\tau\frac{a}{c}dr$; and for the internal point the limits of r are $a \pm c$, while for the external point they are $c \pm a$.

The value of V can also be deduced from the differential equation (γ), Art. 329. For V depends solely on the distance, r, of the internal point from the centre, and not on θ or ϕ. Hence (γ) reduces to

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0,$$

therefore $r^2\frac{dV}{dr} = C =$ constant. But at the centre the force-intensity, $\frac{dV}{dr}$, vanishes, $\therefore\ C = 0$, $\therefore\ \frac{dV}{dr} = 0$ everywhere inside,

$$\therefore\ V \text{ at any point} = V \text{ at centre} = \gamma \,.\, \frac{\text{mass of shell}}{a}.$$

It follows that for a homogeneous spherical shell contained between a sphere of radius a' and a sphere of radius a, the potential at any point inside the inner sphere (a') is $4\pi\gamma\rho\int_{a'}^{a} r\,dr$; i.e.

$$V = 2\pi\gamma\rho(a^2 - a'^2),$$

while at an external point at a distance D from the centre

$$V = \tfrac{4}{3}\pi\gamma\rho\frac{a^3 - a'^3}{D}.$$

At a point inside the matter of the shell, at a distance c from the centre

$$V = 2\pi\gamma\rho(a^2 - c^2) + \tfrac{4}{3}\pi\gamma\rho\frac{c^3 - a'^3}{D}.$$

6. To find V at any point for a solid homogeneous sphere. If the point is outside the sphere (radius a),

$$V = \tfrac{4}{3}\pi\gamma\rho\frac{a^3}{D}.$$

If it is inside, at a distance c from the centre, add the potential due to the shell contained between the surface of the given sphere and that of the sphere of radius OP', to the potential due to the solid sphere of radius OP'. Thus

$$\begin{aligned} V &= 2\pi\gamma\rho\,(a^2 - c^2) + \tfrac{4}{3}\,\pi\gamma\rho c^2 \\ &= 2\pi\gamma\rho a^2 - \tfrac{2}{3}\pi\gamma\rho c^2. \end{aligned}$$

V can also be obtained from the differential equation (γ), Art. 329. Thus, at any internal point this equation gives, since V is a function of r only,

$$\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dV}{dr}\Big) = -4\,\pi\gamma\rho\,;$$

$$\therefore\ \ r^2\frac{dV}{dr} = -\tfrac{4}{3}\,\pi\gamma\rho r^3 + C.$$

Now $\dfrac{dV}{dr} = 0$ at the centre, $\therefore\ C = 0$, and $V = -\tfrac{2}{3}\pi\gamma\rho\, r^2 + C'$.

But at the centre V is easily seen to be $2\pi\gamma\rho\, a^2$, $\therefore$ this $= C$.

7. To find V for an infinite homogeneous circular cylinder. If P is outside, $\Delta^2 V = 0$. Use cylindrical co-ordinates. Then V is simply a function of ζ (p. 282), so that

$$\frac{d^2 V}{d\zeta^2} + \frac{1}{\zeta}\frac{dV}{d\zeta} = 0. \qquad (1)$$

Therefore by integration

$$\frac{dV}{d\zeta} = \frac{C}{\zeta}.$$

To determine C, suppose P to be very distant from the cylinder, so that the latter may be treated as a thin bar. Then $\dfrac{dV}{d\zeta}$ is the force-intensity at P, which

$$= -\frac{2\gamma k\rho}{\zeta};\ \ \therefore\ C = -2\,\gamma k\rho = -2\,\pi\gamma\rho a^2,$$

if $a =$ radius of cylinder. Hence

$$\frac{dV}{d\zeta} = -\frac{2\pi\gamma\rho a^2}{\zeta}, \tag{2}$$

which shows that the intensity of attraction at any point outside the cylinder varies inversely as the distance from the axis. Integrating,

$$V = -2\pi\gamma\rho a^2 \log_e \zeta + C;$$

and to determine C, let the point P be supposed so far from the cylinder that the latter may be taken as a mere bar, or wire. Now in this case V is given in example (2), and since A and B are both zero, $V = \infty$, therefore $C = \infty$.

When none of the attracting matter is at infinity, V is, as has been explained, the work done in bringing a condensed unit mass from infinity to the position P, but it ceases to have this meaning when attracting matter is contemplated as existing at infinity. The summation $\int \frac{dm}{r}$ for an infinitely long bar is, in every position of P, really infinite. But if we are concerned only with the amount of work done in bringing the unit mass from one *finite* position, Q, to another, P, we can easily show that the difference

$$\left(\int \frac{dm}{r}\right)_P - \left(\int \frac{dm}{r}\right)_Q$$

is finite, notwithstanding that each integral itself is of infinite magnitude (see example 3).

Moreover, the supposition itself on which the equation for V is (1) falls to the ground; for it is only for points *finitely* distant from the cylinder that V depends simply on ζ.

Hence instead of choosing infinity as the zero position of P we must choose some other. We may choose a position on the surface of the cylinder, and define the potential at P as the work done by the attraction in conveying a gramme mass from P to the surface of the cylinder. With this definition, we have

$$V = 2\pi\gamma\rho a^2 \int_a^\zeta \frac{d\zeta}{\zeta}$$

$$= 2\pi\gamma\rho a^2 \log_e \frac{\zeta}{a},$$

and now $\frac{dV}{d\zeta}$ will be the force-intensity in the negative sense of ζ.

If P is inside the substance of the cylinder, (1) must, by Art. 329, be replaced by

$$\frac{1}{\zeta}\frac{d}{d\zeta}\left(\zeta\frac{dV}{d\zeta}\right) = -4\pi\gamma\rho;$$

$$\therefore \quad \zeta\frac{dV}{d\zeta} = -2\pi\gamma\rho\zeta^2 + C,$$

and since $\frac{dV}{d\zeta} = 0$ on the axis, $C = 0$, $\therefore \quad \frac{dV}{d\zeta} = -2\pi\gamma\rho\zeta$,

$$\therefore \quad V = -\pi\gamma\rho\zeta^2 + C' = \pi\gamma\rho(a^2 - \zeta^2).$$

8. To find the potential and attraction-intensity at any point produced by a spherical shell whose thickness (or density) varies inversely as the cube of the distance from a given point.

Let the thickness of the shell at any point Q (Fig. 281) vary inversely as the cube of the distance of Q from the given external point O, and let the potential at any internal point, P, be required.

Produce OP to S so that $OP \,.\, OS = OQ \,.\, OR =$ square of tangent from O. Imagine the spherical surface broken up into elements by cones described about S, each with very small conical angle, $d\omega$. One of these cones is represented by the line RSU, which must be understood to stand for a slender double cone of rays with vertex S. Joining O to all points on the contours of the small areas at R and U by lines represented by OR and OU, we obtain small elements of area at Q and T; joining the contours of these to S, we obtain two small frustums of cones at Q and T.

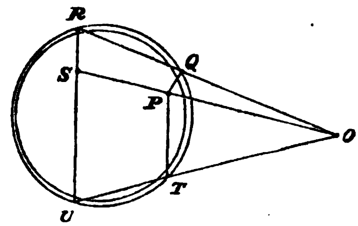

Fig. 281.

It is the sum of the potentials at P produced by these frustums that we shall take; and it is clear that the assemblage of such frustums as those at Q and T will exhaust the whole shell.

Let a be the radius of the sphere. Then (Art. 316) the cone of rays through S cuts off an area $\frac{2a}{RU} \cdot SR^2 . d\omega$ at R. Also, since the elements of area at R and Q are equally inclined to the line OQR, they are to each other in the ratio $OR^2 : OQ^2$; therefore the element of area at Q is

$$\frac{OQ^2}{OR^2} \cdot \frac{2a}{RU} \cdot SR^2 d\omega.$$

But $\frac{SR}{RO} = \frac{QP}{PO}$ (from the similar triangles ROS and POQ); therefore this element of area $= OQ^2 . \frac{PQ^2}{PO^2} \cdot \frac{2a}{RU} d\omega$. Let the thickness at Q be $\frac{k}{OQ^3}$, where k is a constant. Then the potential of this element at P is $2\gamma a k\rho \frac{PQ . d\omega}{RU . OQ . PO^2}$; and similarly the potential at P due to the element at T is $2\gamma a k\rho \frac{PT . d\omega}{RU . OT . PO^2}$; and the sum of these

$$= \frac{2\gamma a k\rho d\omega}{RU \,.\, PO^2} \cdot \frac{PQ . OT + PT . OQ}{OQ . OT} = \frac{2\gamma a k\rho d\omega}{RU . PO} \cdot \frac{QT}{OQ . OT}$$

(by Ptolemy's theorem for the quadrilateral $OQPT$, which is inscribable in a circle). But the triangles OQT and OUR are similar,

$$\therefore \quad \frac{QT}{OQ} = \frac{RU}{UO};$$

hence the above sum $= \frac{2\gamma ak\rho d\omega}{t^2 . OP}$, where $t^2 = OT . OU =$ square of tangent.

Now the whole shell is exhausted by summing $d\omega$ from 0 to 2π, and as the multiplier of $d\omega$ is constant, we have

$$V = \frac{4\pi\gamma ak\rho}{D^2 - a^2} \cdot \frac{1}{OP}, \tag{1}$$

where D is the distance of O from the centre. Hence the remarkable result that the potential at any internal point varies inversely as its distance from O.

For a reason to be given hereafter, we shall call O the *inducing point.*

The mass of the shell is easily found. For (p. 258) the area of the belt generated by the revolution of the element of length at Q about the line joining O to the centre is $2\pi \frac{a}{D} r dr$, where $r = QO$. Hence the mass of this $= \frac{2\pi ak\rho}{D} \frac{dr}{r^2}$, and if $M =$ mass of shell

$$M = \frac{4\pi k\rho a^2}{D(D^2 - a^2)}, \tag{2}$$

since the limits of r are $D \pm a$.

Hence from (1) and (2)

$$V = \gamma \frac{\frac{D}{a} \cdot M}{OP}, \tag{3}$$

which shows that *the potential at any internal point is the same as if a mass greater than that of the shell in the ratio $\frac{D}{a}$ were concentrated at the inducing point.*

Of course it follows that the attraction of the shell on a particle at P acts in the line PO, and is equal to

$$\gamma \frac{\frac{D}{a} \cdot M}{OP^2},$$

per unit mass at P.

The inducing point being still external, let the attracted particle be also external to the shell—at P', suppose.

Take the inverse point P, which will be internal. Then since $\frac{QP'}{QP}$ is constant, $= \frac{R}{a}$, where R is the distance of P' from the centre, it follows that V at $P' = V$ at P multiplied by $\frac{a}{R}$,

$$\therefore \; V = \gamma \frac{\frac{D}{R} \cdot M}{OP}.$$

But instead of OP we can put $\frac{D}{R} O'P$, where O' is the inverse of O, on account of similar triangles. Hence at any external point

$$V = \gamma \frac{M}{O'P}, \tag{4}$$

so that *for an external point the mass of the shell may be concentrated at the internal point which is the inverse of the inducing point, and the attraction is directed towards this inverse point.*

Finally, consider the case in which the inducing point is inside. This is at once reducible to the case in which the inducing point is outside, by taking the inverse point. Let O be the inducing point, and O' its inverse. Let the attracted particle, P, be inside, and let the thickness at any point, Q, of the shell be $\frac{k}{OQ^3}$; thus it will also be $\frac{kD^3}{a^3}\cdot\frac{1}{O'Q^3}$, where D is the distance of O' from the centre; so that the values of V and M are given by (1) and (2) in which we replace k by $\frac{kD^3}{a^3}$; and we have the result (3), viz.

$$V=\gamma\frac{\frac{D}{a}\cdot M}{O'P}, \tag{5}$$

which shows that the attraction is directed to O'.

Let P be external, while O is internal. Take the inverses of both, so that P' is internal and O' external.

If V' is the potential at P', we have by (3),

$$V'=\gamma\frac{\frac{D}{a}\cdot M}{O'P'}.$$

But if V is the potential at P, we have $\frac{V'}{V}=\frac{R}{a}$, where R is the distance of P from the centre; also, as we are finally concerned with P and not with P', we shall substitute OP for $O'P'$ by the equation $\frac{O'P'}{D}=\frac{OP}{R}$. Hence

$$V=\gamma\frac{M}{OP}. \tag{6}$$

Four different cases may therefore arise, viz. inducing and attracted point both on same side of surface, or on opposite sides; and summarising the results, we may say that the effect on the attracted particle is always the same as if a certain mass were condensed at a point on the opposite side of the surface; this mass is always *equal* to that of the shell when the attracted particle is outside, and always greater than that of the shell when the particle is inside. The point at which the shell may be condensed is always either the given inducing point or its inverse.

The solution of this question by the ordinary application of the Integral Calculus would be very much more difficult than the simple and elegant solution here given, which is due to Sir William Thomson. (See his *Papers on Electrostatics and Magnetism*, pp. 60, &c.; or Thomson and Tait's *Nat. Phil.*, vol. 1, part II.)

Newton also made use of the relation between inverse points in discussing the attraction of a sphere (see Book I of the *Principia*, Prop. 82).

9. To find the attraction of a thin circular plate of uniform thickness and density on a particle in its plane, the law of attraction being that of the inverse cube of the distance.

Let P (Fig. 282) be the position of the attracted particle, whose mass may be supposed to be one unit.

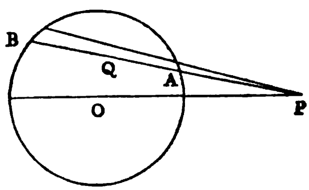

Fig. 282.

From P draw two very close radii vectores intercepting a narrow strip of the plate between them.

Let O be the centre of the plate, let θ be the angle OPA made by one of the radii vectores, and let $\theta+d\theta$ be the angle made by the other, with OP. Let Q be a point on PA, and $PQ=r$. Then the mass of the element at Q included between circles of radii r and $r+dr$ described with P as centre is

$$k\rho r\,dr\,d\theta,$$

k and ρ being the thickness and density of the plate.

The attraction of this element on P resolved along PO is

$$\gamma\frac{k\rho\,dr\,d\theta}{r^2}\cos\theta;$$

hence the resultant attraction is

$$\gamma k\rho\iint\frac{dr\,d\theta}{r^2}\cos\theta,$$

the integrations in r being performed from $r=PA$ to $r=PB$, and those in θ from $\theta=-\sin^{-1}\frac{a}{c}$ to $\theta=\sin^{-1}\frac{a}{c}$, where a is the radius of the plate and $c=OP$, the extreme values of θ corresponding to the two tangents that can be drawn from P to the circle.

Now denoting PA by r_1 and PB by r_2, and integrating first with respect to r, the attraction is

$$\gamma k\rho\int\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\cos\theta\,d\theta.$$

The values of r_1 and r_2 are given by the equation

$$r^2-2\,cr\cos\theta+c^2-a^2=0,$$

$$\therefore\ \frac{1}{r_1}-\frac{1}{r_2}=\frac{2\sqrt{a^2-c^2\sin^2\theta}}{c^2-a^2}.$$

Hence the attraction is

$$\frac{2k\rho\gamma}{c(c^2-a^2)}\int_{-a}^{a}\sqrt{a^2-t^2}\,dt,\ \text{or}\ \frac{\pi k\rho\gamma a^2}{c(c^2-a^2)},$$

where t is put for $c\sin\theta$.

In this case we might have found the attraction from the potential. The latter is easily found by dividing the plate into rings with O as centre. If r is the radius of one of these rings, we have

$$V=\frac{\gamma k\rho}{2}\iint\frac{r\,d\theta\,dr}{r^2-2cr\cos\theta+c^2}.$$

Integrating first from $\theta=0$ to $\theta=\pi$, and doubling the result, we have

$$V=\pi k\rho\gamma\int\frac{r\,dr}{c^2-r^2},$$

in which r runs from 0 to a. Hence

$$V=\frac{\pi k\rho\gamma}{2}\log\frac{c^2}{c^2-a^2}.$$

But V may also be easily found from the attraction, thus:

$$\frac{dV}{dc}=-\frac{\pi k\rho\gamma a^2}{c(c^2-a^2)},$$

$$\therefore\quad V=\frac{\pi k\rho\gamma}{2}\log\frac{c^2}{c^2-a^2}+\text{const.}$$

Now, since $V=\dfrac{\gamma}{2}\displaystyle\int\frac{dm}{r^2}$, it is clear that at infinity $V=0$, or $V=0$ when $c=\infty$. This gives the const. $=0$,

$$\therefore\quad V=\frac{\pi k\rho\gamma}{2}\log\frac{c^2}{c^2-a^2}.$$

10. If V_n and V_{n-2} denote the potentials of an attracting mass when the law of attraction is the n^{th} and $(n-2)^{\text{th}}$ power of the distance, respectively, prove that

$$V_{n-2}=\frac{\nabla^2 V_n}{(n-1)(n+2)},$$

where $\nabla^2\equiv\dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}+\dfrac{d^2}{dz^2}$, the co-ordinates of the attracted particle being x, y, z.

We have
$$V_n=-\frac{\lambda}{n+1}\int r^{n+1}dm,$$

where λ is a constant. Therefore

$$\frac{dV_n}{dx}=-\lambda\int(x-x')r^{n-1}dm,$$

and
$$\frac{d^2V_n}{dx^2}=-\lambda\int\{r^{n-1}+(n-1)(x-x')^2r^{n-3}\}\,dm.$$

Adding to this the similar values of $\dfrac{d^2V}{dy^2}$ and $\dfrac{d^2V}{dz^2}$, we have

$$\nabla^2V_n=(n-1)(n+2)V_{n-2}.$$

This equation enables us, generally, to find the potential for the $(n-2)^{\text{th}}$ power of the distance when that for the n^{th} is known; but it fails in two most important cases, namely, when $n=1$ and when $n=-2$.

If the attracting mass is a plate, $r^2 = (x-x')^2+(y-y')^2$, and the result is easily proved to be

$$\nabla^2 V_n = (n^2-1)\,V_{n-2}.$$

In the last example we find the Potential of a circular plate for the inverse third power; hence we have at once the Potentials, and therefore the attractions for the inverse fifth, seventh, &c., powers of the distance.

11. Calculate the attraction of a uniform spherical shell of small thickness on an external particle when the attraction varies as the n^{th} power of the distance.

Using the expression (A), Art. 320, for the element of surface, and assuming the law of attraction to be λr^n, we have

$$V = -\frac{2\pi\lambda\rho\tau}{n+1}\frac{a}{D}\int_{D-a}^{D+a} r^{n+2}dr$$

$$= -\frac{2\pi\lambda\rho\tau a}{(n+1)(n+3)D}\left[(D+a)^{n+3}-(D-a)^{n+3}\right],$$

where D is the distance of the point from the centre.

If we wish to find the attraction of a full sphere of radius r, we observe that τ is da, and we integrate this expression from $a = 0$ to $a = r$.

In each case the attraction towards the centre is $-\dfrac{dV}{dD}$.

12. From the theorem of Gauss (p. 287) deduce the following result—*the mean Potential over a spherical surface due to matter entirely outside the sphere is equal to the Potential of this matter at the centre of the sphere.* (Gauss, Papers on Forces varying inversely as the square of the distance, Taylor's *Scientific Memoirs*, vol. iii. part x.)

For, let mass of uniform density ρ and small uniform thickness, τ, be supposed to be distributed on the sphere; let dS be an element of its surface at any point P, V the Potential at P due to the external attracting mass, and a the radius of the sphere. Then, since the Potential of a shell at an external point whose distance from the centre is r

$$= \frac{4\pi\gamma\rho\tau a^2}{r},$$

it follows that if dm is an element of the attracting matter,

$$\rho\tau\int V dS = 4\pi\gamma\rho\tau a^2\int\frac{dm}{r} = 4\pi\rho\tau a^2 V_0,$$

if V_0 is the Potential at the centre of the sphere. Hence

$$\frac{\int V dS}{4\pi a^2} = V_0,$$

which proves the proposition, since $\int VdS$ divided by the whole surface of the sphere is the mean value of the Potential over its surface.

More elementary proof. Let there be a particle of mass m outside

a spherical surface of radius a at a distance D from its centre. The mean value of the Potential over the sphere is $\frac{\gamma m}{4\pi a^2}\int\frac{dS}{r}$, where r is the distance from m of the element dS of surface. But (p. 258) $dS = 2\pi\frac{a}{D}rdr$, and the limits of r are $D\pm a$. Hence this mean value is

$$\frac{\gamma m}{D},$$

i.e. the Potential at the centre; and the result therefore holds for any assemblage of external particles.

13. Find an approximate value of the Potential of any solid mass at a very distant point.

Let G be the centre of mass of the solid body, P the distant point, P' any point in the mass at which the element of mass is dm. Take G as origin and GP as axis of x; let $GP = r$, $GP' = r'$, and let the x of P' be x'.

Then
$$\frac{1}{\gamma}V = \int\frac{dm}{\sqrt{r^2-2rx'+r'^2}} = \frac{1}{r}\int\left(1-2\frac{x'}{r}+\frac{r'^2}{r^2}\right)^{-\frac{1}{2}}dm$$
$$= \frac{1}{r}\int\left(1+\frac{x'}{r}-\frac{r'^2}{2r^2}+\frac{3}{2}\cdot\frac{x'^2}{r^2}\right)dm,$$

neglecting all higher powers of $\frac{r'}{r}$ than the second.

Now $\int x'dm = 0$, and if we denote by λ and λ' the radii of gyration of the solid about the axes of y and z, and by k its radius of gyration about GP, we have

$$\int r'^2 dm = M\frac{\lambda^2+\lambda'^2+k^2}{2},\quad \int x'^2 dm = M\frac{\lambda^2+\lambda'^2-k^2}{2},$$

where M = mass of body.

Hence
$$V = \frac{M}{r}\left(1+\frac{\lambda^2+\lambda'^2-2k^2}{2r^2}\right).$$

But if k_1, k_2, k_3 are the principal radii of gyration at G, we have $\lambda^2+\lambda'^2+k^2 = k_1^2+k_2^2+k_3^2$; therefore

$$V = \frac{\gamma M}{r}\left(1+\frac{k_1^2+k_2^2+k_3^2-3k^2}{2r^2}\right).$$

By differentiating this with respect to x, y, and z separately, we find the components of attraction in the directions of the principal axes at G on a unit mass at P. For very distant points $V = \frac{\gamma M}{r}$ to a high degree of accuracy.

14. If $V \equiv f(x, y, z)$ be a function satisfying Laplace's equation, $\nabla^2 V = 0$, show that the function $\frac{a}{r}f\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right)$ will also satisfy it (where $r^2 = x^2+y^2+z^2$).

If O is the origin, P the point (x, y, z), Q a point on OP produced such that $OQ = \frac{a^2}{OP}$, the co-ordinates of Q are $\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}$. Let $OQ = \rho$, let (ξ, η, ζ) be the co-ordinates of Q, and let

$$U = \frac{a}{r} f\left(\frac{a^2x}{r^2}, \frac{a^2y}{r^2}, \frac{a^2z}{r^2}\right) = \frac{\rho}{a} f(\xi, \eta, \zeta).$$

Then $\frac{aU}{\rho}$ satisfies the equation

$$\sin\theta \frac{d}{d\rho}\left(\rho^2 \frac{d\frac{U}{\rho}}{d\rho}\right) + \frac{d}{d\theta}\left(\sin\theta \frac{d\frac{U}{\rho}}{d\theta}\right) + \frac{1}{\sin\theta} \frac{d^2\frac{U}{\rho}}{d\phi^2} = 0.$$

But $\rho^2 \frac{d}{d\rho} = -a^2 \frac{d}{dr}$; therefore this equation becomes

$$r\sin\theta \frac{d^2(Ur)}{dr^2} + \frac{d}{d\theta}\left(\sin\theta \frac{dU}{d\theta}\right) + \frac{1}{\sin\theta} \frac{d^2U}{d\phi^2} = 0.$$

The first term being the same as $\sin\theta \frac{d}{dr}\left(r^2 \frac{dU}{dr}\right)$, this equation is, by Art. 329, the equivalent of

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0.$$

15. A homogeneous fluid, self-attracting according to the law of nature, completely fills the space between two spherical non-concentric surfaces one of which entirely surrounds the other; find the resultant attraction at any point of the fluid, and also the level surfaces.

Let O be the centre of the larger and O' the centre of the smaller sphere; P any point in the fluid; $OO' = c$; radius of smaller sphere $= b$; $OP = r$, $O'P = r'$; $\rho =$ density of fluid.

To calculate the resultant force at P, imagine that the place of the smaller sphere is occupied with fluid; then the larger is completely full, and there is a force $\frac{4}{3}\pi\gamma\rho r$ in the line PO towards O. Now let the effect of the fluid which we have introduced be annulled by combining with the above force the force exercised at P by a repulsive fluid of same density filling the smaller sphere. This latter force would be $\frac{4\pi\gamma\rho b^3}{r'^2}$; and this would act in the line $O'P$ from O'.

The resultant of these forces is the resultant force at P. If V is the Potential at P,

$$\frac{1}{\gamma} dV = -\frac{4}{3}\pi\rho r dr + \frac{4\pi\rho b^3}{3r'^2} dr' \text{ [p. 299]};$$

$$\therefore \quad \frac{V}{\gamma} = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r'} + \text{const.}$$

This value is otherwise evident, since the Potential at a point due

to any attracting bodies is the sum of their separate Potentials at the point. If a is the radius of the larger sphere (see p. 282),

$$\frac{V}{\gamma} = -\frac{2}{3}\pi\rho r^2 - \frac{4\pi\rho b^3}{3r} + 2\pi\rho a^2.$$

The level surfaces are given by the equation

$$r^2 + \frac{2b^3}{r} = \text{const.}$$

16. If two different masses have the same external level surfaces, the values of their Potentials on any one common surface of level are directly proportional to the quantities of the two masses.

Let M and M' be the two masses; let V be the Potential of the first and V' that of the second at any point P outside both. Then

$$\nabla^2 V = 0,\ \nabla^2 V' = 0. \tag{1}$$

Now since when V is constant, V' is also constant, V' must be some function of V. Let $V' = \phi(V)$. Performing the operation ∇^2 on both sides of this equation, we have

$$\nabla^2 V' = \phi'(V).\nabla^2 V + \phi''(V)\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}, \tag{2}$$

which (1) reduces to $\phi''(V) = 0$, since the coefficient of $\phi''(V)$ cannot vanish.

Hence $\phi(V) = cV + c'$, $\therefore\ V' = cV + c'$; but since at infinity $V = V' = 0$ (if none of the attracting matter is at infinity),

$$c' = 0; \quad \therefore \quad V' = cV.$$

Again, for very distant points (example 13),

$$V = \gamma\frac{M}{r} \text{ and } V' = \gamma\frac{M'}{r}.$$

Hence, finally,
$$\frac{V'}{M'} = \frac{V}{M}.$$

17. If X_n and X_{n-2} denote the component attractions of a given solid at a given point along a given line when the law of attraction is that of the n^{th} power, and that of the $(n-2)^{\text{th}}$ power, of the distance, respectively, prove that

$$X_{n-2} = \frac{\nabla^2 X_n}{(n-1)(n+2)}.$$

18. Find the attraction of a circular plate of uniform thickness and density on an external particle of unit mass in its plane, the law of attraction being that of the inverse distance.

Ans. The mass of the plate divided by the distance of the particle from its centre, multiplied by a constant.

19. Prove that if a material lamina attract according to the law of the inverse distance and if N is its attraction on a unit mass at any

point of a closed curve, measured outwards along the normal, we shall have

$$\int N ds = 0, \text{ or } = -2\pi\gamma m_i,$$

according as there is no mass or mass m_i inside the closed curve, and hence that $\nabla^2 V = 0$, or $= -2\pi\gamma\rho$.

20. Prove that the values of $\nabla^2 V$ calculated for external points and for internal points do not agree for points on the surface of a solid sphere.

21. Prove that neither Laplace's nor Poisson's equation holds for points on the bounding surface of an attracting solid.

22. If a number of uniform bars of the same section and density form any closed polygon with no re-entrant angle, prove that they produce the same Potential (for the law of the inverse square) at any point inside the polygon as a polygon of bars formed by joining the feet of the perpendiculars from the given point on the sides of the given polygon.

Extend this proposition to any curve.

(See equation (a), p. 295.)

23. If a self-attracting sphere of uniform density and radius a changes to one of uniform density and radius a', find the amount of work done by its mutual attractive forces.

Ans.
$$\frac{3}{5}\gamma M^2\left(\frac{1}{a}-\frac{1}{a'}\right),$$

where M is the mass of the sphere, and γ, as usual, the gravitation constant.

24. Two equal uniform bars of given sections and densities are placed parallel to each other and at right angles to the lines joining their extremities; find the amount of work done against their mutual attraction in drawing them a given distance asunder.

Ans. If y is the distance between the bars in any position, l the length of each, m and m' are their masses, the work done in changing the distance from y_1 to y_2 will be the difference of the values of the expression

$$\frac{\gamma m m}{l^2}\left(y - \sqrt{l^2+y^2} - l\log\frac{\sqrt{l^2+y^2}-l}{y}\right),$$

when y_1 and y_2 are successively put for y.

25. The gravitation Potential of an attracting mass cannot have a maximum or minimum value in empty space.

[Let it have a maximum value at A. Then round A, and indefinitely near it, can be described a closed surface, at every point of which V is less than it is at A. Therefore if dn is an elementary length along the normal (measured outwards) to this surface, $\frac{dV}{dn}$ is negative all over the surface; but $N = \frac{dV}{dn}$; hence equation (2), Art. 324, is contradicted.]

333.] **Earnshaw's Theorem.** *If a particle is in equilibrium under the action of forces varying according to the law of inverse square of distance, its equilibrium is unstable.*

If its equilibrium were stable for all displacements, positive work would have to be done against the attractive forces, i.e. these forces would for every small displacement do negative work, or, in other words, V must decrease in all directions from the point. V is therefore a maximum at the point—which, by last example, it cannot be. Therefore, etc.

334.] **Method of Inversion.** Supposing that for any distribution of mass forming either a continuous solid body M (Fig. 283), or a thin shell of any shape, or a series of isolated particles, we know the value of the Potential at any point, A, we may by the aid of an analytical transformation deduce another analogous mass, M', whose Potential at any point, A' may be deduced from the previous Potential.

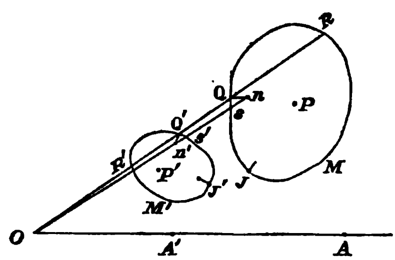

Fig. 283.

Thus, suppose the mass M' to be deduced from the mass M in the following way.

Take any fixed point, O; join it to P, and take the inverse point P', so that

$$rr' = k^2, \tag{α}$$

where $r = OP$, $r' = OP'$. Round P let any very small closed surface be described, and take round P' all the points corresponding to those on this surface. We shall thus get a very small closed surface at P'. Denote the volumes of these elements by $d\Omega$ and $d\Omega'$, respectively. Now $d\Omega$ is filled with a quantity dm of matter belonging to the mass M, and it remains with us to fill $d\Omega'$ with matter according to any law we please. Fill $d\Omega'$ with a quantity dm' bearing to dm the relation

$$\frac{dm'}{dm} = \frac{k}{r}, \text{ or } = \frac{r'}{k}, \tag{β}$$

and let this be done for all the elements, $d\Omega'$, of volume in the derived body M', related as above to the corresponding elements

$d\Omega$ of M. Given the volume-density, ρ, at each point, P, of M; we must now find the volume-density, ρ', at each derived point P'.

Now $dm = \rho d\Omega = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi = -\rho r^2 dr \, d\mu \, d\phi$, in the usual notation of polar formulæ. Similarly, dr' being taken positively without reference to dr,

$$dm' = -\rho' r'^2 dr' \, d\mu \, d\phi;$$

and from (α), if dr and dr' are taken connectedly, we have $r dr' + r' dr = 0$; hence (β) gives

$$\rho' = \rho \frac{k^5}{r'^5}, \qquad (\gamma)$$

so that if ρ is constant, ρ' will vary inversely as the fifth power of the distance, OP', from O.

Let A be any point, at which the Potential of M is V, and take the inverse point A'. It is required to find V', the Potential of M' at A'.

If dV is the Potential at A produced by the element dm at P,

$$dV = \gamma \frac{dm}{AP},$$

where γ is the gravitation constant. Also if dV' is the Potential at A' due to dm' at P',

$$dV' = \gamma \frac{dm'}{A'P'}.$$

Hence $\dfrac{dV'}{dV} = \dfrac{k}{r} \cdot \dfrac{AP}{A'P'}$. But the triangles PAO and $A'P'O$ are similar, $\therefore$ if $OA = D$, $\dfrac{AP}{A'P'} = \dfrac{D}{r'}$; hence $\dfrac{dV'}{dV} = \dfrac{D}{k}$; and this constant relation holds between the Potentials of all corresponding elements, and therefore between the whole Potentials, so that

$$V' = \frac{D}{k} \cdot V. \qquad (\delta)$$

In this transformation the angle at which any two curves in the original system M intersect is equal to that at which the two derived curves intersect. For, let Q and n be any two very close points in the system M, and let Q' and n' be their inverses. Then the quadrilateral $Qnn'Q'$ is inscribable in a circle, so that Qn and $Q'n'$ are ultimately tangents to the circle and therefore equally inclined to OQ. Similarly, if s and s' are two corresponding points very close to Q and Q', the arcs Qs and $Q's'$ are equally inclined to OQ; therefore the angle between the arcs

Qn and Qs is equal to that between $Q'n'$ and $Q's'$. Hence if the contour M is the outer surface of a shell whose thickness varies in any manner, being Qn at the point Q, the inverse points will trace out the contour M' of another shell, and if n' is the inverse of n, $Q'n'$ will be normal to the new shell, and will, of course, be its thickness at Q'.

By similar triangles $\frac{Qn}{Q'n'} = \frac{OQ}{On'}$, or $= \frac{OQ}{OQ'}$ since Q' and n' are nearly coincident. Hence if τ and τ' are the thicknesses of the shells at corresponding points,

$$\frac{\tau'}{\tau} = \frac{r'}{r}; \tag{ϵ}$$

and hence by (γ)

$$\rho'\tau' = \rho\tau \,.\, \frac{k^3}{r'^3}, \tag{ζ}$$

so that if $\rho\tau$ is constant all over the shell M, $\rho'\tau'$ will vary inversely as the cube of the distance from O at every point on the derived shell.

If the mass M forms a spherical shell of uniform thickness and density, its Potential at A is at once known. Hence is known also the potential at A' (any point) due to a spherical shell in which the product $\rho'\tau'$ (which is the mass per unit area of its surface) varies inversely as the cube of the distance from a fixed point, O—a case which has been already discussed (p. 301). Supposing M to be a spherical surface whose centre is P, the inverse point, P', is not the centre of the sphere M', but is the inverse of O with respect to the sphere M'. For if J and J' are any corresponding points on the contours, $\frac{JP}{PO} = \frac{P'J'}{J'O}$, and since M is a sphere with centre P, the left-hand side is constant, therefore the right side is constant, and the two points O and P' are well known to be inverse with respect to the spherical locus of J'.

Again, (δ) gives, since $M' = k\int\frac{dm}{r} = k\frac{M}{OP}$

$$V' = \gamma\frac{M'}{P'A'}, \tag{η}$$

which shows that the level surfaces of M' are spheres round P' as centre; and the result (η) holds both for the case in which M is a uniform spherical shell, and therefore M' a spherical shell

in which the surface-density, $\rho'\tau'$, varies inversely as the cube of the distance from O, and for the case in which M is a solid uniform sphere, and therefore M' a solid sphere in which the density varies inversely as the fifth power of the distance from O.

In this method of transformation we may notice that *the mass of the derived distribution, M', is proportional to the Potential of the given mass at the origin of inversion.* (It is equal to this Potential multiplied by $\frac{k}{\gamma}$.)

335.] **Continuity of the Potential.** The gravitation Potential of any attracting solid mass varies in a continuous manner from point to point in space, whether the points chosen be inside any portion of the mass or outside it.

For, if r be the distance of any element of mass, dm, of the attracting body from P, the point at which the Potential is required, $V = \gamma \int \frac{dm}{r}$. Let P be taken as origin, and let the position of the element dm be defined by the radius vector, r, and two angles, θ and ϕ, and let ρ be the density of the element. Then $dm = \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$, and

$$V = \gamma \iiint \rho r \sin\theta \, dr \, d\theta \, d\phi.$$

This form of V shows that even if r is zero, i.e. if P is inside the mass, the value of the Potential is finite, no infinite term being introduced by the infinitely close proximity of P to some of the (infinitely small) elements of mass.

Hence the Potential varies continuously throughout space, and diminishes from the vicinity of the attracting mass towards the space very remote from it in all directions.

The field of attraction of any matter, according to the Newtonian law, may therefore be compared with a country consisting of hills and valleys which vary *gradually*, even though they may rise or fall rapidly in certain places,—precipices and chasms being wholly absent; and in the field of attraction the Potential at each point is the *gravitation level* of the point, and is the analogue of the height above the sea (or other arbitrary) level in the other.

336.] **Continuity of the First Differential Coefficients of Potential.** In a field of attraction *in which every attracting element is one of finite volume-density*, there is likewise a complete continuity of the first differential coefficients of V from points

within to points without the attracting masses. For these first differential coefficients, $\frac{dV}{dx}$, $\frac{dV}{dy}$, $\frac{dV}{dz}$, are simply the components of force-intensity; and if in (3), (4), (5) of Art. 325, we put $\phi(r) = \frac{\gamma}{r^2}$: the elements under the sign of integration never at any point contain r in the denominator, and are therefore never infinite, even when $r = 0$, i.e. when P is inside the mass. Evidently the case would be different for a law of attraction according to a power higher than that of the inverse square.

And the case is different again, even for the Newtonian law, when the attracting matter forms an *infinitely* thin shell with (necessarily) infinitely great volume-density. In this case the force components in *some* directions vary abruptly for a small change of position of the attracted particle P, although in other directions they vary continuously. Of this more hereafter; but the fact is already sufficiently clear in the case (Art. 322) of the normal component of a thin shell.

337.] **Discontinuity of its Second Differential Coefficients.**

Since $V = \gamma\int\frac{dm}{r}$, we have $\frac{d^2V}{dx^2} = \gamma\int\frac{d^2\left(\frac{1}{r}\right)}{dx^2}dm$, the co-ordinates of the point, P, at which the Potential is V, being x, y, z.

Now if (x', y', z') are the co-ordinates of dm, and (x, y, z) those of P, we have
$$\frac{d^2r}{dx^2} = \frac{1}{r} - \frac{(x-x')^2}{r^3};$$
and since
$$\frac{1}{\gamma}\frac{d^2V}{dx^2} = \int\left\{\frac{2}{r^3}\left(\frac{dr}{dx}\right)^2 - \frac{1}{r^2}\frac{d^2r}{dx^2}\right\}dm,$$
$$\therefore\quad \frac{1}{\gamma}\frac{d^2V}{dx^2} = \int\left\{\frac{3(x-x')^2}{r^5} - \frac{1}{r^3}\right\}dm. \tag{1}$$
Similarly
$$\frac{1}{\gamma}\frac{d^2V}{dy^2} = \int\left\{\frac{3(y-y')^2}{r^5} - \frac{1}{r^3}\right\}dm, \tag{2}$$
$$\frac{1}{\gamma}\frac{d^2V}{dz^2} = \int\left\{\frac{3(z-z')^2}{r^5} - \frac{1}{r^3}\right\}dm. \tag{3}$$

If in these expressions we substitute for $x-x'$, $y-y'$, $z-z'$, and dm, as in last Article, we have
$$\frac{1}{\gamma}\frac{d^2V}{dx^2} = \int(3\sin^2\theta\cos^2\phi - 1)\,\frac{\rho}{r}\sin\theta\, dr\, d\theta\, d\phi;$$

hence, when $r = 0$, i.e. when I is inside the attracting mass, the expression under the integral sign becomes infinite, and the value of $\frac{d^2V}{dx^2}$ ceases to be continuous from points inside to points outside the mass.

Fig. 284 represents the values of $V, \frac{dV}{dx}$, and $\frac{d^2V}{dx^2}$, when the attracting solid is that contained between two concentric spherical surfaces whose radii are Oa' and Oa, and the point P occupies positions along a fixed diameter, Ox, varying from O to infinity. The distance of P from O is here denoted by x, and the values of V are given by the ordinates (distances from Ox) of the continuous curve $ABCD$, of which the portion AB is a right line corresponding to the constant potential within the inner surface.

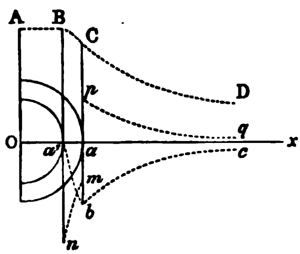

Fig. 284.

The values of $\frac{dV}{dx}$ are given by the ordinates of the continuous curve $Oa'bc$, of which Oa' corresponds to the constant zero value within the inner surface.

The values of $\frac{d^2V}{dx^2}$ are given by the ordinates of the discontinuous curve $Oa'nmpq$.

From Ex. 5, p. 299, when P is completely outside the mass, we have $\frac{d^2V}{dD^2} = \frac{8\pi\gamma\rho(a^3-a'^3)}{3D^3}$, and when P is inside the shell between the two surfaces, $\frac{d^2V}{dD^2} = -\frac{4\pi\gamma\rho}{3}\left(1+\frac{2a'^3}{D^3}\right)$.

By putting $D = a$ in the first of these values we have the value, ap, of $\frac{d^2V}{dD^2}$ when P comes to the outer surface from the outside; and putting $D = a$ in the second we have the (negative) value, am, of $\frac{d^2V}{dD^2}$ when P comes to this surface from the inside.

The above figure is copied from Thomson and Tait's *Nat. Phil.*

338.] **Lines and Tubes of Force.** If at any point, I, in the field of attraction an elementary length is drawn in the direction of the resultant attraction at P, and if this is prolonged at each point $P', I'', \ldots$ so as to be in the direction of the resultant attraction at all points, $P', P'', \ldots$ along it, we obtain a continuous curve which is called a *line of force*. A line of force,

then, is a curve such that its tangent at every point coincides with the direction of resultant attraction at that point.

If the field is mapped out by a series of equipotential surfaces (Art. 328), every line of force will cut every equipotential surface which it meets at right angles, since (Art. 328) at every point on such a surface the resultant force acts in the normal to the surface.

Let P be any point in the field; at P describe any very small closed curve whatsoever; through each point on this curve draw the line of force and prolong it indefinitely. We thus get what is called a *tube of force*.

These terms are due to Faraday.

339.] **Surface-integral for a Tube of Force.** Let $PAQB$ represent any portion of a tube of force, P and Q being elements of two level surfaces intercepted by the tube. Then the attraction on a unit mass at P is normal to the section P, and the attraction on a unit mass at Q is normal to the section Q, while at every point, A or B, on every portion of the lateral surface of the tube the attraction is wholly tangential to the surface.

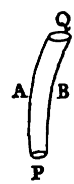

Fig. 285.

Let F be the force at P, F' that at Q, and ω and ω' the areas of the sections P and Q. Then, supposing that the tube contains none of the attracting matter, equation (2) of Art. 324 gives

$$F\omega - F'\omega' = 0, \tag{1}$$

since the only portions of the closed surface $PAQB$ which contribute elements to the surface-integral of normal attraction are the sections P and Q.

Hence, *at all points in empty space on a given line of force the resultant attraction-intensities are inversely proportional to the normal sections of the same tube of force at these points.*

This simple theorem gives the law of attraction very readily in certain cases. For example, let the attracting body be a sphere whose density is the same at the same distance from its centre. Then the lines of force are obviously right lines drawn from its centre; the tubes are therefore cones whose vertices are the centre, and since the normal sections of these cones are directly as the squares of their distances from the centre, the attraction of the sphere at any external point is inversely proportional to the square of its distances from the centre.

Again, let the attracting body be an infinite cylinder whose density is the same at the same distance from its axis. Here the lines of force are right lines emanating from the axis perpendicularly, the tubes become wedges, and the areas of their normal sections are directly proportional to their distances from the axis; hence the attraction of an infinite cylinder at an external point is inversely proportional to its distance from the axis.

Finally, for an infinite attracting plate, the tubes are cylinders and the attraction is constant at all points in empty space.

If the tube of force contain within it a quantity of the attracting matter whose mass is dq, we have by (2) of Art. 324

$$F\omega - F'\omega' = 4\pi\gamma dq. \tag{2}$$

This equation can in like manner be employed to find the resultant force inside a sphere, a cylinder, or a plate.

In the case of a sphere of uniform density, let the tube be contained between the spheres of radii r and $r+dr$. Then $dq = \rho\omega dr$, ρ being the density at the attracted point, and (2) becomes

$$d(F\omega) = 4\pi\gamma\rho\omega dr,$$

or

$$d(Fr^2) = 4\pi\gamma\rho r^2 dr,$$

since ω is proportional to r^2. Integrating this last equation,

$$Fr^2 = \frac{3}{4}\pi\gamma\rho r^3 + C.$$

Now F is evidently zero at the centre, therefore $C = 0$, and

$$F = \frac{4}{3}\pi\gamma\rho r.$$

For a point inside an infinite cylinder at a distance r from the axis we have, since ω is ultimately a rectangle of breadth proportional to r,

$$d(Fr) = 4\pi\gamma\rho r dr,$$

$$\therefore \quad F = 2\pi\gamma\rho r.$$

In general, if the tube is terminated by two level surfaces whose distance measured along the lines of force forming the tube is ds, we have $dq = \rho\omega ds$, and (2) gives for the determination of F

$$d(F\omega) = 4\pi\gamma\rho\omega ds.$$

340.] **Unit Tube of Force.** If at any point P we draw a tube of force such that the product of the force-intensity and the area of the normal section is unity, the tube is called a *unit tube.* Thus, in C. G. S. measures, if the product of the force-intensity,

expressed in dynes per gramme mass, and the area of the normal section, expressed in square centimètres, is numerically unity, the tube is a unit tube.

Section III.

The Attraction of Ellipsoids. [*Method of Chasles.*]

341.] **Shell bounded by Similar Surfaces.** Let $vr'p'$ and rqp be two concentric, similar, and similarly situated surfaces whose normal distance from each other is at all points very small. Suppose the space between these surfaces to be filled by attracting matter of uniform density, and let O be an attracted particle in the interior of the shell. With O as vertex let any slender cone be described, intercepting on the shell two frustums whose thicknesses measured along the generator pr of the cone are pp' and rr'. Then, since by the property of similar, similarly situated, and concentric surfaces of the second degree, the intercepts pp' and rr' are equal whatever be the direction of the line pr, we see by Art. 318 that the attractions of these frustums on O are equal and opposite. Hence the corresponding frustums of all such cones exert equal and opposite attractions on O; and the resultant attraction of the shell on any internal particle is therefore zero.

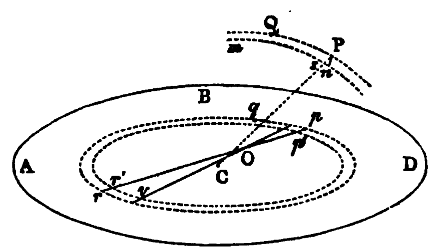

Fig. 286.

Hence, if the law of attraction is that of nature, *every shell of uniform density and small thickness, bounded by similar, similarly situated, and concentric ellipsoidal surfaces produces a constant Potential at all points in its interior, and exerts, therefore, at these points no attraction.*

The same is true for a solid of uniform density and any thickness bounded by two similar, similarly situated, and concentric ellipsoidal surfaces, since the thicknesses of the frustums intercepted between its bounding surfaces will still be equal.

342.] **Corresponding Points on Confocal Ellipsoids.** Let rpq and PQ (Fig. 286) be two confocal ellipsoids, let the axes of the first be α', β', γ', and those of the second α, β, γ, let the co-ordinates of a point p on the first be x', y', z', and those of a point P on the second x, y, z. Then, if

$$\frac{x}{\alpha}=\frac{x'}{\alpha'},\quad \frac{y}{\beta}=\frac{y'}{\beta'},\quad \frac{z}{\gamma}=\frac{z'}{\gamma'},$$

the points P and p are called *corresponding* points on the ellipsoids. Also, let Q and q be two other corresponding points. Then it is very easy to prove that the distance Pq is equal to the distance Qp. (Salmon's *Geometry of Three Dimensions*, Art. 181.)

343.] **External Potential of an Ellipsoidal Shell.** Let it be required to find the Potential at an external point, P, of a shell bounded by the similar, similarly situated, and concentric ellipsoids $vr'p'$ and rqp. Through the point P describe an ellipsoid, PQ, confocal with rqp, and describe also an ellipsoid, msn, confocal with $vr'p'$ and similar to PQ. This latter surface is completely determinate, since its axes must be $\mu\alpha, \mu\beta, \mu\gamma$, and since $\mu^2(\alpha^2-\beta^2)$ must be equal to $\mu'^2(\alpha'^2-\beta'^2)$, where $\mu'\alpha', \mu'\beta'$, $\mu'\gamma'$ are the (given) axes of the ellipsoid $vr'p'$; or $\mu=\mu'$, since $\alpha^2-\beta^2=\alpha'^2-\beta'^2$.

Now at q draw the normal distance, dn', which separates the surfaces rqp and $vr'p'$, and about q describe on the ellipsoid rqp any small closed curve whose area is dS'. Round Q, on the surface QP describe the small closed curve, of area dS, which consists of points corresponding to those forming dS'; and let dn be the normal distance between the surfaces QP and msn. We shall now prove that the elements of volume $dn \,.\, dS$ and $dn' .\, dS'$, which we may denote by $d\omega$ and $d\omega'$, respectively, are connected by the equation

$$\frac{d\omega}{\alpha\beta\gamma}=\frac{d\omega'}{\alpha'\beta'\gamma'}. \tag{1}$$

Let x', y', z' be the co-ordinates of q with reference to the principal axes of the ellipsoids, and let $dx'dy'$ be the projection of dS' on the plane xy. Then, since the cosine of the angle between the normal at q and the axis of z is $\frac{p'z'}{\gamma'^2}$, where p' is the perpendicular from C on the tangent plane at q, we have

$$dS'=\frac{\gamma'^2}{p'z'}dx'dy'.$$

Now since the surfaces rqp and $v'r'p'$ are similar, we have

$$\frac{dn'}{p'} = 1-\mu,$$

$$\therefore \quad dn'.dS' = (1-\mu)\frac{\gamma'^2}{z'}dx'dy'. \tag{2}$$

Similarly $$dn.dS = (1-\mu)\frac{\gamma^2}{z}dx\,dy, \tag{3}$$

where x, y, z are the co-ordinates of Q. But since

$$\frac{dx}{\alpha} = \frac{dx'}{\alpha'},\quad \frac{dy}{\beta} = \frac{dy'}{\beta'},$$

these equations give (1) at once by division. Moreover the Potential at P due to the element of mass $\rho\,d\omega'$ at q is proportional to $\dfrac{\rho\,d\omega'}{Pq}$, while the Potential at p due to the element $\rho\,d\omega$ at Q is proportional to $\dfrac{\rho\,d\omega}{Qp}$; and since $Pq = Qp$,

$$\frac{\text{Potential at } P \text{ due to element of mass at } q}{\text{Potential at } p \text{ due to corresponding element at } Q} = \frac{d\omega'}{d\omega}$$

$$= \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} = \frac{\text{mass of shell } rqp}{\text{mass of shell } PQ}.$$

Now the shell rqp can be broken up into elements of mass formed as $d\omega'$ has been formed, and the corresponding elements, $d\omega$, will completely exhaust the shell PQ; hence, taking all the elements of the inner shell, and all the corresponding elements of the outer, and thus exhausting both shells, we see that

$$\frac{\text{the Potential of the inner shell at } P}{\text{the Potential of the outer shell at } p} = \frac{\text{mass of inner shell}}{\text{mass of outer shell}}.$$

Now since these shells are bounded each by similar surfaces, the Potential of the outer shell is constant at all internal points, and (in virtue of the continuity of the Potential) this Potential is the same as the Potential of the outer shell at P.

Hence the Potential of an ellipsoidal shell bounded by similar surfaces is constant at all points on the surface of any ellipsoid confocal with the surface of the shell—that is, the level surfaces of an ellipsoidal shell are confocal ellipsoids, and its attraction at any point is therefore normal to the confocal ellipsoid through the point.

Let V and V' be the Potentials of the shells PQ and rqp at P; then

$$V' = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}V. \tag{4}$$

We shall show in the next section (example 4) that these shells produce at *all* points outside both Potentials which are proportional simply to the masses of the shells, i.e., related as in (4); so that at all such points their attraction-intensities also bear this relation to each other. Hence at any point outside both shells—even though it is just on the outer surface of PQ—we have

$$\frac{dV'}{dx} = \frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma} \cdot \frac{dV}{dx}. \tag{5}$$

For this reason the calculation of the attraction of an ellipsoidal shell at an external point is reduced to that of a shell at a point on its surface.

344.] **Attraction of an Ellipsoid at an External Point.** Let ABD (Fig 286) be a solid homogeneous ellipsoid, and let it be required to find its attraction on a unit mass condensed at P. Break the ellipsoid up into an infinite number of thin shells bounded by ellipsoids similar to each other and to the surface ABD; let one of these shells be that between the surfaces $vr'p'$ and rqp. Denote this shell by (s); and describe the ellipsoids PQ and msn, similar to each other and confocal with the surfaces of (s), as in the preceding Articles. Denote this shell by (σ).

Let the axes of ABD be a, b, c; let those of rqp be ka, kb, kc, and let those of $vr'p'$ be $(k+dk)a, (k+dk)b, (k+dk)c$. Also, let the axes of the ellipsoid PQ be $k\sqrt{a^2+\lambda^2}, k\sqrt{b^2+\lambda^2}, k\sqrt{c^2+\lambda^2}$; then, by Art. 343, those of msn will be $(k+dk)\sqrt{a^2+\lambda^2}, (k+dk)\sqrt{b^2+\lambda^2}, (k+dk)\sqrt{c^2+\lambda^2}$. Now (Art. 322), the attraction of the shell (σ) on a unit mass at P is

$$4\pi\gamma\rho . Pn^*,$$

where Pn is the normal thickness of the shell at P. This attraction acts in the direction of the normal Pn, whose direction cosines are

$$\frac{px}{k^2(a^2+\lambda^2)}, \quad \frac{py}{k^2(b^2+\lambda^2)}, \quad \frac{pz}{k^2(c^2+\lambda^2)},$$

p being the length of the perpendicular from C, the centre of the ellipsoid on the tangent plane at P, and x, y, z the coordinates of P. Hence the attraction of (σ) on P parallel to the axis of x, in the positive direction, is

$$-\frac{4\pi\gamma\rho p x}{k^2(a^2+\lambda^2)} \cdot Pn. \tag{1}$$

* The curious compensation of errors involved in the usual proof of this is well noticed by Collignon (*Dynamique*, p. 403).

Draw the line CP meeting the inner surface of (σ) in s. Then $\frac{Pn}{Ps} = \frac{p}{CP}$, therefore $Pn = p \cdot \frac{Ps}{CP}$. But $\frac{Cs}{CP} = \frac{\text{axis of } mn}{\text{axis of } PQ}$ $= \frac{k+dk}{k}$; therefore $\frac{Ps}{CP} = -\frac{dk}{k}$, and $Pn = -\frac{pdk}{k}$.

Substituting this value in (1), we find the attraction of (σ) parallel to the axis of x to be

$$\frac{4\pi\gamma\rho p^2 x dk}{k^3(a^2+\lambda^2)}.$$

Multiplying this by the ratio of the mass of (s) to that of (σ), we have the component of the attraction of (s). Denoting this latter by dX, we have

$$dX = \frac{4\pi\gamma\rho abc p^2 x\, dk}{k^3(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}. \tag{2}$$

Now, by the equation of the surface PQ,

$$\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda^2} + \frac{z^2}{c^2+\lambda^2} = k^2.$$

Differentiating this, regarding k and λ as variables, we have

$$\frac{k^3}{p^2}\lambda d\lambda = -dk,$$

by the well-known value of the perpendicular from the centre on the tangent plane of an ellipsoid.

Substituting this value of dk in (2), we have

$$dX = -\frac{4\pi\gamma\rho abc x\lambda\, d\lambda}{(a^2+\lambda^2)^{\frac{3}{2}}\sqrt{(b^2+\lambda^2)(c^2+\lambda^2)}}.$$

To find the limits of λ, we observe that when the shell (s) is taken at the centre, $k = 0$; but the axes of (σ) must be finite; and as they are $k\sqrt{a^2+\lambda^2}$, &c., the value of λ corresponding to a vanishing shell at the centre is ∞. Again, if $k = 1$, or (s) is a shell at the surface ABD, we have $a^2+\lambda^2 = a_1^2$, where a_1 is the semi-axis of the ellipsoid confocal with ABD, and passing through P. Denote this value of λ by λ_1. Then, if M be the mass of the solid ellipsoid ABD, we have

$$X = 3\gamma Mx\int_{\infty}^{\lambda_1}\frac{\lambda d\lambda}{\sqrt{(a^2+\lambda^2)^3(b^2+\lambda^2)(c^2+\lambda^2)}}; \tag{3}$$

and in the same way for the other components, Y and Z,

$$\left.\begin{aligned} Y &= 3\gamma My \int_{\infty}^{\lambda_1} \frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)^3(c^2+\lambda^2)}}, \\ Z &= 3\gamma Mz \int_{\infty}^{\lambda_1} \frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)^3}}. \end{aligned}\right\} \qquad (4)$$

If $L = \int_{\infty}^{\lambda_1} \frac{\lambda\, d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}$, we have evidently

$$X = -6\gamma Mx \frac{dL}{d(a^2)}, \quad Y = -6\gamma My \frac{dL}{d(b^2)}, \quad Z = -6\gamma Mz \frac{dL}{d(c^2)}.$$

The expressions for X, Y, Z may be put into other forms which are useful in practice, by putting

$$\lambda = \frac{c\sqrt{1-u^2}}{u}.$$

Then

$$\left.\begin{aligned} X &= -\frac{3\gamma Mx}{c^3} \int_0^{\frac{c}{c_1}} \frac{u^2\, du}{\sqrt{(1+e^2u^2)^3(1+e'^2u^2)}}, \\ Y &= -\frac{3\gamma My}{c^3} \int_0^{\frac{c}{c_1}} \frac{u^2\, du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)^3}}, \\ Z &= -\frac{3\gamma Mz}{c^3} \int_0^{\frac{c}{c_1}} \frac{u^2\, du}{\sqrt{(1+e^2u^2)(1+e'^2u^2)}}, \end{aligned}\right\} \qquad (5)$$

where $e^2 = \frac{a^2-c^2}{c^2}$, and $e'^2 = \frac{b^2-c^2}{c^2}$, the least semi-axis being c.

If the attracted particle is *on* the surface ABD of the attracting ellipsoid, the limits of u are 0 and 1, since $c_1 = c$.

If the attracted point is inside the ellipsoid, let an ellipsoid be described through it concentric with and similar to the surface ABD, and the portion between these two surfaces exerts no attraction at the point (Art. 320).

Equations (5) show that the components along the principal axes of the attraction of a homogeneous ellipsoid on a particle placed anywhere on its surface or inside its mass are of the forms

$$Ax, \quad By, \quad Cz, \qquad (6)$$

where A, B, C are constant quantities.

345.] **Potential of an Ellipsoid.** *Potential of a homogeneous Ellipsoid at its centre.* Let ABD (Fig. 286) be a homogeneous Ellipsoid of density ρ, whose semi-axes are a, b, c. The polar element of volume being $r^2 \sin\theta\, dr\, d\theta\, d\phi$, the Potential of this

at C is $\gamma\rho r \sin\theta dr d\theta d\phi$; and integrating this from C to the bounding surface, we get $\frac{1}{2}\gamma\rho R^2 \sin\theta d\theta d\phi$, so that

$$V_0 = \tfrac{1}{2}\gamma\rho\int_0^{\pi}\int_0^{2\pi}\frac{\sin\theta d\theta d\phi}{\dfrac{\sin^2\theta\cos^2\phi}{a^2}+\dfrac{\sin^2\theta\sin^2\phi}{b^2}+\dfrac{\cos^2\theta}{c^2}},$$

where V_0 is the Potential at C. This, again, is the same as

$$V_0 = 4\gamma\rho\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta d\phi}{\dfrac{\sin^2\theta\cos^2\phi}{a^2}+\dfrac{\sin^2\theta\sin^2\phi}{b^2}+\dfrac{\cos^2\theta}{c^2}}.$$

Integrating with respect to ϕ, we have

$$V_0 = 2\pi\gamma\rho\int_0^{\frac{\pi}{2}}\frac{\sin\theta d\theta}{\sqrt{\left(\dfrac{\sin^2\theta}{a^2}+\dfrac{\cos^2\theta}{c^2}\right)\left(\dfrac{\sin^2\theta}{b^2}+\dfrac{\cos^2\theta}{c^2}\right)}}. \quad (1)$$

Putting $\tan\theta = t$, we have

$$V_0 = 2\pi\gamma\rho abc^2\int_0^{\infty}\frac{tdt}{\sqrt{(1+t^2)(a^2+c^2t^2)(b^2+c^2t^2)}};$$

or, finally, putting $c^2t^2 = \lambda^2$, we have the symmetrical form

$$V_0 = 2\pi\gamma\rho abc\int_0^{\infty}\frac{\lambda d\lambda}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}. \quad (2)$$

Potential of Ellipsoid at any internal point. Let p (Fig. 286) be the internal point the Potential at which we desire to find. Drawing the ellipsoid pqr, which is similar to the bounding surface DBA, the values of X, Y, Z at p are due entirely to the matter within pqr. Hence if the axes of pqr are ka, kb, kc, we are to put $\lambda_1 = 0$ in equations (3), (4), p. 323, and

$$M = \tfrac{4}{3}\pi k^3\rho abc.$$

Thus we have

$$X = 4\pi\gamma\rho k^3 abcx\int_{\infty}^{0}\frac{\lambda d\lambda}{\sqrt{(k^2a^2+\lambda^2)^3(k^2b^2+\lambda^2)(k^2c^2+\lambda^2)}}.$$

Putting $\lambda^2 = k^2\mu^2$,

$$X = 4\pi\gamma\rho abc\,.\,x\int_{\infty}^{0}\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)^3(b^2+\mu^2)(c^2+\mu^2)}}. \quad (3)$$

Similarly

$$Y = 4\pi\gamma\rho abc\,.\,y\int_{\infty}^{0}\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)^3(c^2+\mu^2)}}, \quad (4)$$

$$Z = 4\pi\gamma\rho abc\,.\,z\int_{\infty}^{0}\frac{\mu d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)^3}}. \quad (5)$$

Denote these values of X, Y, Z by Ax, By, Cz, the quantities A, B, C being obviously the same for all internal points. Then if V is the Potential of the whole ellipsoid at p,

$$dV = Axdx + Bydy + Czdz.$$

Integrating,

$$V = V_0 + \tfrac{1}{2}(Ax^2 + By^2 + Cz^2). \qquad (6)$$

Substituting the values of V_0, A, B, C, just found, we have

$$V = 2\pi\gamma\rho abc\int_0^\infty \left(1 - \frac{x^2}{a^2+\mu^2} - \frac{y^2}{b^2+\mu^2} - \frac{z^2}{c^2+\mu^2}\right) \frac{\mu\, d\mu}{\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)}}. \qquad (7)$$

The integrals involved in these several coefficients are easily reduced to the ordinary forms of elliptic integrals. Thus, assuming that the axes in order of descending magnitudes are a, b, c, assume

$$\mu^2 = b^2\tan^2\phi - c^2\sec^2\phi. \qquad (8)$$

Denoting $\sqrt{(a^2+\mu^2)(b^2+\mu^2)(c^2+\mu^2)}$ by $f(\mu)$, we have

$$\frac{d\mu}{f(\mu)} = \frac{d\phi}{\sqrt{a^2 - c^2 - (a^2-b^2)\sin^2\phi}} \qquad (9)$$

$$= \frac{1}{\sqrt{a^2-c^2}}\cdot\frac{d\phi}{\Delta\phi}, \qquad (10)$$

in the ordinary notation of elliptic integrals. Also

$$\overline{\mu}\Big|_0^\infty = \overline{\phi}\Big|_{\sin^{-1}\frac{c}{b}}^{\frac{\pi}{2}}.$$

Hence, denoting $\sin^{-1}\dfrac{c}{b}$ by ω, we have

$$V_0 = \frac{2\pi\gamma\rho abc}{\sqrt{a^2-c^2}}\int_\omega^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi}, \qquad (11)$$

$$X = \frac{4\pi\gamma\rho abc}{(a^2-c^2)^{\frac{3}{2}}}\int_\omega^{\frac{\pi}{2}} \frac{\cos^2\phi\, d\phi}{\Delta^3\phi} \times x, \qquad (12)$$

$$Y = \frac{4\pi\gamma\rho abc}{(b^2-c^2)\sqrt{a^2-c^2}}\int_\omega^{\frac{\pi}{2}} \frac{\cos^2\phi\, d\phi}{\Delta\phi} \times y, \qquad (13)$$

$$Z = \frac{4\pi\gamma\rho abc}{(b^2-c^2)\sqrt{a^2-c^2}}\int_\omega^{\frac{\pi}{2}} \frac{\cot^2\phi\, d\phi}{\Delta\phi} \times z. \qquad (14)$$

The integral in (12) is reduced to elliptic integrals of the first

and second kinds by the formula (Hymers's *Integral Calculus*, p. 219)

$$\int \frac{d\phi}{\Delta^3\phi} = \frac{1}{k'^2} E(k, \phi) - \frac{k^2 \sin\phi\cos\phi}{k'^2 \Delta\phi},$$

while that in (14) is reduced thus:

$$\int \frac{\cot^2\phi\, d\phi}{\Delta\phi} = -\int \frac{d\cot\phi}{\Delta\phi} - \int \frac{d\phi}{\Delta\phi} = -\frac{\cot\phi}{\Delta\phi} - k'^2 \int \frac{d\phi}{\Delta^3\phi},$$

k' being the complement of k, i.e. $k^2 + k'^2 = 1$.

Potential of an Ellipsoidal Shell at any external point. Let it be required to find the Potential at P (Fig. 286) due to the homogeneous shell contained between pqr and $p'r'v$. By Art. 343, this Potential $= \dfrac{abc}{\sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}}$ times the Potential produced at C by the shell $PQms$; and since the axes of the outer surface, PQ, are $k\sqrt{a^2+\lambda^2}$, &c., it is easily seen that this latter Potential is

$$-8\gamma\rho k\,dk \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta\, d\phi}{\dfrac{\sin^2\theta\cos^2\phi}{a^2+\lambda^2} + \dfrac{\sin^2\theta\sin^2\phi}{b^2+\lambda^2} + \dfrac{\cos^2\theta}{c^2+\lambda^2}}. \quad (15)$$

But we have shown that the double integral in this expression is equal to

$$\frac{\pi}{2} f(\lambda) \int_0^\infty \frac{\mu\, d\mu}{\sqrt{(a^2+\lambda^2+\mu^2)(b^2+\lambda^2+\mu^2)(c^2+\lambda^2+\mu^2)}}, \quad (16)$$

where $f(\lambda) = \sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}$. Hence if dV is the Potential of the shell pqr at P,

$$dV = -4\pi\gamma\rho abc . k\,dk \int_0^\infty \frac{\mu\, d\mu}{\chi(\mu)}, \quad (17)$$

where $\chi(\mu)$ is the denominator under the integral in (16).

We may put $\lambda^2 + \mu^2 = \nu^2$, so that $\overline{\mu}\Big|_0^\infty = \overline{\nu}\Big|_\lambda^\infty$, and

$$dV = -4\pi\gamma\rho abc . k\,dk \int_\lambda^\infty \frac{\nu\, d\nu}{f(\nu)}.$$

But (p. 323) $k\,dk = -\left[\dfrac{x^2}{(a^2+\lambda^2)^2} + \dfrac{y^2}{(b^2+\lambda^2)^2} + \dfrac{z^2}{(c^2+\lambda^2)^2}\right]\lambda\, d\lambda.$

Hence integrating from $\lambda = \infty$ to $\lambda = \lambda_1$, so as to include the whole given ellipsoid,

$$V = 4\pi\gamma\rho abc \int_{\lambda_1} \left\{\left[\frac{x^2}{(a^2+\lambda^2)^2} + \frac{y^2}{(b^2+\lambda^2)^2} + \frac{z^2}{(c^2+\lambda^2)^2}\right] \int_\lambda^\infty \frac{\nu\, d\nu}{f(\nu)}\right\} \lambda\, d\lambda. \quad (18)$$

This is easily reduced to a simpler form thus. Let

$$\int \frac{u\,du}{f(u)} \equiv \phi(u); \text{ then}$$

$$V = 4\pi\gamma\rho abc\int_{\lambda_1}^{\infty}\left[\frac{x^2}{(a^2+\lambda^2)^2} + \frac{y^2}{(b^2+\lambda^2)^2} + \frac{z^2}{(c^2+\lambda^2)^2}\right]$$
$$[\phi(\infty)-\phi(\lambda)]\lambda\,d\lambda.$$

Now taking the term in x only, we have

$$x^2\int_{\lambda_1}^{\infty}\frac{\lambda\,d\lambda}{(a^2+\lambda^2)^2}[\phi(\infty)-\phi(\lambda)] = \frac{x^2}{a^2+\lambda_1^2}\,\frac{\phi(\infty)-\phi(\lambda_1)}{2}$$
$$-\tfrac{1}{2}\int_{\lambda_1}^{\infty}\frac{x^2}{a^2+\lambda^2}\frac{\lambda\,d\lambda}{f(\lambda)}.$$

Adding the terms in y and z, we have, since $\frac{x^2}{a^2+\lambda_1^2}+\ldots = 1$,

$$V = 2\pi\gamma\rho abc\left[\phi(\infty)-\phi(\lambda_1)-\int_{\lambda_1}^{\infty}\left(\frac{x^2}{a^2+\lambda^2}+\frac{y^2}{b^2+\lambda^2}+\frac{z^2}{c^2+\lambda^2}\right)\frac{\lambda\,d\lambda}{f(\lambda)}\right]$$

$$= 2\pi\gamma\rho abc\int_{\lambda_1}^{\infty}\left(1-\frac{x^2}{a^2+\lambda^2}-\frac{y^2}{b^2+\lambda^2}-\frac{z^2}{c^2+\lambda^2}\right)\frac{\lambda\,d\lambda}{f(\lambda)}. \qquad (19)$$

EXAMPLES.

1. Find the attraction of a homogeneous ellipsoid of revolution round the minor axis (oblate spheroid) on a particle placed on its surface.

Here $a = b$, and $e = e'$ in equations (5), p. 324; therefore

$$X = -\frac{3\gamma Mx}{c^3}\int_0^1\frac{u^2\,du}{(1+e^2u^2)^2}.$$

The integral is most easily found by putting $eu = \tan\theta$. We then find

$$X = -\frac{3\gamma Mx}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Y = -\frac{3\gamma My}{2c^3e^3}\left(\tan^{-1}e - \frac{e}{1+e^2}\right);$$

$$Z = -\frac{3\gamma Mz}{c^3e^3}\left(e - \tan^{-1}e\right).$$

These expressions are of importance in the theory of the figure of the Earth.

2. A homogeneous fluid mass, self-attracting according to the law of nature, is acted upon at every element by a force proportional to

the mass of the element and its distance from an axis passing through the centre of mass of the fluid. Prove that an ellipsoid of revolution round the axis is a possible figure of equilibrium of the fluid.

Let kr be the force emanating from the axis on a unit mass at distance r from the axis. Take the axis as axis of z, and assume the surface of the fluid to be an ellipsoid of revolution whose axes are $c\sqrt{1+e^2}$, $c\sqrt{1+e^2}$, c.

Then the x-component of force on a unit mass on the surface is $(-A+k)x$, where A has the value in example 1. Hence if V is the potential at the surface

$$dV = (-A+k)x\,dx + (-A+k)y\,dy - Cz\,dz,$$

which is zero, since if the potential is not constant over the surface of a fluid, there will be a force in the tangent plane causing a flow from one point to another. Also by differentiating the equation of the surface, we have

$$\frac{x\,dx + y\,dy}{1+e^2} + z\,dz = 0.$$

Hence we must have

$$\frac{-A+k}{C} = -\frac{1}{1+e^2}.$$

Substituting the values of A and C from last example, and putting $M = \frac{4}{3}\pi c^3(1+e^2)\rho$, where ρ is the density of the fluid, this equation gives

$$\frac{ke^3}{2\pi\rho} + 3e = (3+e^2)\tan^{-1}e.$$

Put $k = \frac{4}{3}\pi\rho\,.\,q$; then we have

$$\frac{2qe^3+9e}{3(3+e^2)} - \tan^{-1}e = 0,$$

which determines e, the eccentricity, in terms of q; and c, the least axis, is known from M, the whole mass of the fluid.

There is a major limit to the value of q in order that equilibrium in the ellipsoidal form may be possible; but into the discussion of this, which is somewhat tedious, we do not enter. [See the *Mécanique Céleste*, or Besant's *Hydromechanics*.]

3. If from a solid homogeneous ellipsoid there be removed any complete ellipsoid, find the attraction at a point—(a) inside the remaining mass, (b) inside the ellipsoidal cavity.

The attraction is to be found by considering the cavity to be filled with matter of the same density as that of the rest, and then subtracting the results due to the matter which is imagined to fill the cavity.

Let the axes of the complete ellipsoid be taken as those of reference, and let the axes of the cavity make angles $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ with them. Also let the co-ordinates of the attracted particle with reference to these axes be (x, y, z) and (x', y', z'), respectively, and let the components of attraction along these sets of axes be (X, Y, Z) and (X', Y', Z').

Then
$$X = Ax, \quad Y = By, \quad Z = Cz,$$
where A, B, C are constants; and
$$X' = A'x', \quad Y' = B'y', \quad Z' = C'z',$$
where if the attracted particle is outside the cavity, A', B', C' are variables, but if inside, constants.

The whole force parallel to the axis of x on a unit particle is obviously
$$X - (X' \cos a_1 + Y' \cos a_2 + Z' \cos a_3),$$
with similar expressions for the components along the axes of y and z.

If the attracted particle is inside the cavity, the level surface passing through it is easily found. For, the virtual work of the attraction of the whole ellipsoid is $Xdx + Ydy + Zdz$, or $\frac{1}{2}d(Ax^2 + By^2 + Cz^2)$; and that of the attraction of the small ellipsoid is $X'dx' + Y'dy' + Z'dz'$, or $\frac{1}{2}d(A'x'^2 + B'y'^2 + C'z'^2)$. Hence the level surfaces inside the cavity are given by the equation
$$Ax^2 + By^2 + Cz^2 - A'x'^2 - B'y'^2 - C'z'^2 = \text{const.}$$
They are therefore quadrics.

We could in the same way find the effect due to an ellipsoidal mass which contains in its interior another ellipsoidal mass (or nucleus) of density different from that of the remainder. If ρ and ρ' are the densities of the two portions $(\rho' > \rho)$, imagine the whole to consist of a homogeneous mass of density ρ, and add the effect due to the nucleus, supposed of density $\rho' - \rho$.

4. Prove that an oblate spheroid of uniform density cannot have its own surface for one of its level surfaces.

[The condition that its own surface should be a level surface is $\tan^{-1} e = \dfrac{3e}{3 + e^2}$, which cannot be satisfied by any value of e, except zero.]

5. Prove that a prolate spheroid of uniform density cannot have its own surface for a level surface.

[By putting $e = k\sqrt{-1}$ in the last result, the required condition becomes
$$\tfrac{1}{2}\log\frac{1+k}{1-k} = \frac{3k}{3-k^2};$$
which gives by expansion
$$(3-k^2)(1 + \tfrac{1}{3}k^2 + \tfrac{1}{5}k^4 + \ldots) = 3, \quad \text{or} \quad \frac{1}{3 \cdot 5} + \frac{2k^2}{5 \cdot 7} + \ldots = 0,$$
which is, of course, quite impossible.]

6. Prove that in the spheroid considered in example 2 the resultant attraction at any point on the surface is proportional to the length of the normal between that point and the axis of revolution.

7. Express gravity on the surface of such a spheroid in terms of the latitude.

[The latitude of a point on the surface is the angle made with the plane of the equator by the normal at the point.

If E denotes the value of gravity at the equator, G the value in latitude λ, and ϵ the eccentricity of the generating ellipse,

$$G = \frac{E}{\sqrt{1-\epsilon^2 \sin^2\lambda}};$$

so that if ϵ is small, the increase of gravity at any point above the equatorial value is proportional to $\sin^2$ (latitude).]

8. The components of attraction of a homogeneous ellipsoid at an internal point (x, y, z) being Ax, By, Cz (as in p. 324), prove that

$$A+B+C = -4\pi\gamma\rho,$$

where ρ is the density at the point.

9. From a continuous mass, M, a portion M' is removed and reduced to a state of infinite diffusion; show that the work thus done is

$$\int V\,dm' - \tfrac{1}{2}\int V'\,dm',$$

the integrals being extended throughout the volume of M' (while it forms part of M), V being the Potential at any point of M' due to the complete mass, V' the Potential due to M' alone, and dm' an element of M'.

10. A homogeneous ellipsoid of density ρ and semi-axes a, b, c, contains a concentric spherical cavity of radius r; prove that the work done in filling the cavity with homogeneous matter of density ρ, brought from a state of diffusion, is

$$\frac{8}{15}\gamma\pi^2\rho^2 r^3 \left\{5\,abc\int_0^\infty \frac{\lambda\, d\lambda}{f(\lambda)} - 3r^2\right\},$$

where $f(\lambda) \equiv \sqrt{(a^2+\lambda^2)(b^2+\lambda^2)(c^2+\lambda^2)}$, and verify this result for the case in which the ellipsoid is a sphere (example 23, p. 310).

(Use the value of V in (6), p. 326, and observe that

$$A+B+C = -4\pi\gamma\rho,$$

and also that $\int x^2 dm' = \int y^2 dm' = \int z^2 dm' = \frac{4}{15}\pi\rho r^5$.)

11. If the external level surfaces of any attracting system are confocal ellipsoids defined by the parameter λ in the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1,$$

shew that the potential is given by the equation

$$V = \gamma\frac{M}{2}\int \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where M is the mass of the attracting system.

(Transform the equation $\nabla^2 V = 0$, which holds for all points outside the mass, into a differential equation in which λ is the independent variable. Thus

$$\nabla^2 V = \frac{d^2V}{d\lambda^2}\left\{\left(\frac{d\lambda}{dx}\right)^2 + \left(\frac{d\lambda}{dy}\right)^2 + \left(\frac{d\lambda}{dz}\right)^2\right\} + \frac{dV}{d\lambda}\nabla^2\lambda = 0.$$

But if p is the central perpendicular on the tangent plane to the ellipsoid at any point, we easily find by differentiating the equation connecting λ with x, y, z, that

$$\frac{d\lambda}{dx}=\frac{2p^2x}{a^2+\lambda}, \text{ and that } \left(\frac{d\lambda}{dx}\right)^2+\ldots=4p^2,$$

while $\nabla^2\lambda=2p^2\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)$. Therefore, &c.)

Section IV.—*Green's Equation and Spherical Harmonics.*

346.] Green's Equation. Let U and V be any finite and continuous functions of the co-ordinates of a point in space, and let ∇^2 stand, as usual, for the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

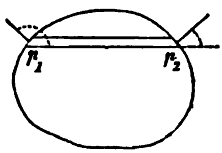

Fig. 287.

Take any closed surface (Fig. 287, or Fig. 274, p. 238); let $d\Omega$ represent an element of volume of the space inside this surface, and let dS represent any element of area of the surface. Then we shall have the following equation, which is due to Green:

$$\int U\nabla^2 V d\Omega=\int U\frac{dV}{dn}dS-\int\left(\frac{dU}{dx}\frac{dV}{dx}+\frac{dU}{dy}\frac{dV}{dy}+\frac{dU}{dz}\frac{dV}{dz}\right)d\Omega, \quad (a)$$

dn in this equation being an element of the normal to the surface drawn outwards, as in Art. 329.

For, $d\Omega=dx\,dy\,dz$, so that the left-hand side is

$$\iiint U\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dx\,dy\,dz.$$

Consider the term $U\frac{d^2V}{dx^2}$ separately. Taking $\int U\frac{d^2V}{dx^2}dx$, and integrating between the extreme values of x, considering y and z both constant—i.e. in the figure, performing a summation along the line $p_1 p_2$ parallel to the axis of x—we get

$$\int U\frac{d^2V}{dx^2}dx=\left(U\frac{dV}{dx}\right)_2-\left(U\frac{dV}{dx}\right)_1-\int\frac{dU}{dx}\frac{dV}{dx}dx, \quad (1)$$

in which the suffixes denote the values of the quantities in brackets at the extreme points p_2 and p_1 of the integration. Fig. 287 represents the line $p_2 p_1$ as meeting the given closed

surface in only two points, but our result holds whatever be the number of these points—observing that it must be an *even* number, since the surface is closed. Fig. 274, p. 238, will represent the more general case if we imagine the line OQ to be parallel to the axis of x; and with this figure the terms outside the sign of integration in (1) would be

$$\left(U\frac{dV}{dx}\right)_2 - \left(U\frac{dV}{dx}\right)_1 + \left(U\frac{dV}{dx}\right)_4 - \left(U\frac{dV}{dx}\right)_3. \qquad (2)$$

The integration along $p_1\,p_2$ is performed in reality along a very slender parallelopiped whose transverse section is $dy\,dz$, and not along a line. Multiplying the different terms of (1) by $dy\,dz$, we have the right-hand side equal to

$$\left(U\frac{dV}{dx}\right)_2 dy\,dz - \left(U\frac{dV}{dx}\right)_1 dy\,dz - dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}dx. \qquad (3)$$

Now if dS_2 is the element of surface cut off by the parallelopiped at p_2 and if λ_2 is the angle (represented by the dotted line) made with the axis of x by the outward-drawn normal at p_2, we have

$$dy\,dz = \cos\lambda_2 . dS_2; \qquad (4)$$

and if dS_1 is the element of area cut off at p_1, while λ_1 is the direction angle of the outward-drawn normal at p_1, *measured in the same sense as at* p_2, we have

$$dy\,dz = -\cos\lambda_1 . dS_1; \qquad (5)$$

with exactly like results in the general figure, Fig. 274, p. 238, the cosines being negative at the points P_1, P_3, and positive at P_2, P_4. Hence (3) becomes

$$\left(U\frac{dV}{dx}\cos\lambda\,dS\right)_2 + \left(U\frac{dV}{dx}\cos\lambda\,dS\right)_1 - dy\,dz\int\frac{dU}{dx}\frac{dV}{dx}dx; \qquad (6)$$

and hence

$$\int U\frac{d^2V}{dx^2}d\Omega = \int U\frac{dV}{dx}\cos\lambda\,dS - \iiint\frac{dU}{dx}\frac{dV}{dx}dx\,dy\,dz, \qquad (7)$$

λ denoting the angle made by the normal at any point with the axis of x.

In the same way, if μ and ν are the angles made by the normal with the axes of y and z, we have

$$\int U\frac{d^2V}{dy^2}d\Omega = \int U\frac{dV}{dy}\cos\mu\,dS - \iiint\frac{dU}{dy}\frac{dV}{dy}dx\,dy\,dz, \qquad (8)$$

$$\int U\frac{d^2V}{dz^2}d\Omega = \int U\frac{dV}{dz}\cos\nu\,dS - \iiint\frac{dU}{dz}\frac{dV}{dz}dx\,dy\,dz. \qquad (9)$$

Adding (7), (8), and (9) together, we obtain the equation (α).

Writing down the value of $\int V\nabla^2 U d\Omega$, and subtracting the result from (α), we obtain

$$\int(U\nabla^2 V - V\nabla^2 U)d\Omega = \int\left(U\frac{dV}{dn} - V\frac{dU}{dn}\right)dS. \qquad (\beta)$$

If U is taken identical with V, we have the result

$$\int V\nabla^2 V d\Omega = \int V\frac{dV}{dn}dS - \int\left[\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right]d\Omega. \quad (\gamma)$$

Green's Equation is probably the most remarkable and powerful analytical result in the whole range of Mathematical Physics. It is put by Sir W. Thomson into the following somewhat generalised form

$$\int U\left(\frac{d\,.\,\phi\frac{dV}{dx}}{dx} + \frac{d\,.\,\phi\frac{dV}{dy}}{dy} + \frac{d\,.\,\phi\frac{dV}{dz}}{dz}\right)d\Omega$$

$$= \int U\phi\frac{dV}{dn}dS - \int\phi(U_1V_1 + U_2V_2 + U_3V_3)d\Omega, \quad (\delta)$$

where ϕ is any function whatever and

$$U_1 = \frac{dU}{dx},\ V_1 = \frac{dV}{dx},\ \text{\&c.}$$

This is at once deducible as before.

Green's equation holds also for the space included between any closed surface S (Fig. 288) and any closed surfaces, M_1, M_2, included by S. In this case the boundary of the space considered is not continuous—that is, starting from any one point, P_1, on the boundary, it is not possible to reach every other point (such as P_4) on the boundary by travelling merely over the boundary itself.

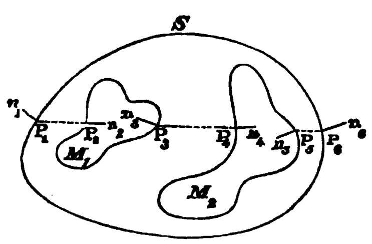

Fig. 288.

The figure represents a line $P_1P_2...P_6$ parallel to the axis of x along which the integration $\int U\frac{d^2V}{dx^2}dx$ is performed, and the lines P_1n_1, P_2n_2, P_3n_3, ... are the elements of the normals drawn outwards from the space considered, i.e. the space included between the contours of S, M_1, and M_2.

The functions U and V may be any whatever—subject to the conditions of being finite, continuous, and (as we shall assume for the present) single-valued.

Take $U = C =$ any constant, for example, and the equation (a) becomes

$$\int \nabla^2 V d\Omega = \int \frac{dV}{dn} dS. \tag{10}$$

If V is the Potential due to any attracting matter,

$$\nabla^2 V = -4\pi\rho\gamma$$

(p. 280), and we have at once the equation

$$-4\pi\gamma M_i = \int N dS, \tag{11}$$

as in p. 264; and if the surface S has all the attracting matter outside it, we have in the same way $\int NdS = 0$ (p. 264); for in Fig. 288 let the contour of M_2 represent any closed surface, let M_1 represent any attracting matter outside this surface, and let S be any surface completely surrounding both. Applying Green's equation (10) to the space included between the surfaces S and M_2 and the contours of these surfaces, we have

$$-4\pi\gamma M_1 = \int NdS + \int N'dS',$$

where N and dS refer to the surface S, and N' and dS' to the surface of M_2. But, ignoring the surface of M_2 altogether, (11) gives

$$-4\pi\gamma M_1 = \int NdS.$$

Hence $\int N'dS' = 0$.

The quantity $-\nabla^2 V$ is called by Clerk Maxwell the *concentration* of V. Hence (10) asserts that if a function has no concentration at any point inside a closed surface, the surface-integral of the normal variation of this function over the surface is zero.

As another example, take $U = V$, and let V be the Potential due to any attracting matter. Then the equation becomes

$$4\pi\gamma \int V dm = -\int V \frac{dV}{dn} dS + \int R^2 d\Omega, \tag{ϵ}$$

where R is the resultant force intensity at any point inside the closed surface, $dm = \rho d\Omega =$ element of mass at any point inside, and γ, as usual, the gravitation constant. Now (Art. 331) the left-hand side of (ϵ) is $8\pi\gamma$ times the Potential Work of the attractive forces of the system, or, in other words, $8\pi\gamma$ times the amount of work done by these forces in bringing the system

from a state of infinite diffusion to its present configuration. Hence the right-hand side is another expression for the same thing. A simpler expression is obtained by taking the closed surface S of infinite size, i.e. every point of it at infinity. Now if none of the attracting matter is infinitely distant, $V=0$ at every point of this infinitely distant surface; nevertheless the integral $\int \frac{dV}{dn} dS$ is finite and $=-4\pi\gamma M$, where M is the quantity of the given matter. Hence $\int V\frac{dV}{dn} dS$ over this surface must be zero, and we have

$$4\pi\gamma\int V dm = \int R^2 d\Omega, \qquad (\zeta)$$

the integral on the right-hand side being taken all through the attracting matter and through infinite space outside the attracting matter, and the work required to reduce the given self-attracting system to a state of infinite diffusion is

$$\frac{1}{8\pi\gamma}\int R^2 d\Omega, \qquad (\eta)$$

the integration extending *through all space outside the matter, and through the matter itself.*

EXAMPLES.

1. Take the case of a homogeneous solid sphere of radius a. Then at any point inside $V=2\pi\gamma\rho(a^2-\frac{1}{3}r^2)$, r being the distance of the point from the centre. We may take $dm=4\pi\rho r^2 dr$, and we find $\frac{1}{2}\int V dm = \frac{3}{5}\gamma\frac{M^2}{a}$, where $M=$ mass of sphere.

At any external point $R=\frac{\gamma M}{r^2}$; therefore

$$\int R^2 d\Omega = \gamma^2 M^2 \iiint \frac{1}{r^2} dr\, d\mu\, d\phi = 4\pi\gamma^2 M^2 . \frac{1}{a}.$$

At any internal point $R=-\frac{4}{3}\pi\gamma\rho r$, $\therefore$ $\int R^2 d\Omega = \frac{4}{5}\pi\gamma^2 M^2 . \frac{1}{a}$; and the sum of these two integrals divided by $8\pi\gamma$ gives the same value of the Potential work as before. (See p. 310.)

2. Supposing that a sphere of water is brought together by mutual attractions of particles from a state of infinite diffusion, find its radius if the amount of work done by these forces is sufficient to raise its temperature 1°C.

Let a centimètres be its radius. Then the number of ergs done by the forces is $\frac{3}{5}\gamma M^2 . \frac{1}{a}$, where $M=$ its mass in grammes $=\frac{4}{3}\pi a^3$.

But 1 water-gramme-centigrade degree is equivalent to 42×10^6 ergs (Joule's *Dynamical Equivalent* of heat). Hence the heat, in ergs, required to raise M grammes through 1° is $42\times10^6\times M$. Therefore

$$\tfrac{3}{5}\gamma M^2\cdot\frac{1}{a}=42\times10^6\times M;$$

and we know that $\gamma=\dfrac{1}{1543\times10^4}$ dynes (Art. 321);

$$\therefore\ a=\tfrac{1}{2}\sqrt{\frac{210\times1543}{\pi}}\times10^5 \text{ centimètres}$$

$$=16\times10^6 \text{ (roughly).}$$

Now the Earth's radius $=637\times10^6$ cms.; therefore the diameter of the required water sphere $=\dfrac{\text{Earth's diameter}}{40}$, roughly.

3. If any surface, S, enclosing a given distribution of mass is a surface of zero potential for this mass, the potential of the system is constantly zero at all points outside S.

Draw an infinitely distant sphere enclosing the system, and apply Green's equation, taking $U=V$, to the space between this sphere and the given surface S. The volume-integral $\int V\nabla^2V\,.\,d\Omega$ taken through this space is zero, since ∇^2V is everywhere zero. Also the two surface-integrals $\int V\dfrac{dV}{dn}dS$, one taken over S and the other over the infinitely distant sphere, both vanish—the former evidently, the latter because V is of the order $\dfrac{dm}{r}$ while $\dfrac{dV}{dn}$ is of the order $\dfrac{dm}{r^2}$, and dS is of the type $r^2d\mu d\phi$, so that the infinitely great value of r reduces to zero each term $V\dfrac{dV}{dn}dS$. Hence Green's equation reduces to $\int R^2d\Omega=0$, where, as in (ϵ), Art. 346, R is the resultant force-intensity at any point in the space considered; $\therefore\ R=0$ at each point, i.e. V is constant, and equal to zero everywhere.

4. If for each of two different material systems, M and M', a certain surface, S, which encloses both, is a surface of constant potential, all the external level surfaces of M are also level surfaces of M'.

For, let A be the constant value of the potential of M on S, and let A' be the constant value for M' on S. Then, if we increase the density at every point of M' in the constant ratio $\dfrac{A}{A'}$, we obtain a mass system occupying the position of M', whose total quantity is $\dfrac{A}{A'}M'$ and whose potential on S is A. Reverse the sign of every element of this new mass, and take this reversed system conjointly with M. We then have a mass system, $M-\dfrac{A}{A'}M'$, producing constant zero potential over the surface S, and therefore at every point

outside S, by last example. Hence every level surface of M between S and infinity is also a level surface of M', and the ratio of the potentials is, on all, $\frac{A}{A'}$.

Thus is proved the equivalence of the ellipsoidal shell, qp', Fig. 286, with the shell Qs so far as attraction at all points outside both, or at the outer surface of the latter, is concerned.

5. If two different masses of equal amounts have the same external level surfaces, prove that $\iiint \rho\, U\, dx\, dy\, dz$ is the same for both, where U is any function satisfying Laplace's equation.

By example 16, p. 309, we see that their Potentials must be identical at all external points. Let V be the Potential on any common level surface. Then applying Green's equation (β), p. 334, to the volume and surface of this level surface, we have for one of the masses

$$-4\pi\gamma\int U\rho\, d\Omega = \int U\frac{dV}{dn}\,dS - V\int\frac{dU}{dn}\,dS.$$

Now since U has no concentration inside the surface (p. 335) we have $\int\frac{dU}{dn}\,dS = 0$; also $\rho\, d\Omega = dm =$ the element of mass; therefore

$$\int U\,dm = -\frac{1}{4\pi\gamma}\int U\frac{dV}{dn}\,dS.$$

For the other mass $\frac{dV}{dn}$ is the same as for the first, since their Potentials are equal at all points. Hence for it

$$\int U\,dm' = -\frac{1}{4\pi\gamma}\int U\frac{dV}{dn}\,dS,$$

which gives $\int U\,dm = \int U\,dm'$, as required.

If the two masses are not equal, these integrals are proportional to their amounts; or

$$\frac{\int U\,dm}{M} = \frac{\int U\,dm'}{M'}. \tag{a}$$

6. If two different masses have the same external level surfaces, they have the same centre of mass and the same principal axes at this point, and their Ellipsoids of gyration are confocal.

For, let $U = x$ in (a), and we have $\bar{x} = \bar{x}'$. Similarly if $U = y$, and $U = z$, we obtain $\bar{y} = \bar{y}'$, &c. We may take the centre of mass as origin.

Secondly, let $U = xy$ (which satisfies $\nabla^2 U = 0$); then

$$\frac{1}{M}\int xy\,dm = \frac{1}{M'}\int xy\,dm';$$

so that if the products of inertia round the axes of co-ordinates vanish for the first mass, they also vanish for the second. Take the principal axes as axes of co-ordinates.

Thirdly, let $U = y^2 + z^2 - 2x^2$, and if $A, \ldots A', \ldots$ are the principal moments of inertia, we have

$$\frac{1}{M}(B + C - 2A) = \frac{1}{M'}(B' + C' - 2A').$$

Two similar equations also follow. Hence

$$\frac{A}{M}=\frac{A'}{M'}+\lambda;\ \frac{B}{M}=\frac{B'}{M'}+\lambda;\ \frac{C}{M}=\frac{C'}{M'}+\lambda.$$

7. Prove that the mean value of any continuous function, ϕ, taken over a sphere of radius a exceeds the value which the function has at the centre of the sphere by

$$\frac{1}{4\pi}\int\left(\frac{1}{r}-\frac{1}{a}\right)\nabla^2\phi\,.\,d\Omega,$$

this integral being taken through the volume of the sphere, and r being the distance of any point from the centre.

Round the centre of the sphere describe a circle of extremely small radius, b, and apply Green's equation to the space between the two spheres. This space has for boundary the surfaces of the two spheres. Let $\frac{1}{r}-\frac{1}{a}$ be taken as U. Then from (β), Art. 346, since $\nabla^2 U=0$,

$$\int\left(\frac{1}{r}-\frac{1}{a}\right)\nabla^2\phi\,.\,d\Omega=\int_1\left(\frac{1}{r}-\frac{1}{a}\right)\frac{d\phi}{dr}dS-\int_2\left(\frac{1}{r}-\frac{1}{a}\right)\frac{d\phi}{dr}dS$$
$$-\int\phi\,\frac{d\frac{1}{r}}{dr}dS+\int_2\phi\,\frac{d\frac{1}{r}}{dr}dS,$$

the integrals with suffix 1 referring to the surface of the outer sphere (for which $dn=dr$), and those with suffix 2 to the surface of the inner (for which $dn=-dr$). Now the first integral on the right-hand side is zero, $\because\ r=a$; the third integral $=\frac{1}{a^2}\int\phi dS$; the fourth $=-\frac{1}{b^2}\int\phi dS=-4\pi\phi_0$ (where ϕ_0 is the value of ϕ at the centre) because ϕ at every point on the surface of the small sphere is very nearly constant, and $\int dS=4\pi b^2$. Also the second integral is zero, because $\frac{d\phi}{dr}$ is very nearly the same at all points on the small sphere, and $r=b$ at all points, so that this integral

$$=-\left(\frac{1}{b}-\frac{1}{a}\right).4\pi b^2.\left(\frac{d\phi}{dr}\right)_0,$$

which is infinitely small since b is so. Hence we have

$$\int\left(\frac{1}{r}-\frac{1}{a}\right)\nabla^2\phi d\Omega=\frac{1}{a^2}\int\phi\,dS-4\pi\phi_0,$$

which gives the desired result.

If ϕ is the Potential of matter wholly external to the sphere, we have the result in example 12, p. 306.

If there is matter internal as well as external to the sphere, it can be shown at once that the mean value of the Potential on the surface is equal to the Potential at the centre due to the external mass, plus

the Potential which would be produced at the centre by distributing the internal mass as a shell over the surface; in other words,

$$\frac{1}{4\pi a^2}\int V\,dS = V_0^{(e)} + \gamma\frac{M^{(i)}}{a}.$$

8. If ϕ is any function of the co-ordinates of a point, P, and round P as centre a small sphere, of radius r, be described, prove that if $\bar{\phi}$ is the mean value of ϕ (i.e., mean volume-value) for all points within the sphere,

$$\bar{\phi} = \phi + \tfrac{1}{10}r^2\nabla^2\phi.$$

347.] **Remarkable Consequence of Green's Equation.** The first result that we shall deduce from Green's Equation is the following, which is of fundamental importance in the theory of Attraction—

There cannot be two different functions which both satisfy Laplace's equation at every point of a closed region of space and which have both the same value at **every** *point of the surface or surfaces bounding this region.*

If possible, let there be two different functions V and U such that at every point in the region enclosed by the surface in Fig. 287, p. 332, or at every point in the region included between the surfaces of S, M_1, and M_2 in Fig. 288, we have

$$\nabla^2 V = 0 \text{ and } \nabla^2 U = 0,$$

and also such that $V = U$ at *every* point on the bounding surface in Fig. 287, and at every point on the surface S, every point on the surface of M_1, and every point on the surface of M_2 in Fig. 288,

Then our theorem is that V and U must be identical.

For, by Green's equation, if ϕ is *any* function,

$$\int\phi\nabla^2\phi\,d\Omega = \int\phi\,\frac{d\phi}{dn}dS - \int\Big[\Big(\frac{d\phi}{dx}\Big)^2 + \Big(\frac{d\phi}{dy}\Big)^2 + \Big(\frac{d\phi}{dz}\Big)^2\Big]d\Omega, \quad (1)$$

in which $d\Omega$ is any element of volume of the space considered and dS an element of area of the boundary.

Let $\phi \equiv V - U$. Then by hypothesis $\nabla^2\phi = 0$ at every point in the volume, and $\phi = 0$ at every point on the boundary; hence (1) becomes

$$\int\Big[\Big(\frac{d\phi}{dx}\Big) + \Big(\frac{d\phi}{dy}\Big)^2 + \Big(\frac{d\phi}{dz}\Big)^2\Big]d\Omega = 0.$$

Now this asserts that a summation of a sum of squares is zero, which cannot be unless every term in the summation $= 0$. Hence at *every* point in the volume considered we must have

$$\frac{dV}{dx} = \frac{dU}{dx}; \quad \frac{dV}{dy} = \frac{dU}{dy}; \quad \frac{dV}{dz} = \frac{dU}{dz};$$

and these require $V \equiv U$ at every point of the included space.

The application of this result to the theory of Potential is obvious. M_1, M_2 may be any distributions of attracting matter and S an infinitely distant surface. If no portion of the attracting matter is contemplated as at infinity, the Potential has a zero value at every point on S. Then the Theorem just proved, when applied to the region included between the infinitely distant surface and the contours of M_1 and M_2—i.e. to the whole of the space external to the masses M_1 and M_2—comes to this: if we know any function, V, of the co-ordinates (x, y, z) which vanishes for all points at infinity, which at every point on the contours of M_1 and M_2 has the value of the Potential of these masses at the point, and which at every point outside these masses satisfies Laplace's equation $\nabla^2 V = 0$; then V is the Potential produced by the masses at any point (x, y, z) of the space external to them.

For, the Potential satisfies all these conditions, and as there is only one function which can do so, the given function, V, must be the Potential.

348.] **Central Solid of Revolution. Theorem of Legendre.** For the case in which the attracting matter forms any central solid of revolution we shall now prove the following remarkable result which was first proved by Legendre: *If in the case of any body which is symmetrical, both as to shape and to density, about an axis, we know a Potential function (of x, y, z or any other co-ordinates which determine the position of a point) which for all points on the axis outside the body is the Potential of the body at these points, this function is the Potential at every point outside the body.*

[The expression 'Potential function' is here used for brevity to signify one satisfying Laplace's equation, $\nabla^2\phi = 0$.]

Legendre's proof of this theorem (which is that commonly employed) will be subsequently given. The following seems to be more simple and elementary.

The Potential for this case must be simply a function of the two cylindrical co-ordinates z, ζ (Art. 329). Hence if V is the Potential at any point,

$$\frac{d^2 V}{dz^2} + \frac{d^2 V}{d\zeta^2} + \frac{1}{\zeta}\frac{dV}{d\zeta} = 0. \tag{1}$$

Let U be the function which we know, and which satisfies the conditions above enunciated. Then U also satisfies (1). Let $\phi = V - U$; then ϕ also satisfies the equation (1), or

$$\zeta\left(\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{d\zeta^2}\right) + \frac{d\phi}{d\zeta} = 0. \tag{2}$$

Now all along the axis of z we have $\phi = 0$, and therefore

$$\frac{d\phi}{dz} = 0, \quad \frac{d^2\phi}{dz^2} = 0, \quad \frac{d^3\phi}{dz^3} = 0, \ldots.$$

With these conditions, and with the condition that (2) holds for all values of z and ζ we wish to show that all the differential coefficients of ϕ, such as $\frac{d^{m+n}\phi}{dz^m d\zeta^n}$, vanish at all points on the axis of z.

Firstly, at all points on the axis of z (since $\zeta = 0$) we have by (2)

$$\frac{d\phi}{d\zeta} = 0.$$

Again, differentiating (2) with respect to ζ and putting $\zeta = 0$, we have

$$\frac{d^2\phi}{d\zeta^2} = 0.$$

Differentiating (2) n times with respect to ζ, we have by Leibnitz's Theorem

$$\zeta\left(\frac{d^{n+2}\phi}{dz^2 d\zeta^n} + \frac{d^{n+2}\phi}{d^{n+2}\zeta}\right) + n\frac{d^{n+1}\phi}{dz^2 d\zeta^{n-1}} + (n+1)\frac{d^{n+1}\phi}{d\zeta^{n+1}} = 0,$$

so that at all points on the axis of z

$$n\frac{d^2}{dz^2}\left(\frac{d^{n-1}\phi}{d\zeta^{n-1}}\right) + (n+1)\frac{d^{n+1}\phi}{d\zeta^{n+1}} = 0. \tag{3}$$

Hence if at all points on the axis $\frac{d^{n-1}\phi}{d\zeta^{n-1}} = 0$, we shall have $\frac{d^{n+1}\phi}{d\zeta^{n+1}} = 0$. But $\frac{d\phi}{d\zeta}$ and $\frac{d^2\phi}{d\zeta^2}$ have both been proved to vanish at all points on the axis, and therefore all the differential coefficients of ϕ with respect to ζ vanish on the axis; and hence also, on account of the independence of the order of differentiation, all of the form $\frac{d^{m+n}\phi}{dz^m d\zeta^n}$ also vanish on the axis.

Now, by Maclaurin's Theorem, if $\phi = f(z, \zeta)$, we have

$$\phi = \phi_0 + z\left(\frac{d\phi}{dz}\right)_0 + \zeta\left(\frac{d\phi}{d\zeta}\right)_0 + \frac{1}{1.2}\left\{z^2\left(\frac{d^2\phi}{dz^2}\right)_0 + 2z\zeta\left(\frac{d^2\phi}{dz\,d\zeta}\right)_0 + \zeta^2\left(\frac{d^2\phi}{d\zeta^2}\right)_0\right\} + \ldots$$

where ϕ_0, $\left(\frac{d\phi}{dz}\right)_0$, ... mean the values of ϕ and its differential coefficients when $(0, 0)$ are put for (z, ζ). Hence, by what has

just been proved, ϕ is zero everywhere—that is, V is identical with U.

The same proof shows that if we know a Potential function, U, which at every point inside the attracting mass satisfies the equation $\nabla^2 U = -4\pi\gamma\rho$, and if U for all points on the axis of symmetry is the Potential, it is the Potential for *all* points in the mass. For, putting $\phi \equiv V - U$, we have still the equation (2), with all its consequences, for ϕ; and, as before, we prove $\phi = 0$ for all points.

349.] **Laplacians.** Let O be any fixed origin, P a point whose polar co-ordinates are (r, θ, ϕ) and P' a point whose co-ordinates are (r', θ', ϕ'). Then, denoting, as previously, $\cos\theta$ by μ and $\cos\theta'$ by μ', the reciprocal of the distance between P and P' is

$$\frac{1}{\sqrt{r^2 - 2rr'\{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')\} + r'^2}}. \qquad (a)$$

Now since the reciprocal of the distance between P and any other point is the type of a Potential function ($\frac{m}{PP'}$ is, in fact, the Potential at P due to a mass m condensed at P'), it follows that the expression (a) satisfies the equation $\nabla^2\left(\frac{1}{PP'}\right) = 0$ where $\nabla^2 \equiv \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, or its equivalent operation in (r, μ, ϕ), or in (z, ζ, ϕ); and again that $\nabla'^2\left(\frac{1}{PP'}\right) = 0$, where ∇'^2 signifies the same operations with reference to the co-ordinates of P'.

Again, the expression (a) may be developed in an infinite series proceeding by powers of the ratio $\frac{r'}{r}$, or of the ratio $\frac{r}{r'}$, the coefficients of these successive powers being functions of μ, μ', ϕ, ϕ'. Moreover, *any one coefficient*—as, for instance, that of $\left(\frac{r'}{r}\right)^i$—is *a rational integral function of* μ, $\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$, *and is the very same function of* μ', $\sqrt{1-\mu'^2}\cos\phi'$, $\sqrt{1-\mu'^2}\sin\phi'$. It is, again, obviously the same whether $\frac{1}{PP'}$ is developed in powers of $\frac{r}{r'}$ or of $\frac{r'}{r}$.

Let this development be

$$\frac{1}{PP'} = \frac{1}{r}\left\{L_0 + L_1\frac{r'}{r} + L_2\frac{r'^2}{r^2} + \dots + L_i\frac{r'^i}{r^i} + \dots\right\}. \qquad (\beta)$$

The coefficients of this development possess very remarkable properties, and we shall call them *Laplacians*, after Laplace, to whom their employment is due.

Thus L_i is the Laplacian of the i^{th} degree. We may speak of it as the Laplacian of the i^{th} degree for the two points P, P', whose angular co-ordinates are involved in it.

If, regarding r', μ', ϕ' as constant, we perform the operation ∇^2 on the right-hand side of (β), since the result is zero for all values of r and r', the coefficients of the several powers must all separately vanish. Thus we must have

$$\nabla^2\frac{L_i}{r^{i+1}} = 0. \qquad (\gamma)$$

Similarly, if, regarding r, μ, ϕ as constant, we perform the operation ∇'^2, we must have $\nabla'^2(r'^i L_i) = 0$, and therefore of course, by symmetry, $\nabla^2(r^i L_i) = 0.$ (δ)

Substituting $\frac{L_i}{r^{i+1}}$ for V in (δ), p. 281, we have the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dL_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2L_i}{d\phi^2} + i(i+1)L_i = 0, \qquad (\epsilon)$$

and the substitution of r^iL_i for V gives exactly the same equation.

The value of L_i can, of course, be found by simple binomial expansion of (a); but such a method is very tedious, and we shall adopt a different one.

Let λ be put for $\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$, and let

$$(1-2\lambda h + h^2)^{\frac{1}{2}} = 1 - xh. \qquad (1)$$

This gives

$$x = \lambda + h\frac{x^2-1}{2},$$

from which we can expand x in ascending powers of h by Lagrange's Theorem (Williamson's *Diff. Cal.*, chap. VII.).

Thus

$$x = \lambda + \frac{h}{1}\frac{\lambda^2-1}{2} + \frac{h^2}{1\,.\,2}\frac{d}{d\lambda}\left(\frac{\lambda^2-1}{2}\right)^2 + \dots$$
$$+ \frac{h^i}{\underline{|i}}\frac{d^{i-1}}{d\lambda^{i-1}}\left(\frac{\lambda^2-1}{2}\right)^i + \dots. \qquad (2)$$

Now from (1) we have $\frac{dx}{d\lambda} = (1 - 2\lambda h + h^2)^{-\frac{1}{2}}$, which by hypothesis, when expanded in powers of h is

$$L_0 + L_1 h + L_2 h^2 + \dots + L_i h^i + \dots.$$

Differentiating (2) with respect to λ, and identifying the coefficients of h^i in the two values of $\frac{dx}{d\lambda}$, we have

$$L_i = \frac{1}{2^i \lfloor i} \frac{d^i (\lambda^2 - 1)^i}{d\lambda^i}. \qquad (\zeta)$$

By actually expanding $(\lambda^2 - 1)^i$ and differentiating, we have

$$L_i = \frac{1}{2^i \lfloor i}[2i(2i-1)\dots(i+1)\lambda^i - \text{\&c.}],$$

which shows that L_i is a rational integral function of the i^{th} degree of μ, $\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$, and the very same function of μ',

$$\sqrt{1-\mu'^2}\cos\phi', \ \sqrt{1-\mu'^2}\sin\phi'.$$

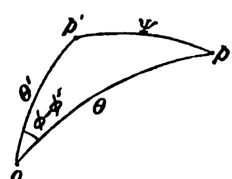

Fig. 289.

In the figure (Fig. 289) let the spherical triangle be that in which a sphere is intersected by the axis of z (from which θ and θ' are measured), and the lines OP and $O'P'$; these lines meeting the surface in o, p, p', respectively. The point o being the pole from which angles are measured, the function $\frac{1}{PP'}$ satisfies the differential equation

$$\left[\frac{d}{dr}\cdot r^2 \frac{d}{dr} + \frac{d}{d\mu}\cdot(1-\mu^2)\frac{d}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2}{d\phi^2}\right]\frac{1}{PP'} = 0;$$

but if p' is taken as pole, the expression for PP' involves only OP, OP', and $\cos\psi$ (or λ), without any term in longitude. Hence we have

$$\left[\frac{d}{dr}\cdot r^2 \frac{d}{dr} + \frac{d}{d\lambda}\cdot(1-\lambda^2)\frac{d}{d\lambda}\right]\frac{1}{PP'} = 0;$$

and putting here for $\frac{1}{PP'}$ the development (β), and equating to zero the coefficients of the several powers of r, we have

$$\frac{d}{d\lambda}\left\{(1-\lambda^2)\frac{dL_i}{d\lambda}\right\} + i(i+1)L_i = 0. \qquad (\eta)$$

L_i being given by (ζ), and satisfying (η), we conclude generally that any function, X, of the form

$$X = a\,\frac{d^i (x^2-1)^i}{dx^i}, \tag{3}$$

where a does not involve x, will satisfy the equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dX}{dx}\right\} + i(i+1)X = 0. \tag{4}$$

The value of L_i as given by (ζ) is not of much practical use. To make it useful, it must be exhibited as a series of cosines of multiples of $\phi - \phi'$ (which we may denote by ω) thus:

$$L_i = M_0 + M_1 \cos\omega + M_2 \cos 2\omega + \ldots + M_i \cos i\omega, \tag{5}$$

the series ending with $\cos i\omega$, because the highest power of $\cos\omega$ in L_i is the i^{th}, and we know by elementary Trigonometry that

$$2^{i-1}\cos^i\omega = \cos i\omega + i\cos(i-2)\omega + \frac{i(i-1)}{1\,.\,2}\cos(i-4)\omega + \ldots.$$

Laplace deduces L_i in the desired form (5) by elementary algebraic processes; but as we prefer to present it in a more succinct form than that given by Laplace, we shall turn in the next Article to the consideration of functions, generally, which satisfy the equation $\nabla^2 V = 0$.

It is to be observed that if $\mu = \mu'$ and $\phi = \phi'$, the points p and p' (Fig. 289) coincide, and $PP' = r - r'$, so that every Laplacian becomes equal to unity—as is verified by putting $\lambda = 1$ in (ζ).

In general, any Laplacian for two points, p and p', has reference to a certain fixed point or *pole*, o, and is a function of the position-angles (θ, ϕ, θ', ϕ') of these points with regard to the pole. If either point, as p, is taken as pole, the Laplacian (being always simply a function of $\cos\psi$) will reduce to a function of μ' alone, and its value is then obtained by taking $\mu = 1$ and $\phi - \phi' = 0$ in its *general* expression. In this case—i.e. when one of the two related points in the Laplacian is the pole—the Laplacian is called a *Legendre's coefficient*, which therefore expresses exactly the same thing as the Laplacian, but by a transformation of co-ordinates. In this special form these functions were employed by Legendre before Laplace used them in the general form.

350.] Spherical Harmonics. *To determine a homogeneous function of x, y, z, of the most general form, which satisfies the equation* $\nabla^2 V = 0$.

Firstly, such a function involves $2i+1$ arbitrary constants, because it contains $\frac{1}{2}(i+1)(i+2)$ terms; and $\nabla^2 V$, being a rational integral function of degree $i-2$, will contain $\frac{1}{2}i(i-1)$ separate terms. The condition that $\nabla^2 V$ should vanish for all values of x, y, z is that the coefficient of each of these $\frac{1}{2}i(i-1)$ is zero; so that we have $\frac{1}{2}i(i-1)$ equations between the $\frac{1}{2}(i+1)(i+2)$ coefficients. This leaves $2i+1$ of them independent.

Changing from Cartesian to polar co-ordinates, such a function will be of the form $r^i Y_i$, where Y_i is, of course, a rational, integral, and homogeneous function of μ, $\sqrt{1-\mu^2}\cos\phi$, and $\sqrt{1-\mu^2}\sin\phi$, and Y_i will satisfy the equation (ϵ), p. 344, or

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 Y_i}{d\phi^2} + i(i+1)\,Y_i = 0. \qquad (1)$$

We can now show that a value of Y_i which is the product of a function of μ only and a function of ϕ only can be found to satisfy this equation *. Let $Y_i = M\Phi$, where M is a function of μ only and Φ a function of ϕ only. Then we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dM}{d\mu}\right\} + i(i+1)M + \frac{M}{1-\mu^2}\cdot\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \qquad (2)$$

Assume
$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \qquad (3)$$
where n is a constant. Then

$$\Phi = A\cos n\phi + B\sin n\phi. \qquad (4)$$

Equation (2) for M now becomes, putting k for $i(i+1)$,

$$(1-\mu^2)\frac{d^2 M}{d\mu^2} - 2\mu\frac{dM}{d\mu} + \left(k - \frac{n^2}{1-\mu^2}\right)M = 0. \qquad (5)$$

Now if $v \equiv \dfrac{d^i(\mu^2-1)^i}{d\mu^i}$, we have shown (last Art.) that

$$(1-\mu^2)\frac{d^2 v}{d\mu^2} - 2\mu\frac{dv}{d\mu} + kv = 0; \qquad (6)$$

and we proceed to show that χ can be determined so that the value $M = \chi\dfrac{d^n v}{d\mu^n}$ will satisfy the equation (5).

---

* This method is found in Ferrers's *Spherical Harmonics*, p. 78, a work which ought to be studied by the student who desires to pursue this subject further.

For brevity denote $\dfrac{d^n v}{d\mu^n}$ by v_n, $\dfrac{d^{n+1} v}{d\mu^{n+1}}$ by v_{n+1}, $\dfrac{d\chi}{d\mu}$ by χ', &c. Then (5) becomes

$$(1-\mu^2)\chi v_{n+2}+2\{(1-\mu^2)\chi'-\mu\chi\}v_{n+1}$$
$$+\left\{(1-\mu^2)\chi''-2\mu\chi'+\left(k-\frac{n^2}{1-\mu^2}\right)\chi\right\}v_n=0. \quad (7)$$

Differentiate (6) n times, employing the theorem of Leibnitz. Then

$$(1-\mu^2)v_{n+2}-2(1+n)\mu v_{n+1}+\{k-n(n+1)\}v_n=0. \quad (8)$$

Now identifying (7) and (8), if possible, we have

$$(1-\mu^2)\chi'+n\mu\chi=0, \quad (9)$$

$$(1-\mu^2)\chi''-2\mu\chi'+n\left\{n+1-\frac{n}{1-\mu^2}\right\}\chi=0; \quad (10)$$

and since (10) is deducible from (9) by differentiation, the identification of (7) and (8) is possible. From (9) we have

$$\chi=a(1-\mu^2)^{\frac{n}{2}},$$

where a is any constant. Hence

$$M=a(1-\mu^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}, \quad (\alpha)$$

and the function

$$(1-\mu^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}(A\cos n\phi+B\sin n\phi), \quad (\beta)$$

where n may obviously be *any integer from* 0 *to* i, satisfies the equation (1); and this function when multiplied by r^i is the type of rational integral functions of x, y, z satisfying Laplace's equation $\nabla^2 V=0$.

All such functions are called *Spherical Harmonics.*

The coefficients of the various powers of $\dfrac{r'}{r}$ in the expansion of $\dfrac{1}{PP'}$, which we have spoken of as *Laplacians*, are, of course, Spherical Harmonics particularised.

The function $r^i Y_i$ (which is a homogeneous function of x, y, z) is called a Solid Spherical Harmonic, or simply a Solid Harmonic, of the i^{th} degree; while the portion Y_i, which is a function of μ and ϕ, is called a Surface Harmonic.

Again (see Art. 329) corresponding to a Solid Spherical Harmonic $r^i Y_i$ of *positive* degree, i, there is a Solid Spherical Harmonic, $\dfrac{Y}{r^{i+1}}$, of negative degree, $-(i+1)$.

Any expression of the form (β) is called a *Tesseral Surface Harmonic* of degree i and order n.

When $n = 0$, the Tesseral Harmonic becomes

$$\frac{d^i(\mu^2-1)^i}{d\mu^i},$$

multiplied by a factor independent of μ, and this is called a *Zonal Harmonic* of the i^{th} degree. The Zonal Harmonic of the i^{th} degree becomes identical in form with the Laplacian when p' is taken as pole (Fig. 289), and in order that it may assume the value unity when $\mu = 1$, we take

$$P_i = \frac{1}{2^i \lfloor i} \frac{d^i(\mu^2-1)^i}{d\mu^i}, \qquad (\gamma)$$

(p. 346) where we use P_i to denote the Zonal Harmonicof the i^{th} degree.

It is evident that the sum of all such terms as (β), each multiplied by an arbitrary constant, n receiving all values from 0 to i both inclusive, will satisfy (1); and that this sum of terms gives us a function, Y_i, involving $2i+1$ arbitrary constants. It is, therefore, the function sought.

It is thus seen that the Zonal Harmonic P_i is the base or *source* of the general Spherical Harmonic of the i^{th} degree. Thus, for example, the Spherical Harmonic of the 3rd degree is derived from the source

$$\frac{d^3(\mu^2-1)^3}{d\mu^3},$$

i.e. from $120\mu^3 - 72\mu$, or, neglecting a numerical factor, from $5\mu^3 - 3\mu$; and this Harmonic will be the sum of the terms obtained by giving n the values 0, 1, 2, 3 in the expression

$$(1-\mu^2)^{\frac{n}{2}} \frac{d^n}{d\mu^n}(5\mu^3 - 3\mu) \, . \, (A\cos n\phi + B\sin n\phi).$$

It is therefore of the general form

$$A_0(5\mu^3 - 3\mu) + (1-\mu^2)^{\frac{1}{2}}(5\mu^2-1)(A_1\cos\phi + B_1\sin\phi)$$
$$+ (1-\mu^2)\mu(A_2\cos 2\phi + B_2\sin 2\phi)$$
$$+ (1-\mu^2)^{\frac{3}{2}}(A_3\cos 3\phi + B_3\sin 3\phi),$$

the coefficients $A_0, A_1, \ldots$ being all arbitrary constants.

The corresponding Solid Harmonic is obtained by multiplying this by r^3.

The homogeneity of the expression for Y_i as a function of μ,

$\sqrt{1-\mu^2}\cos\phi$, $\sqrt{1-\mu^2}\sin\phi$ may not be at once apparent. For example, the term $A_0(5\mu^3-3\mu)$ comes from the function

$$A_0\{5z^3-3z(x^2+y^2+z^2)\}, \text{ or } A_0(2z^2-3x^2-3y^2)z,$$

from which the term in ϕ disappears in consequence of the relation $\sin^2\phi+\cos^2\phi=1$.

We are now in a position to express the Laplacian L_i. Since it is a spherical surface harmonic of the i^{th} degree both in the co-ordinates (μ, ϕ) and in the co-ordinates (μ', ϕ') and involves both in identically the same way, its general term must be

$$A_n(1-\mu^2)^{\frac{n}{2}}(1-\mu'^2)^{\frac{n}{2}}\frac{d^nP_i}{d\mu^n}\frac{d^nP_i'}{d\mu'^n}\cos n(\phi-\phi'), \tag{δ}$$

where A_n is a factor i dependent of μ, μ', ϕ, ϕ'; and L_i is the sum of all such terms obtained by giving n values from 0 to i, inclusive.

For the purpose of actual calculation, it will be better to write the coefficient of $\cos n(\phi-\phi')$ in the form

$$C_n(1-\mu^2)^{\frac{n}{2}}(1-\mu'^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}\frac{d^{i+n}(\mu'^2-1)^i}{d\mu'^{i+n}}. \tag{ϵ}$$

Since the determination of C_n is merely the analytical process of identifying the expression (ϵ) with the coefficient of $\cos n(\phi-\phi')$ in the value of L_i given in (ζ), Art. 349, we may obviously suppose $\mu=\mu'$. In this case (ϵ) becomes

$$C_n(1-\mu^2)^n\left[\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}\right]^2; \tag{11}$$

and the highest term in μ in this expression is

$$C_n(-1)^n(2i\,.\,2i-1\ldots i-n+1)^2\mu^{2i}. \tag{12}$$

Now using ω for $\phi-\phi'$, in this case

$$\lambda=\mu^2+(1-\mu^2)\cos\omega;$$

$$\therefore\ \lambda^2-1=\left(2\sin\frac{\omega}{2}\right)^2(\mu^2-1)\left(\mu^2\sin^2\frac{\omega}{2}+\cos^2\frac{\omega}{2}\right)$$

$$=\left(2\sin\frac{\omega}{2}\right)^2\left(\xi^2\sin^2\frac{\omega}{2}+\xi\right),$$

if we put $\mu^2-1=\xi$.

Again, $\dfrac{d}{d\lambda}=\dfrac{1}{2\sin^2\frac{\omega}{2}}\cdot\dfrac{d}{d\xi}$. Hence the value of L_i becomes

$$\frac{1}{\lfloor i}\frac{d^i}{d\xi^i}\left(\xi^2\sin^2\frac{\omega}{2}+\xi\right)^i. \tag{13}$$

We shall derminine C_n by equating the coefficient of the highest power of μ (or of ξ) in the coefficient of $\cos n\omega$ in (13) to the expression (12). Now obviously the highest term in ξ in (13) is

$$\frac{2i\,.\,2i-1\ldots i+1}{\underline{|i}}\sin^{2i}\frac{\omega}{2}\cdot\xi^i,$$

so that the highest term in μ is

$$\frac{2i\,.\,2i-1\ldots i+1}{\underline{|i}}\sin^{2i}\frac{\omega}{2}\cdot\mu^{2i}; \tag{14}$$

and the coefficient of $\cos n\omega$ in this must be identical with (12).

But, by elementary Trigonometry,

$$(-1)^i\,.\,2^{2i}\sin^{2i}\theta = 2\cos 2i\theta + \ldots$$
$$+(-1)^p 2\,\frac{2i\,.\,2i-1\ldots 2i-p+1}{\underline{|p}}\cos(2i-2p)\,\theta + \ldots. \tag{15}$$

Hence if $p = i-n$, we have

$$(-1)^i\,.\,2^{2i}\sin^{2i}\frac{\omega}{2} = 2\cos i\omega + \ldots$$
$$+(-1)^{i-n}2\,\frac{2i\,.\,2i-1\ldots i+n+1}{\underline{|i-n}}\cos n\omega + \ldots;$$

therefore by (14),

$$C_n(2i\,.\,2i-1\ldots i-n+1)^2$$
$$= \frac{1}{2^{2i-1}}\cdot\frac{2i\,.\,2i-1\ldots i+1}{\underline{|i}}\cdot\frac{2i\,.\,2i-1\ldots i+n+1}{\underline{|i-n}},$$

or

$$C_n = \frac{2}{(2^i\,\underline{|i})^2}\,\frac{\underline{|i-n}}{\underline{|i+n}}. \tag{η}$$

As before said, n is to receive all values from 0 to i, and when $n = i$, the expression $\underline{|i-n}$ is to be taken as unity.

When $n = 0$, the value of C given by (η) must be halved. because there is a middle term in (15), which is independent of ω, and it is not multiplied by the 2 which affects all the other terms.

Hence for the Laplacian of the i^{th} order, we have

$$L_i = \frac{2}{(2^i\,\underline{|i})^2}\sum_{n=0}^{n=i}\frac{\underline{|i-n}}{\underline{|i+n}}(1-\mu^2)^{\frac{n}{2}}(1-\mu'^2)^{\frac{n}{2}}\frac{d^{i+n}(\mu^2-1)^i}{d\mu^{i+n}}$$
$$\frac{d^{i+n}(\mu'^2-1)^i}{d\mu'^{i+n}}\cos n(\phi-\phi'),\ldots \tag{θ}$$

in which the first term (corresponding to $n = 0$) must be halved.

351.] **Fundamental Property of Spherical Harmonics.** If Y_i and $Z_{i'}$, are any two Spherical Harmonics of degrees i and i',

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_i Z_{i'}\, d\mu\, d\phi = 0, \tag{a}$$

or, in other words, $\int Y_i Z_{i'} dS$ extended over a sphere of unit radius is zero—that is, *the spherical surface-integral of the product of any two Spherical Harmonics of different degrees is zero.*

For, $\nabla^2 (r^i Y_i) = 0$, which gives (1) Art. 350; and (Art. 329)

$$\nabla^2 Y_i = \frac{1}{r^2}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2 Y_i}{d\phi^2}\right] = -i(i+1)\frac{Y_i}{r^2}.$$

Similarly $\nabla^2 Z_{i'} = -i'(i'+1)\frac{Z_{i'}}{r^2}$. Now in Green's Equation, (β), Art 346, let $V = Y_i$, $U = Z_{i'}$, and let the integrations be extended through the volume and over the surface of a sphere of radius a. Then, the centre of this sphere being the origin of the co-ordinates (r, μ, ϕ), it is clear that $\frac{dY_i}{dn} = 0 = \frac{dZ_{i'}}{dn}$. Hence we have

$$[i(i+1) - i'(i'+1)]\int Y_i Z_{i'} \frac{d\Omega}{r^2} = 0.$$

But $d\Omega = r^2 dr\, d\mu\, d\phi$, and $Y_i Z_{i'}$ does not involve r; therefore we have

$$[i(i+1) - i'(i'+1)]\, a \int\int Y_i Z_{i'} d\mu\, d\phi = 0,$$

which gives the result (a) *except when* $i = i'$.

We postpone for a moment the investigation of the value of the double integral when i and i' are the same.

352.] **Spherical Harmonic Expansion of a Function of μ and ϕ.** Let P (Fig. 277, p. 257), be any point outside a spherical surface of radius a, at a distance R from the centre, and let r be the distance, PQ, between P and any point on the surface. Then if dS is an element of surface at Q, we have

$$\int \frac{dS}{r^n} = \frac{2\pi a}{(n-2)R}\left\{\frac{1}{(R-a)^{n-2}} - \frac{1}{(R+a)^{n-2}}\right\},$$

as is easily found by using for dS the expression (A), p. 258.

Hence wherever P is, we have

$$(R^2 - a^2)^{n-2}\int \frac{dS}{r^n} = \frac{2\pi a}{(n-2)R}\{(R+a)^{n-2} - (R-a)^{n-2}\}. \tag{a}$$

If P is at A, i.e. on the surface, its distance from one of the surface elements becomes zero, and $R = a$; so that the left-hand

side of (a) assumes an apparently indeterminate form. (See, however, the remarks, p. 254.) But it is really finite and, as the right-hand side shows, equal to $\frac{2^{n-1}\pi}{n-2}a^{n-2}$.

Of course, of the whole surface of the sphere it is only an infinitely small element of the tangent plane at A that contributes to the integral, each element, $(R^2-a^2)\frac{dS}{r^n}$, of the integral being zero when r is appreciable, R being equal to a.

Hence
$$\left[(R^2-a^2)^{n-2}\int\frac{dS}{r^n}\right]_{R=a}=\frac{2^{n-1}\pi}{n-2}a^{n-2}. \qquad (\beta)$$

Again, if U is any function of the co-ordinates of a point, (x, y, z) or (r, μ, ϕ), the value of $(R^2-a^2)^{n-2}\int\frac{UdS}{r^n}$ when P is at A is obviously $U_A\times\left[(R^2-a^2)^{n-2}\int\frac{dS}{r^n}\right]_{R=a}$, assuming that when r is anything different from zero, U is never $=\infty$; because in this case it is only an infinitely small element of the tangent plane at A that contributes to the integral. In other words, if for no point, Q, on the sphere U is ∞, we have
$$\left[(R^2-a^2)^{n-2}\int\frac{UdS}{r^n}\right]_{R=a}=\frac{2^{n-1}\pi}{n-2}a^{n-2}U_A, \qquad (\gamma)$$
where U_A denotes the value of U at A.

We may assume U to be, definitely, a function of μ and ϕ, which does not become infinite at any point. Take the case $n=3$. Then denoting by dashes the values of functions at variable points on the sphere, such as p' (Fig. 289, p. 345), the functions without dashes belonging to a fixed point, p, on the sphere, we have
$$\left[(R^2-a^2)\int\frac{U'dS}{r^3}\right]_{R=a}=4\pi aU. \qquad (\delta)$$

Now taking $r^2=R^2-2aR\lambda+a^2$, with the same meaning of λ as in Art. 349, we have $\frac{dr}{dR}=\frac{R-a\lambda}{r}=\frac{R^2-a^2+r^2}{2Rr}$; and
$$\frac{d\frac{1}{r}}{dR}=-\frac{1}{r^2}\cdot\frac{dr}{dR}=-\frac{R^2-a^2}{2Rr^3}-\frac{1}{2Rr}; \text{ therefore}$$
$$\frac{R^2-a^2}{r^3}=-2R\frac{d\frac{1}{r}}{dR}-\frac{1}{r}. \qquad (1)$$

Now $$\frac{1}{r} = \frac{1}{R}\left(L_0 + L_1\frac{a}{R} + L_2\frac{a^2}{R^2} + \dots + L_i\frac{a^i}{R^i} + \dots\right),$$

therefore (1) becomes

$$\frac{R^2-a^2}{r^3} = \frac{1}{R}\left[L_0 + 3L_1\frac{a}{R} + 5L_2\frac{a^2}{R^2} + \dots + (2i+1)L_i\frac{a^i}{R^i} + \dots\right]. \quad (2)$$

Multiplying both sides of this equation by $U'dS$, that is by $U'a^2d\mu'd\phi'$, we have, whatever be the value of R,

$$(R^2-a^2)\int\frac{U'dS}{r^3} = \frac{a^2}{R}\left[\iint L_0U'd\mu'd\phi' + 3\frac{a}{R}\iint L_1U'd\mu'd\phi' + \dots\right.$$
$$\left. + (2i+1)\frac{a^i}{R^i}\iint L_iU'd\mu'd\phi' + \dots\right]; \quad (\epsilon)$$

the limits of μ being 1 and -1, and those of ϕ being 0 and 2π.

Now put $R = a$ in this equation, and we have, by (δ),

$$\iint L_0U'd\mu'd\phi' + 3\iint L_1U'd\mu'd\phi' + \dots$$
$$+ (2i+1)\iint L_iU'd\mu'd\phi' + \dots = 4\pi U. \quad (\zeta)$$

As a particular case let $U = Y_i$ = any Spherical Harmonic of the i^{th} degree. Then, since L_i is also a Spherical Harmonic of the same degree, every term except one in (ζ) vanishes by last Article, and we have

$$\int_{-1}^{1}\int_0^{2\pi} L_iY_i'd\mu'd\phi' = \frac{4\pi}{2i+1}Y_i, \quad (\eta)$$

which expresses a most remarkable property of a Laplacian, namely—*If over a sphere there be taken the surface-integral of the product of any Spherical Harmonic and the Laplacian of the same degree, i, with reference to any fixed point on the sphere, the result is the value of the given Spherical Harmonic at this fixed point, multiplied by* $\frac{4\pi}{2i+1}$.

This result enables us to express any function of μ and ϕ which does not become infinite for any values of μ and ϕ in the form of a series of Spherical Harmonics.

Thus, let U be the given function, which belongs to the fixed point p, Fig. 289, p. 345, and let

$$U = Y_0 + Y_1 + Y_2 + \dots + Y_i + \dots, \quad (3)$$

the quantities Y_0, Y_1, ... to be determined.

To determine Y_i, substitute running co-ordinates μ', ϕ' (those of p') in both sides of (3), multiply by L_i, where L_i is the Laplacian of the i^{th} degree for p and p', and integrate.

Then by (η) and last Article, we have

$$Y_i = \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} L_i U' d\mu' d\phi', \tag{4}$$

and by giving i all values from 0 upwards, we find the series of Y's.

The above method is due to Ivory (see Todhunter's *History of the Theories of Attraction*, &c., Vol. II, p. 261).

It is scarcely necessary to observe that a given function of μ and ϕ can be expanded in only one way in a series of Spherical Harmonics; for every Harmonic of the series is perfectly and uniquely determined by (4).

There is, however, another method by which a function of μ and ϕ can be expanded in a series of surface Harmonics without integration and the employment of Laplacians. To explain this method, suppose $F(x, y, z)$ to be any rational, integral, and homogeneous function of x, y, z of the n^{th} degree. Then this function can be expressed in the form

$$F(x, y, z) = S_n + r^2 S_{n-2} + r^4 S_{n-4} + \ldots, \tag{5}$$

where $S_n, S_{n-2}, \ldots$ are solid Harmonics of degrees $n, n-2, \ldots$, the last term being $r^n S_0$ if n is even, and $r^{n-1} S_1$ if n is odd.

Terms involving odd powers of r cannot appear in (5); for we can easily prove that

$$\nabla^2 . r^p S_q = p(p+2q+1) r^{p-2} S_q, \tag{6}$$

S_q being a solid Harmonic of degree q. Now if a term rS_{n-1} occurred in (5) and we performed the operation ∇^2 on both sides, this term would give rise to the only term in $\frac{1}{r}$ in the equation. Hence this term must be absent. Similarly a term $r^3 S_{n-3}$ could not occur; for, after performing ∇^2 twice on each side of (5), we should have the same result as before. Successive performances of the operation ∇^2 on (5) will give the required Harmonics $S_0, S_2, S_4, \ldots$ if n is even, or $S_1, S_3, S_5, \ldots$ if n is odd, in this order.

For example, to express xyz^3 in the form (5). Let

$$xyz^3 = S_5 + r^2 S_3 + r^4 S_1, \tag{7}$$

$$\therefore \quad 6xyz = 18 S_3 + 28 r^2 S_1,$$

$$0 = S_1,$$

by performing ∇^2 twice. Hence $S_3 = \frac{1}{3} xyz$, and (7) gives

$$S_5 = \frac{1}{3} xyz (2z^2 - x^2 - y^2).$$

Now this enables us to exhibit $\sin^2\theta\cos^3\theta\sin\phi\cos\phi$ as a series of surface Harmonics; for when this is multiplied by r^5, it becomes xyz^3, and we have

$$r^5\sin^2\theta\cos^3\theta\sin\phi\cos\phi = \tfrac{1}{3}xyz(2z^2-x^2-y^2)+\tfrac{1}{3}r^2xyz,$$

so that the given expression in θ and ϕ is of the form Y_5+Y_3,

where $\quad Y_5 = \dfrac{1}{3r^5}xyz(2z^2-x^2-y^2)$, and $Y_3 = \dfrac{1}{3r^3}xyz$,

$$\therefore \quad Y_5 = \mu(1-\mu^2)(\mu^2-\tfrac{1}{5})\sin\phi\cos\phi,$$

and
$$Y_3 = \tfrac{1}{5}\mu(1-\mu^2)\sin\phi\cos\phi.$$

353.] **Value of** $\int_{-1}^{1}\int_0^{2\pi} Y_i Z_i\, d\mu\, d\phi$. The spherical surface-integral of the product of two Spherical Harmonics of the *same* degree is found by Laplace very simply from the results of last Article.

Denoting by M_n the factor in μ in (β), p. 348, we may write

$$Y_i = A_0M_0+M_1(A_1\cos\phi+B_1\sin\phi)+\dots$$
$$+M_n(A_n\cos n\phi+B_n\sin n\phi)+\dots, \quad (1)$$
$$Z_i = a_0M_0+M_1(a_1\cos\phi+b_1\sin\phi)+\dots$$
$$+M_n(a_n\cos n\phi+b_n\sin n\phi)+\dots; \quad (2)$$

the two functions differing simply in their constants A's, B's, a's, b's.

Now since in the integration ϕ runs from 0 to 2π, it is obvious that the integrals of all products will vanish except those of the type

$$M_n^2(A_n\cos n\phi+B_n\sin n\phi)(a_n\cos n\phi+b_n\sin n\phi),$$

and the integral of this is

$$\pi(A_na_n+B_nb_n).M_n^2, \quad (a)$$

but for the first term the integral will be

$$2\pi A_0a_0M_0^2. \quad (a')$$

We have therefore to find $\int_{-1}^{1}M_n^2\,d\mu$, which Laplace finds as follows. With the notation of Art. 350, write

$$L_i = C_0M_0M_0'+C_1M_1M_1'\cos(\phi-\phi')+\dots$$
$$C_nM_nM_n'\cos n(\phi-\phi')+\dots. \quad (3)$$

Put running co-ordinates into (1), multiply by (3) and take the surface-integral over a sphere. Then we have simply a sum of terms of the type

$$C_nM_n\int_{-1}^{1}\int_0^{2\pi}M_n'^2(A_n\cos n\phi'+B_n\sin n\phi')\cos n(\phi-\phi')\,d\mu'\,d\phi'.$$

Performing the integration in ϕ', this becomes

$$\pi C_n M_n (A_n \cos n\phi + B_n \sin n\phi) \cdot \int_{-1}^{1} M_n'^2 d\mu. \qquad (\beta)$$

The sum of all terms of the type (β) is therefore the value of $\int\int L_i Y_i' \, d\mu' \, d\phi'$. But (last Article) this $= \frac{4\pi}{2i+1} Y_i$; therefore by identification of coefficients of like terms,

$$C_n \int_{-1}^{1} M_n'^2 d\mu' = \frac{4}{2i+1}.$$

Putting for C_n its value given in (η), p. 351,

$$\int_{-1}^{1} M_n^2 d\mu = \frac{2(2^i \lfloor i)^2}{2i+1} \cdot \frac{\lfloor i+n}{\lfloor i-n}, \qquad (\gamma)$$

which holds, without change, for the case $n = 0$, notwithstanding that the value of C_n (Art. 350) must be halved when $n = 0$; because in the product of (1) and (3) the term independent of ϕ' is $C_0 A_0 M_0 M_0'^2$, which in the integration will give

$$2\pi C_0 A_0 M_0 \int_{-1}^{1} M_0'^2 d\mu'.$$

Hence we have

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_i Z_i d\mu d\phi = \frac{2^{2i+1}}{2i+1} \pi (\lfloor i)^2 \sum_0^i \frac{\lfloor i+n}{\lfloor i-n} (A_n a_n + B_n b_n), \qquad (\delta)$$

the first term (that corresponding to $n = 0$) being doubled, by (α').

Putting $a_n = A_n$, $b_n = B_n$, we obtain the value of

$$\int_{-1}^{1} \int_{0}^{2\pi} Y_i^2 d\mu d\phi.$$

354.] **Table of Laplacians.** For convenience of reference the following table of the Laplacians as far as L_4 is given; but, to save space, we give in the coefficients of

$$\cos(\phi - \phi'), \cos 2(\phi - \phi'), \ldots$$

only the portion which depends on μ. This portion is to be multiplied by exactly the same function of μ'. Thus, for example, in L_3 the coefficient of $\cos 2(\phi - \phi')$ is $\frac{15}{4} \mu (1 - \mu^2) \cdot \mu' (1 - \mu'^2)$, of which only the part $\frac{15}{4} \mu (1 - \mu^2)$ is given in the column under $\cos 2(\phi - \phi')$; in L_4 the term involving $\cos 3(\phi - \phi')$ is

$$\frac{35}{8} \mu (1 - \mu^2)^{\frac{3}{2}} \cdot \mu' (1 - \mu'^2)^{\frac{3}{2}} \cos 3(\phi - \phi'); \text{ \&c.}$$

Values of i.	Term in μ only.	Coefficient of $\cos(\phi-\phi')$	Coefficient of $\cos 2(\phi-\phi')$	Coefficient of $\cos 3(\phi-\phi')$	Coefficient of $\cos 4(\phi-\phi')$
0	1				
1	μ	$(1-\mu^2)^{\frac{1}{2}}$			
2	$\frac{1}{4}(3\mu^2-1)$	$3\mu(1-\mu^2)^{\frac{1}{2}}$	$\frac{3}{4}(1-\mu^2)$		
3	$\frac{1}{4}(5\mu^3-3\mu)$	$\frac{3}{8}(1-\mu^2)^{\frac{1}{2}}(5\mu^2-1)$	$\frac{15}{4}\mu(1-\mu^2)$	$\frac{5}{8}(1-\mu^2)^{\frac{3}{2}}$	
4	$\frac{1}{64}(35\mu^4-30\mu^2+3)$	$\frac{5}{8}(1-\mu^2)^{\frac{1}{2}}(7\mu^3-3\mu)$	$\frac{5}{16}(1-\mu^2)(7\mu^2-1)$	$\frac{35}{8}\mu(1-\mu^2)^{\frac{3}{2}}$	$\frac{35}{64}(1-\mu^2)^2$

The column of terms in μ only gives the values of the first five Legendre's Coefficients, with the numerical coefficient squared; thus,

$$P_0 = 1,$$
$$P_1 = \mu,$$
$$P_2 = \tfrac{1}{2}(3\mu^2-1),$$
$$P_3 = \tfrac{1}{2}(5\mu^3-3\mu),$$
$$P_4 = \tfrac{1}{8}(35\mu^4-30\mu^2+3).$$

The Solid Zonal Harmonics play exactly the same part with regard to the Potential of a body symmetrical about an axis (see example 2 following) as the variables x, y, z do with respect to the equation of a plane surface, the equation of such a surface consisting of the sum of these co-ordinates each multiplied by a constant, and these constants depending on the position of the plane. Similarly, the Potential of a symmetrical body at any point consists of the sum of a number of these Harmonics each multiplied by a coefficient which depends on the shape and law of density of the body and not on the position of the attracted particle. In fact, Zonal Harmonics may be considered as the *running co-ordinates of the Potential* of such a body.

Of the Zonal Surface Harmonics P_1, P_3, P_5, ... are all of the form $\mu f(\mu^2)$, and P_2, P_4, P_6, ... are all of the form $f(\mu^2)$. For, in the identity

$$(1-2\mu x+x^2)^{-\frac{1}{2}} = P_0 + P_1 x + P_2 x^2 + \ldots + P_i x^i + \ldots \qquad (\alpha)$$

change μ to $-\mu$, and we get

$$(1+2\mu x+x^2)^{-\frac{1}{2}} = P_0 + P_1' x + P_2' x^2 + \ldots + P_i' x^i \ldots, \qquad (\beta)$$

where P_1', P_2', ... denote the values of the Harmonics when μ is changed to $-\mu$. Again, changing only the sign of x,

$$(1+2\mu x+x^2)^{-\frac{1}{2}} = P_0 - P_1 x + P_2 x^2 - P_3 x^3 + \ldots. \qquad (\gamma)$$

Identifying the results (β) and (γ), we see that P_1, P_3, ... all change sign with μ; while P_2, P_4, ... do not. Therefore, &c.

EXAMPLES.

1. Find $\int_{-1}^{1}\int_{0}^{2\pi} L_i^2 d\mu' d\phi'$.

Let P be any point outside a sphere of radius a, at a distance R from the centre, and Q any point on the surface; find $\int \frac{dS}{r^2}$ over the surface, where $r = PQ$.

Now $$\frac{1}{r} = \frac{1}{R}(L_0 + L_1 \frac{a}{R} + \ldots + L_i \frac{a^i}{R^i} + \ldots),$$

and $\frac{1}{r^2}$ will involve such terms as $L_m L_i$ which will vanish (Art. 351) in the integration. Hence, obviously, since $dS = -a^2 d\mu' d\phi'$, we have
$$\int \frac{dS}{r^2} = \ldots \frac{a^{2i+2}}{R^{2i+2}} \iint L_i^2 d\mu' d\phi' + \ldots.$$

But $dS = 2\pi \frac{a}{R} r dr$ (see p. 258); therefore the left-hand side is $2\pi \frac{a}{R} \log \frac{R+a}{R-a}$. Develope this in a series ascending by powers of $\frac{a}{R}$, and equate the coefficients of $(\frac{a}{R})^{2i+2}$ on both sides (since the development holds for all values of R), and we have
$$\int_{-1}^{1}\int_{0}^{2\pi} L_i^2 d\mu' d\phi' = \frac{4\pi}{2i+1}.$$

The result is therefore quite independent of the pole o (Fig. 289, p. 345) from which μ and μ' are measured, and is the same as if the line OP (or Op) is the axis of θ, or p the pole of the Laplacian.

2. Prove the theorem of Legendre (Art. 348) by Spherical Harmonics.

Taking the centre of the solid as origin, and axis of revolution as that from which θ is measured, let (R, μ, ϕ) be the co-ordinates of the attracted particle, P, (r', μ', ϕ') those of any point, P', inside the solid, ρ the density of the solid at P', and γ the constant of gravitation. Then V, the Potential at P, is given by the equation
$$V = \gamma \iiint \frac{\rho r'^2 dr' d\mu' d\phi'}{PP'}.$$

Now, assuming the distance of P from the centre to be greater than that of every point P' in the solid, $\frac{1}{PP'}$ may be developed in the convergent series (β), p. 344. Hence
$$V = \gamma \iiint (\frac{L_0}{R} + L_1 \frac{r'}{R^2} + \ldots + L_i \frac{r'^i}{R^{i+1}} + \ldots)\rho r'^2 dr' d\mu' d\phi'. \qquad (1)$$

But by hypothesis ρ is a function of r' and μ' only; and if when r' is produced out to meet the surface of the solid its value is R', this latter will be simply a function of μ', and will not involve ϕ'.

Take the general term of the series (1), and first perform the integration in r' from 0 to R', taking the term

$$\int_0^{R'} \rho r'^{i+2} dr' = \chi(\mu'),$$

where the form of χ is unknown if the shape of the surface and the law of density are not given. Then we have

$$V = \dots \frac{\gamma}{R^{i+1}} \int_{-1}^{1} \int_0^{2\pi} L_i \chi(\mu') d\mu' d\phi' + \dots. \tag{2}$$

Now perform the integration in ϕ'. We shall have simply

$$\int_0^{2\pi} L_i d\phi',$$

which, of course, reduces to the first term of L_i, and is therefore (Art. 350) $\dfrac{2\pi}{(2^i \lfloor i)^2} \dfrac{d^i(\mu^2-1)^i}{d\mu^i} \dfrac{d^i(\mu'^2-1)^i}{d\mu'^i}$. Hence

$$V = \dots \frac{2\pi}{(2^i \lfloor i)^2} \frac{d^i(\mu^2-1)^i}{d\mu^i} \frac{\gamma}{R^{i+1}} \int_{-1}^{1} \frac{d^i(\mu'^2-1)^i}{d\mu'^i} \chi(\mu') d\mu' + \dots. \tag{3}$$

Let v be the Potential at a point on the axis distant z from the centre. Then (3) gives

$$v = \dots \frac{2\pi}{(2^i \lfloor i)^2} \cdot 2^i \lfloor i \cdot \frac{\gamma}{z^{i+1}} \int_{-1}^{1} \frac{d^i(\mu'^2-1)^i}{d\mu'^i} \chi(\mu') d\mu' + \dots. \tag{4}$$

But supposing, as we do, that v is known for all points on the axis, let it be expanded from the given form in a series, so that

$$v = \frac{a_0}{z} + \frac{a_1}{z^2} + \dots + \frac{a_i}{z^{i+1}} + \dots. \tag{5}$$

Then identifying (4) and (5), we have

$$\frac{2\pi}{(2^i \lfloor i)^2} \gamma \int_{-1}^{1} \frac{d^i(\mu'^2-1)^i}{d\mu'^i} \chi(\mu') d\mu' = \frac{a_i}{2^i \lfloor i},$$

so that the unknown coefficient in (3) is thus known. Hence

$$V = \gamma \frac{M}{R} + \dots \frac{a_i}{2^i \lfloor i} \frac{d^i(\mu^2-1)^i}{d\mu^i} \cdot \frac{1}{R^{i+1}} + \dots, \tag{6}$$

the first term being easily seen to be $\dfrac{\gamma M}{R}$, where M is the mass of the solid. If $P_0, P_1, \dots$ denote, as before, the several Zonal Harmonics, or Legendre's coefficients, for the attracted point with reference to the axis of the solid, we may write, by (γ), p. 349,

$$V = \frac{1}{R} \left\{ a_0 P_0 + \frac{a_1 P_1}{R} + \frac{a_2 P_2}{R^2} + \dots + \frac{a_i P_i}{R^i} + \dots \right\}. \tag{7}$$

The components of attraction at P are of course known from this value of V.

Thus, then, the Potential of a solid symmetrical about an axis, both

as regards shape and density, is in all cases given by a series of Solid Zonal Harmonics (of either positive or negative degrees, according as the point considered is internal or external), in which series the only things unknown are the coefficients a_0, a_1, ... and the values of these depend on the nature of the particular attracting body.

3. Application of this method to the case of a uniform circular ring.

The Potential at a point distant z from the centre on the axis of the ring (that is, the line through its centre perpendicular to its plane) is given by the equation $v = 2\pi\gamma\rho k a \dfrac{1}{\sqrt{z^2+a^2}}$ where ρ, k, a are the density, area of transverse section, and radius of the ring. If M is the mass of the ring, $M = 2\pi\rho k a$; and if the point is at a distance $> a$ from the centre, we have

$$V = \frac{M}{z}\left\{1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{1\,.\,3}{2\,.\,4}\frac{a^4}{z^4} - \ldots + (-1)^i \frac{1\,.\,3\,.\,5\ldots 2i-1}{2\,.\,4\,.\,6\ldots 2i}\frac{a^{2i}}{z^{2i}} + \ldots\right\}.$$

Hence, by last example, if the attracted particle is anywhere off the axis, at a distance r from the centre ($r > a$),

$$V = \frac{M}{r}\left\{1 - \tfrac{1}{2}P_2\frac{a^2}{r^2} + \ldots + (-1)^i\frac{1\,.\,3\ldots 2i-1}{2\,.\,4\ldots 2i}P_{2i}\frac{a^{2i}}{r^{2i}} + \ldots\right\}.$$

If z is $< a$, the radical $(z^2+a^2)^{-\frac{1}{2}}$ must be expanded in direct powers of z, and for a point anywhere at a distance $< a$,

$$V = \frac{M}{a}\left\{1 - \tfrac{1}{2}P_2\frac{r^2}{a^2} + \ldots + (-1)^i\frac{1\,.\,3\ldots 2i-1}{2\,.\,4\ldots 2i}P_{2i}\frac{r^{2i}}{a^{2i}} + \ldots\right\}.$$

If the point is at the distance a from the centre, it is easy to prove that $V = \dfrac{\gamma M}{2\pi a\sqrt{2}}\displaystyle\int_0^{2\pi}\frac{d\phi}{\sqrt{1-\sin\theta\cos\phi}}$, θ being the angle between the axis of the ring and the line joining the point to the centre. This is equivalent to the convergent series

$$V = \gamma\frac{M}{a\sqrt{2}}\left\{1 + \frac{1\,.\,3}{2\,.\,4}\,.\,\tfrac{1}{2}\sin^2\theta + \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}\,.\,\frac{1\,.\,3}{2\,.\,4}\sin^4\theta + \ldots\right.$$
$$\left. + \frac{1\,.\,3\,.\,5\ldots 4n-1}{2\,.\,4\,.\,6\ldots 4n}\,.\,\frac{1\,.\,3\,.\,5\ldots 2n-1}{2\,.\,4\,.\,6\ldots 2n}\sin^{4n}\theta + \ldots\right\}.$$

Of course in this and in all similar examples, the value of V for a general position of the attracted particle, P, can be written down *in virtue of Legendre's Theorem solely* (Art. 348) by first calculating v, and in its expression replacing any such term as $\dfrac{k}{z^i}$ by $\dfrac{kP_{i-1}}{r^i}$, because this latter satisfies the equation $\nabla^2 U = 0$, and it coincides with the former when P is on the axis, since $\mu = 1$, $P_1 = P_2 = \ldots = P_i = 1$. The expression thus obtained (somewhat tentatively) can, by Legendre's Theorem, be none other than the Potential sought.

4. Application to a uniform circular plate.

The position of the attracted particle being at a distance z from the centre on the axis of the plate, $v = 2\gamma \frac{M}{a^2}(\sqrt{z^2+a^2}-z)$. When $z>a$, we have

$$v = 2\gamma \frac{M}{a^2}\left\{\frac{1}{2}\cdot\frac{a^2}{z} - \frac{1}{2^2}\cdot\frac{1}{\lfloor 2}\frac{a^4}{z^3} + \dots + (-1)^{i-1}\frac{1}{2^i}\cdot\frac{1.3.5\dots 2i-3}{\lfloor i}\frac{a^{2i}}{z^{2i-1}} + \dots\right\}.$$

Hence

$$V = 2\gamma \frac{M}{a^2}\left\{\frac{1}{2}\frac{a^2}{r} - \frac{1}{2^2}\frac{1}{\lfloor 2}\frac{P_2 a^4}{r^3} + \dots + (-1)^{i-1}\frac{1}{2^i}\frac{1.3\dots 2i-3}{\lfloor i}\frac{P_{2i-2}\,a^{2i}}{r^{2i-1}} + \dots\right\}.$$

When $z<a$, we easily find

$$V = 2\gamma \frac{M}{a^2}\left\{a - P_1 r + \tfrac{1}{2}P_2\frac{r^2}{a} + \dots + (-1)^{i-1}\frac{1}{2^i}\frac{1.3\dots 2i-3}{\lfloor i}P_{2i}\frac{r^{2i}}{a^{2i-1}} + \dots\right\}.$$

5. To find the conical angle subtended at any point, P, by a given circle.

Draw the axis of the circle, i.e. a perpendicular to its plane through its centre, O. Let $OP = r$, a = radius of circle. Now if P were on the axis at a distance z from O, we should have

$$\omega_0 = 2\pi\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right), \tag{1}$$

ω_0 being the conical angle subtended at the point; and since conical angles satisfy all the equations of Potential functions, the theorem of Legendre applies to them.

Developing (1) in powers of $\frac{a}{z}$ or $\frac{z}{a}$, according as z is $>$ or $<a$, we have

$$\omega_0 = 2\pi\left\{\tfrac{1}{2}\frac{a^2}{z^2} - \frac{1.3}{2.4}\frac{a^4}{z^4} + \dots - (-1)^i\frac{1.3\dots 2i-1}{2.4.2i}\frac{a^{2i}}{z^{2i}} + \dots\right\}, \tag{2}$$

$$\omega_0 = 2\pi\left\{1 - \frac{z}{a} + \tfrac{1}{2}\frac{z^3}{a^3} - \dots - (-1)^i\frac{1.3\dots 2i-1}{2.4\dots 2i}\frac{z^{2i+1}}{a^{2i+1}} + \dots\right\}. \tag{3}$$

Hence when P is off the axis we have in these two cases, respectively,

$$\omega = 2\pi\left\{\tfrac{1}{2}\frac{P_1 a^2}{r^2} - \frac{1.3}{2.4}\frac{P_3 a^4}{r^4} + \frac{1.3.5}{2.4.6}\frac{P_5 a^6}{r^6} - \dots\right\}, \tag{4}$$

$$\omega = 2\pi\left\{1 - \frac{Pr}{a} + \tfrac{1}{2}\frac{P_3 r^3}{a^3} - \frac{1.3}{2.4}\frac{P_5 r^5}{a^5} + \dots\right\}. \tag{5}$$

6. Find the conical angle subtended at a point 10 feet distant from the centre of a circle 1 foot in radius, the colatitude of the point with reference to the axis of the circle being $\frac{\pi}{3}$.

Ans. $\pi \times \cdot 0050328$, nearly.

7. If P_i is the Zonal Surface Harmonic of the i^{th} degree (Legendre's coefficient), show that

$$\mu \frac{dP_i}{d\mu} - iP_i = \frac{dP_{i-1}}{d\mu}. \qquad (a)$$

We have by definition

$$\frac{1}{\sqrt{1-2\mu x+x^2}} = P_0 + P_1 x + \ldots + P_{i-1}x^{i-1} + P_i x^i + \ldots . \qquad (1)$$

Denote the radical by T, and differentiate both sides with regard to x. Then

$$\frac{\mu - x}{T^3} = P_1 + \ldots + iP_i x^{i-1} + \ldots . \qquad (2)$$

Differentiate (1) with respect to μ; then

$$\frac{x}{T^3} = \ldots x^{i-1}\frac{dP_{i-1}}{d\mu} + x^i \frac{dP_i}{d\mu} + \ldots . \qquad (3)$$

Multiplying (2) by x and (3) by $\mu - x$, we obtain two series which must be identical; and equating the coefficients of x^i in them, we have at once the result (a).

This result enables us to write down the values of the successive Zonal Harmonics when the first is known. For treating (a) as a linear differential equation for P_i, we have

$$P_i = \mu^i \left\{ C + \int \frac{1}{\mu^{i+1}} \frac{dP_{i-1}}{d\mu} d\mu \right\}. \qquad (4)$$

Then, as $P_0 = 1$, this gives $P_1 = C\mu$, and each P is to be unity when $\mu = 1$; therefore $C = 1$. Similarly P_2 is deduced from P_1; &c.

The expression (γ), p. 349, gives, however, the values of the Harmonics directly, and is the most convenient form for actual calculation.

8. Prove that $\qquad (i+1)P_{i+1} - (2i+1)\mu P_i + iP_{i-1} = 0.$

Divide (1) by (2) and equate coefficients of like powers.

9. Prove that $\qquad (1-\mu^2)\dfrac{dP_i}{d\mu} + i\mu P_i = iP_{i-1}. \qquad (a)$

We have from (a), Example 7,

$$\overline{P_{i-1}}\Big|_{\mu}^{1} = \int_{\mu}^{1} \mu \frac{dP_i}{d\mu} d\mu - i\int_{\mu}^{1} P_i \, d\mu = \mu \overline{P_i}\Big|_{\mu}^{1} - (i+1)\int_{\mu}^{1} P_i \, d\mu;$$

$$\therefore \quad P_{i-1} = \mu P_i + (i+1)\int_{\mu}^{1} P_i \, d\mu. \qquad (1)$$

But from the fundamental equation for P_i,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\}+i(i+1)P_i=0,$$

we have by integration

$$i(i+1)\int_\mu^1 P_i\,d\mu=(1-\mu^2)\,\frac{dP_i}{d\mu}.$$

Substituting in (1), we have the result (a).

10. For an attracting body or system symmetrical about an axis, in shape and density, prove that if the Potential (external) is arranged in the series

$$\frac{a_0}{r}+a_1\frac{P_1}{r^2}+a_2\frac{P_2}{r^3}+\ldots+a_i\frac{P_i}{r^{i+1}}+\ldots$$

of Zonal Harmonics, the lines of force trace out surfaces given by the equation

$$a_0\mu-2\frac{a_1}{r}\int_\mu^1 P_1\,d\mu-3\,\frac{a_2}{r^2}\int_\mu^1 P_2\,d\mu-\ldots-(i+1)\frac{a_i}{r^i}\int_\mu^1 P_i\,d\mu\ldots=C,$$

in which different constant values are assigned to C.

Let O be the origin, and P any point external to the body, OP being r; let S be the radial attraction intensity at P (acting in the direction PO), and T the attraction intensity perpendicular to OP in the sense in which θ increases.

Then, the resultant of S and T acting along the tangent to the line of force at P, we have as the differential equation of this line

$$\frac{-dr}{rd\theta}=\frac{R}{T}. \tag{1}$$

$$\text{But}\quad R=-\frac{dV}{dr}=\frac{1}{r^2}\left(a_0+2\frac{a_1}{r}P_1+\ldots+(i+1)\frac{a_i}{r^i}P_i+\ldots\right), \tag{2}$$

$$\text{and}\qquad S=\frac{dV}{rd\theta}=\frac{1}{r^2}\left(\frac{a_1}{r}\frac{dP_1}{d\theta}+\ldots+\frac{a_i}{r^i}\frac{dP_i}{d\theta}+\ldots\right). \tag{3}$$

Observing that $\dfrac{d}{d\theta}=-\sqrt{1-\mu^2}\,\dfrac{d}{d\mu}$, we get, by substituting from (2) and (3) in (1), the equation

$$a_0d\mu+a_1\left\{2\frac{P_1}{r}\,d\mu+\frac{1-\mu^2}{r^2}\frac{dP_1}{d\mu}dr\right\}+\ldots$$
$$+a_i\left\{(i+1)\frac{P_i}{r^i}\,d\mu+\frac{1-\mu^2}{r^{i+1}}\,\frac{dP_i}{d\mu}\,dr\right\}+\ldots. \tag{4}$$

Now, by example 9, the coefficient of a_i in this equation is

$$(i+1)\left\{\frac{P_i}{r^i}\,d\mu+\frac{i}{r^{i+1}}dr\int_\mu^1 P_i\,d\mu\right\},\ \text{that is,}\ -(i+1)D\left\{\frac{1}{r^i}\int_\mu^1 P_i\,d\mu\right\},$$

where D stands for the total differential of the quantity in brackets (with respect to μ and r). Hence, integrating (4), we have the equation which was to be proved.

In particular, if the series for the Potential stops with P_2, the equation of a line of force is

$$a_0\mu - 2\frac{a_1}{r}\int_\mu^1 \mu d\mu - 3\frac{a_2}{r^2}\int_\mu^1 \tfrac{1}{2}(3\mu^2-1)\,d\mu = C,$$

or

$$a_0\cos\theta - \frac{a_1}{r}\sin^2\theta - \frac{3}{2}\frac{a_2}{r^2}\cos\theta\sin^2\theta = C.$$

11. If the density at any point of a solid sphere is proportional to the distance from a given central plane, find the Potential at any external point, P.

Ans. If $a =$ radius of sphere, $R =$ distance of P from centre, and $\rho = \lambda z'$ where z' is the perpendicular from any point on the plane,

$$V = \gamma\cdot\frac{4\lambda\pi a^5}{15R^3}.z.$$

[Here $V = \gamma\lambda\iiint r'^3\mu'\,(\frac{L_0}{R} + L_1\frac{r'}{R^2} + \ldots)dr'\,d\mu'\,d\phi'$. Integrate first from $r' = 0$ to $r' = a$, and since μ' is a Harmonic of the first degree, the only term not vanishing is that in L_1; therefore

$$V = \frac{\gamma\lambda a^5}{5R^2}\int_{-1}^1\int_0^{2\pi} L_1\mu'\,d\mu'\,d\phi' = \frac{\gamma\lambda a^5}{5R^2}\cdot\frac{4\pi}{3}\mu;\ \text{\&c.}]$$

12. In the same way exactly prove that if the density at any point in a solid sphere of radius a is proportional to any solid Harmonic, S_i, of positive degree in the co-ordinates of the point, the Potential of the sphere at any external point whose distance from the centre is R is

$$\frac{4\lambda\pi a^{2i+3}}{(2i+1)(2i+3)}\cdot\frac{S_i}{R^{2i+1}},$$

the co-ordinates (x, y, z) involved in S_i being those of the given external point, and λ being the constant involved in the density.

Deduce the result also for a spherical shell and any internal point.

13. If the origin of co-ordinates is transferred from O to a point O' along the axis of z (from which θ is measured), calculate the solid Zonal Harmonic of degree i with reference to O' as origin in terms of the solid Zonal Harmonics with reference to O.

Let Z_i be the solid Harmonic of degree i with reference to O, and Z'_i, that with reference to O'. Then, with the notation of Art. 329,

$$Z_i = f(z, \zeta);\ \text{and if } OO' = h,\ Z_i' = f(z+h, \zeta);$$

$$\therefore\ Z_i' = Z_i + h\frac{dZ_i}{dz} + \frac{h^2}{1.2}\frac{d^2Z_i}{dz^2} + \ldots \qquad (1)$$

[Here (z, ζ) are the cylindrical co-ordinates of a point P with reference to O, and $(z+h, \zeta)$ are the co-ordinates of the same point, P, with reference to O'.]

Now $Z_i = r^i P_i$, where P_i is the surface Zonal Harmonic, and $\frac{d}{dz} = \frac{1-\mu^2}{r}\frac{d}{d\mu} + \mu\frac{d}{dr}$. Hence

$$\begin{aligned}\frac{dZ_i}{dz} &= r^{i-1}\left\{(1-\mu^2)\frac{dP_i}{d\mu} + i\mu P_i\right\}\\ &= ir^{i-1}.P_{i-1} \quad \text{(by example 9)}\\ &= iZ_{i-1}.\end{aligned}$$

Hence, again, $\frac{d^2 Z_i}{dz^2} = i(i-1)Z_{i-2}$, &c., and therefore

$$Z_i' = Z_i + ihZ_{i-1} + \frac{i(i-1)}{1.2}h^2 Z_{i-2} + \ldots ih^{i-1}Z_1 + h^n.$$

Let us find in the same way the value of the Solid Harmonic of negative degree, $-(i+1)$. Let this Harmonic, with reference to O be U_i, or $\frac{P_i}{r^{i+1}}$.

Then
$$\begin{aligned}\frac{dU}{dz} &= \frac{1}{r^{i+2}}\left\{(1-\mu^2)\frac{dP_i}{d\mu} - (i+1)\mu P_i\right\}\\ &= \frac{1}{r^{i+2}}\{iP_{i-1} - (2i+1)\mu P_i\} \quad \text{(by example 9)}\\ &= -(i+1)\frac{P_{i+1}}{r^{i+2}} \quad \text{(by example (8)}\\ &= -(i+1)U_{i+1}.\end{aligned}$$

Hence $U_i' = U_i - (i+1)hU_{i+1} + \frac{(i+1)(i+2)}{1.2}h^2 U_{i+2} - ad\ infin.$

14. Arrange the expression $\frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta\cos^2\phi}{a^2} + \frac{\sin^2\theta\cos^2\phi}{b^2}$ as a series of Spherical Harmonics.

Ans. $\frac{1}{3}\left(\frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{1}{3}\left(\frac{1}{c^2} - \frac{1}{2a^2} - \frac{1}{2b^2}\right)(3\mu^2 - 1)$

$$+\frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)(1-\mu^2)\cos 2\phi.$$

15. Express the central radius vector of a nearly spherical ellipsoid by Spherical Harmonics.

Ans. If $\frac{a-c}{c} = k$, $\frac{b-c}{c} = k'$, we have

$$r = c\left\{1 + \tfrac{1}{3}(k+k') - \tfrac{1}{6}(k+k')(3\mu^2-1) - \tfrac{1}{2}(k'-k)(1-\mu^2)\cos 2\phi\right\},$$
which is of the form $r = c\,(Y_0 + Y_2)$.

16. If the expression $(1-2\mu x+x^2)^{-\frac{2m+1}{2}}$ be developed in a series in the form $Q_0+Q_1x+Q_2x^2+\ldots+Q_ix^i+\ldots$, prove that, in analogy with the Legendrian coefficients,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}-2m\mu\frac{dQ_i}{d\mu}+i(2m+i+1)Q_i=0.$$

Differentiate the given identity (α) with regard to x, and we obtain an identity (β); differentiate (α) with regard to μ, and we obtain an identity (γ); from (β) and (γ) we have, by equating the coefficients of x^i,

$$iQ_i=\mu\frac{dQ_i}{d\mu}-\frac{dQ_{i-1}}{d\mu}. \qquad (\delta)$$

Multiply (α) by $(2m+1)(\mu-x)$ and (β) by $1-2\mu x+x^2$, and we have

$$(i+1)Q_{i+1}-(2m+2i+1)\mu Q_i+(2m+i)Q_{i-1}=0. \qquad (\epsilon)$$

Differentiating (ϵ) with regard to μ and eliminating Q_{i-1} by (δ),

$$\frac{dQ_{i+1}}{d\mu}-\mu\frac{dQ_i}{d\mu}-(2m+i+1)Q_i=0. \qquad (\zeta)$$

Replace i by $i+1$ in (δ), combine with (ζ), and we have

$$(i+1)Q_{i+1}=-(1-\mu^2)\frac{dQ_i}{d\mu}+(2m+i+1)\mu Q_i. \qquad (\eta)$$

Differentiate (η) with respect to μ, and subtract the result from (ζ) multiplied by $(i+1)$, and we have the required equation.

17. In this development how far is it true that

$$\int_{-1}^{1}Q_iQ_i'\,d\mu=0,$$

i and i' being different integers?

It is always true if one of the numbers i, i' is even and the other odd. In Green's equation, applied through the interior and over the surface of a sphere, let $U=Q_i$, $V=Q_i'$, and observe that

$$\nabla^2Q_i=\frac{1}{r^2}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}=\frac{1}{r^2}\left\{2m\mu\frac{dQ_i}{d\mu}-i(2m+i+1)Q_i\right\}.$$

18. Exhibit $\cos^2\theta\sin^2\theta\sin\phi\cos\phi$ as a series of Surface Harmonics.

The simplest method is, of course, that at the end of Art. 352, which deduces the result by expressing xyz^2 in the form $S_4+r^2S_2$. Performing ∇^2, we get $S_2=\frac{1}{7}xy$, and $\therefore\ S_4=xy(z^2-\frac{1}{7}r^2)$. Now $S_4=r^4Y_4$ and $S_2=r^2Y_2$; whence Y_4 and Y_2 are at once found. Nevertheless it may be useful to show how to deduce the result by Laplacians and integration.

The given function is $\frac{1}{2}\mu^2(1-\mu^2)\sin 2\phi$, and since the term of highest degree in the quantities μ, $\sqrt{1-\mu^2}\sin\phi$, and $\sqrt{1-\mu^2}\cos\phi$ is obviously of the fourth degree in these quantities, it follows that the given function must be of the form $Y_1+Y_2+Y_3+Y_4$, the term in Y_0 being obviously non-existent.

Now Y_1 is obtained by taking $\frac{1}{2}\mu'^2(1-\mu'^2)\sin 2\phi'$, multiplying it by L_1, and integrating $\overline{\mu'}\Big|_{-1}^{1}$, $\overline{\phi'}\Big|_{0}^{2\pi}$; but as L_1 is of the form $A+B\cos(\phi-\phi')$, and as $\int_0^{2\pi}\sin 2\phi'\cos(\phi-\phi')d\phi'=0$, we see at once that $Y_1=0$.

It is clear that in L_2, L_3, L_4, it is only the terms involving $\cos 2(\phi-\phi')$ that will give any result when multiplied by $\sin 2\phi'$ and integrated. In L_2 occurs the term (see table, p. 358)

$$\tfrac{3}{4}(1-\mu^2)(1-\mu'^2)\cos 2(\phi-\phi'); \text{ hence (Art. 353)}$$

$$\frac{1}{2}\cdot\frac{3}{4}\int_{-1}^{1}\int_0^{2\pi}(1-\mu^2)\cdot\mu'^2(1-\mu'^2)^2\sin 2\phi'\cos 2(\phi-\phi')d\mu'd\phi'=\frac{4\pi}{5}Y_2.$$

But $\int_0^{2\pi}\sin 2\phi'\cos 2(\phi-\phi')d\phi'=\pi\sin 2\phi$; and

$$\int_{-1}^{1}\mu'^2(1-\mu'^2)^2d\mu'=\frac{16}{3.5.7},$$

$$\therefore\quad Y_2=\tfrac{1}{14}(1-\mu^2)\sin 2\phi.$$

Again, in L_3 the only possible term is

$$\tfrac{15}{4}\mu(1-\mu^2)\mu'(1-\mu'^2)\cos 2(\phi-\phi');$$

but the integration in μ' destroys this, since it gives

$$\int_{-1}^{1}f(\mu'^2)\cdot\mu'd\mu',$$

which obviously vanishes. Hence $Y_3=0$.

Finally, in L_4 the only term to be taken is

$$\tfrac{5}{16}(1-\mu^2)(7\mu^2-1)\cdot(1-\mu'^2)(7\mu'^2-1).$$

Hence

$$\tfrac{1}{2}\cdot\tfrac{5}{16}(1-\mu^2)(7\mu^2-1)\int_{-1}^{1}\mu'^2(1-\mu'^2)^2(7\mu'^2-1)d\mu'$$
$$\times\int_0^{2\pi}\sin 2\phi'\cos 2(\phi-\phi')d\phi'=\frac{4\pi}{9}Y_4;$$

$$\therefore\quad Y_4=\tfrac{1}{14}(1-\mu^2)(7\mu^2-1)\sin 2\phi;$$

and therefore

$$\cos^2\theta\sin^2\theta\sin\phi\cos\phi=\tfrac{1}{14}(1-\mu^2)\sin 2\phi+\tfrac{1}{14}(1-\mu^2)(7\mu^2-1)\sin 2\phi,$$

which is of the form Y_2+Y_4.

19. Exhibit $\cos\theta\sin^3\theta\cos^2\phi\sin\phi$ as a series of Harmonics.

Ans. $\{\frac{1}{7}\mu\sqrt{1-\mu^2}\sin\phi\}+\{-\frac{1}{28}\sqrt{1-\mu^2}(7\mu^3-3\mu)\sin\phi$
$+\frac{1}{4}\mu(1-\mu^2)^{\frac{3}{2}}\sin 3\phi\}$,

which is of the form Y_2+Y_4.

[The given expression is $\frac{1}{4}\mu(1-\mu^2)^{\frac{3}{2}}(\sin\phi+\sin 3\phi)$; hence the only terms to attend to in the L's are those in $\cos(\phi-\phi')$ and $\cos 3(\phi-\phi')$. The term in L_1 is destroyed by the integration in μ', which also destroys both the terms in L_3.] Deduce the result also from x^2yz.

20. Why cannot $\sin\theta$, $\sin^3\theta$, or any odd power of $\sin\theta$ be expanded in a finite series of Harmonics?

Because they are of the form $(1-\mu^2)^{\frac{2n+1}{2}}$, which can be developed in an infinite series ascending by powers of μ, and every term, such as μ^m, can be developed in a finite series of Zonals, P_1, P_2, ... Also a function can be expanded in only one way.

355.] **Case of Spheroids.** Any solid body differing little in shape from a sphere is called a *Spheroid*. Supposing the body to be homogeneous, the radius vector from its centre of mass to any point on its surface will be nearly of constant length. Thus (following the notation of Laplace), if a denote a small numerical quantity, and R' any radius vector from the centre of mass to the surface, we shall have

$$R' = a + a\alpha f(\mu', \phi'), \tag{1}$$

where a is a constant length and $f(\mu', \phi')$ some function of the angular co-ordinates depending on the precise shape of the bounding surface. Laplace uses y' for the function $f(\mu', \phi')$, and he assumes that y' is expanded in a series of Spherical Harmonics; thus,

$$R' = a + a\alpha(Y_0 + Y_1' + Y_2' + \ldots + Y_i' + \ldots). \tag{2}$$

If the series stops with Y_2', the bounding surface will be that of an ellipsoid.

External Point. To calculate the Potential at an external point, P, produced by a homogeneous spheroid, the distance of P from the origin O being greater than the greatest radius vector from O to the surface, let $OP = R$, $\rho =$ density of the body, and (r', μ', ϕ') the co-ordinates of any point P' in the body of the spheroid. Then, γ being, as usual, the gravitation constant,

$$V = \gamma\int_0^{2\pi}\int_{-1}^{1}\int_0^{R'}\frac{\rho r'^2 dr'\,d\mu'\,d\phi'}{PP'} \tag{3}$$

$$= \frac{\gamma}{R}\int_0^{2\pi}\int_{-1}^{1}\int_0^{R'}\Big(L_0 + L_1\frac{r'}{R} + L_2\frac{r'^2}{R^2} + \ldots$$

$$+ L_i\frac{r'^i}{R^i} + \ldots\Big)\rho r'^2 dr'\,d\mu'\,d\phi' \tag{4}$$

$$= \frac{\gamma\rho}{R}\int_0^{2\pi}\int_{-1}^{1}\Big(\tfrac{1}{3}L_0 R'^3 + L_1\frac{R'^4}{4R} + \ldots + L_i\frac{R'^{i+3}}{(i+3)R^i} + \ldots\Big)d\mu'\,d\phi'.$$

Now from (2), neglecting higher powers than the first of α,

$$R'^{i+3} = a^{i+3}\{1 + (i+3)\alpha(Y_0' + Y_1' + \ldots + Y_i' + \ldots)\},$$

and by substitution in the last value of V, since it is (Art. 351) only the term $\iint L_i Y_i' \, d\mu' d\phi'$ which does not vanish, we have

$$V = \gamma \frac{4\pi\rho a^3 (1 + 3Y_0)}{3R}$$

$$+ \alpha\gamma \cdot \frac{\rho a^3}{R} \int_0^{2\pi} \int_{-1}^{1} \left(\frac{a}{R} L_1 Y_1' + \frac{a^2}{R^2} L_2 Y_2' + \ldots + \frac{a^i}{R^i} L_i Y_i' + \ldots \right) d\mu' d\phi'.$$

Now the volume of the Spheroid is $\frac{4}{3}\pi a^3 (1 + 3Y_0)$, and if we choose a so that $\frac{4}{3}\pi a^3$ shall be the volume, Y_0 will be zero. Thus, attending to the result (η), Art. 352, we have

$$V = \gamma \frac{M}{R} + 3\alpha\gamma \frac{M}{R} \left(\frac{a}{3R} Y_1 + \frac{a^2}{5R^2} Y_2 + \ldots + \frac{a^i}{(2i+1)R^i} Y_i + \ldots \right), \quad (5)$$

where M is the mass of the Spheroid.

It is very easy to see that, with the origin at the centre of mass of the Spheroid, the term Y_1 is zero.

For if, in general, $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of the centre of mass, we have

$$M\bar{x} = \rho \iiint r'^3 (1 - \mu'^2)^{\frac{1}{2}} \cos\phi' \, dr' d\mu' d\phi'$$
$$= \tfrac{1}{4} \rho \iint R'^4 (1 - \mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' d\phi', \quad (6)$$

and since $(1 - \mu'^2)^{\frac{1}{2}} \cos\phi'$ is a Spherical Harmonic of the first degree, in the expansion of R'^4—viz. $a^4 \{1 + 4\alpha(Y_1' + Y_2') + \ldots\}$—the only term that will not identically vanish in (6) is

$$\alpha\rho a^4 \iint Y_1' (1 - \mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' d\phi'.$$

But this is zero because $\bar{x} = 0$. Hence

$$\iint Y_1' (1 - \mu'^2)^{\frac{1}{2}} \cos\phi' \, d\mu' d\phi' = 0. \quad (7)$$

Similarly, since $\bar{y} = 0$, we must have

$$\iint Y_1' (1 - \mu'^2)^{\frac{1}{2}} \sin\phi' \, d\mu' d\phi' = 0; \quad (8)$$

and since $\bar{z} = 0$, $\quad \iint Y_1' \mu' \, d\mu' d\phi' = 0. \quad (9)$

But Y_1' is (Art. 350) of the form

$$A\mu' + (1 - \mu'^2)^{\frac{1}{2}} (B\cos\phi' + C\sin\phi'),$$

where A, B, C are constants, and the results (7), (8), (9) make $A = B = C = 0$, as is easily seen either by direct integration, or by multiplying the left-hand sides of these equations by A, B, C and adding. We thus get $\iint Y_1'^2 \, d\mu' d\phi' = 0$, which requires Y_1' to vanish identically.

For example, take the case of a nearly spherical ellipsoid of revolution round the smaller axis, c.

In this case (see example 15, p. 366) $k = k'$, and we have

$$R' = c\{1 + \tfrac{2}{3}k - \tfrac{1}{3}k(3\mu'^2 - 1)\}.$$

But the a in (5) is determined from the equation

$$a^3 = c^3(1 + 2k); \qquad \therefore\ a = c(1 + \tfrac{2}{3}k),$$

and since $R' = a(1 + \alpha Y_2')$, we have $\alpha = -\frac{1}{3}k$. Hence (5) gives

$$V = \gamma\frac{M}{R} - \tfrac{1}{5}\gamma k M\frac{a^2}{R^3}(3\mu^2 - 1),$$

in which a or c may be used indifferently in the small term.

If the Spheroid is not homogeneous, but consists of strata of different densities, each stratum differing but little from a sphere, the Potential can still be very easily expressed. Thus, let $r' = a'(1 + \alpha y')$ be the equation of any stratum, a' being the radius of a sphere whose volume is equal to that of the stratum, so that

$$y' = Y_1' + Y_2' + \ldots + Y_i' + \ldots,$$

where the Y''s involve a' as well as μ' and ϕ', unless the strata are all similar.

Now if the Spheroid were homogeneous and of density ρ as far as the stratum a', the Potential of this portion would be given by the equation

$$\frac{V}{\gamma} = \frac{4}{3}\frac{\pi}{R}\cdot\rho a'^3 + \frac{4\alpha\pi}{R}\left\{\frac{1}{3R}\rho Y_1' a'^4 + \frac{1}{5R^2}\rho Y_2' a'^5 + \ldots \right.$$
$$\left. \ldots + \frac{1}{(2i+1)R^i}\rho Y_i' a'^{i+3} + \ldots\right\}.$$

Let $a' + da'$ be the constant of the next stratum outside, and let the value of V due to the whole portion of the Spheroid, supposed homogeneous and still of density ρ, up to and including this stratum, be written down. Subtract the first result from the second and we obtain the Potential due to the shell of density ρ included between the strata a' and $a' + da'$.

The Potential of the homogeneous solid $a' + da'$ being $V + dV$, we have by subtracting that due to the homogeneous solid a',

$$\frac{dV}{\gamma} = \frac{4\pi}{3R}\rho\, d(a'^3) + \frac{4\alpha\pi}{R}\rho\left\{\frac{1}{3R}d(a'^4 Y_1') + \ldots \right.$$
$$\left. \ldots + \frac{1}{(2i+1)R^i}d(a'^{i+3}Y_i') + \ldots\right\},$$

the independent variable in the differentiations being a', the parameter which determines any one stratum of constant density.

Now if the value of a' for the bounding surface of the Spheroid is a, we have by integrating the above

$$\frac{V}{\gamma} = \frac{4\pi}{3R}\int_0^a \rho d(a'^3) + \frac{4\alpha\pi}{R}\int_0^a \rho\left\{\frac{1}{3R}d(a'^4 Y_1') + \dots \right.$$
$$\left. \dots + \frac{1}{(2i+1)R^i}d(a'^{i+3}Y_i') + \dots\right\},$$

or
$$\frac{V}{\gamma} = \frac{M}{R} + \frac{4\alpha\pi}{3R^2}\int_0^a \rho d(a'^4 Y_1') + \dots$$
$$\dots + \frac{4\alpha\pi}{(2i+1)R^{i+1}}\int_0^a \rho d(a'^{i+3}Y_i') + \dots. \quad (10)$$

Internal Point. If the point, P, at which the value of the Potential is desired is inside the Spheroid, we may treat the spheroid as consisting of a sphere and a superficial layer which is everywhere of comparatively small thickness.

The Potential of a solid homogeneous sphere at an internal point has been already found. We must therefore find the Potential at P due to the shell at the surface of this sphere—observing that, according to the shape of the spheroid, the thickness of this shell measured outwards from the surface of the sphere may be positive or negative. If the equation of the surface is $r = a(1 + \alpha y)$, the thickness of the shell at any point is (nearly) $a\alpha y$; or the value of r' ranges from $r' = a$ to $r' = a(1 + \alpha y)$. If v is the Potential at P (internal) due to the shell,

$$\frac{v}{\gamma} = \iiint \rho\left(L_0 + L_1\frac{R}{r'} + \dots + L_i\frac{R^i}{r'^i} + \dots\right) r'\,dr'\,d\mu'\,d\phi'.$$

Performing the integration in r' first, we have

$$\frac{v}{\gamma} = \alpha\rho\iint\left(L_0 a^2 + L_1 Ra + \dots + L_i\frac{Ri}{a^{i-2}} + \dots\right)y'\,d\mu'\,d\phi',$$

which, by Art. 352, is

$$\frac{v}{\gamma} = 4\alpha\pi\rho a^2\left\{Y_0 + \frac{R}{3a}Y_1 + \frac{R^2}{5a^2}Y_2 + \dots + \frac{R^i}{(2i+1)a^i}Y_i + \dots\right\}, \quad (11)$$

in which the Y's belong to the attracted point P. To this must be added $2\pi\rho a^2 - \frac{2}{3}\pi\rho R^2$, which is due to the sphere of radius a, so that

$$\frac{V}{\gamma} = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho R^2 + 4\alpha\pi\rho a^2\left\{\dots + \frac{R^i}{(2i+1)a^i}Y_i + \dots\right\}. \quad (12)$$

As has been already proved, the terms Y_0 and Y_1 may be dispensed with.

The case of a heterogeneous spheroid is treated exactly as before. The point P being internal, let b be the parameter of the stratum of constant density passing through P, and take for V the sum of the Potentials due to the spheroid as far as this stratum and to the portion between this stratum and the bounding surface (of parameter a). The point P is external to the first, and the corresponding part of V is given by (10) in which we have simply to change the limit a to b in the integrations. The Potential due to any stratum (a', ρ) surrounding P can be obtained by subtracting the Potential due to a *solid* homogeneous spheroid, (a', ρ) from that due to a *solid* homogeneous spheroid $(a'+da', \rho)$. Thus by (12) the Potential due to the *stratum* (a', ρ) is

$$2\pi\rho d(a'^2)+4\alpha\pi\rho d\left\{a'^2 Y_0'+\frac{R}{3}a' Y_1'+\ldots+\frac{R^i}{2i+1}\frac{Y_i'}{a'^{i-2}}\text{\&c.}\right\}.$$

Integrating this between $a'=b$ and $a'=a$, we have by addition to the first portion,

$$\frac{V}{\gamma}=\frac{4\pi}{3R}\int_0^b \rho d(a'^3)+\ldots+\frac{4\alpha\pi}{(2i+1)R^{i+1}}\int_0^b \rho d(a'^{i+3}Y_i')+\ldots$$
$$+2\pi\int_b^a \rho d(a'^2)+\ldots+\frac{4\alpha\pi}{2i+1}R^i\int_b^a \rho d\left(\frac{Y_i'}{a'^{i-2}}\right)+\ldots. \quad (13)$$

For the discussion of the figure and law of density of the strata of the earth the reader will, of course, consult the *Mécanique Céleste*. A valuable epitome of Laplace's and other results will be found in Pratt's *Treatise on Attractions, Laplace's Functions, and the Figure of the Earth.*

Miscellaneous Examples.

1. Find the work required to scatter the particles of a uniform circular plate to infinite distances from each other (for the law of nature).

Ans. Let M be the mass of the plate in grammes, a its radius in centimètres, and γ the C. G. S. constant of gravitation; then the work is

$$\frac{8\gamma M^2}{3\pi a}\text{ ergs.}$$

At any distance, x, from the centre, inside the plate

$$V = 4\gamma\rho\tau\int_0^{\frac{\pi}{2}} \sqrt{a^2 - x^2\sin^2\theta}\,d\theta,$$

where $\tau =$ thickness of plate. Hence

$$\tfrac{1}{2}\int V dm = 4\pi\gamma\rho^2\tau^2\int_0^{\frac{\pi}{2}}\int_0^{a} \sqrt{a^2 - x^2\sin^2\theta}\,.\,x dx d\theta.$$

Perform the integration in x first; &c.

2. Considering the attraction-intensity of an infinite plate at a point near its surface, show that it is greater for the law of inverse square than for the law $\frac{1}{r^n}$ when $n < 2$, and less for the law of inverse square than for the law $\frac{1}{r^n}$ when $n > 2$.

The attracted particle having any position on the axis of the plate (assumed circular), the attraction-intensity for the law $\frac{1}{r^n}$ is

$$2\pi\gamma\rho\tau\frac{(1-\cos^n\alpha)\tan^{n-2}\alpha}{n-1}.$$

If the particle is near the plate, $\cos\alpha = x$, where x is very small, $\tan\alpha = \frac{1}{x}$, and the most important part of this expression becomes $\frac{1}{(n-1)x^{n-2}}$; from which the result follows.

3. At a point in the plane of a uniform circular plate outside its circumference, the Potential is

$$4\gamma\rho\tau\left(K - \frac{x^2-a^2}{x^2}E\right).x,$$

where x is the distance of the point from the centre, and K and E are the complete elliptic integrals of the first and second kinds with modulus $\frac{a}{x}$.

[Let P be the point, O the centre, Q any point on the circumference, $\angle OPQ = \theta$; then

$$V = 4\gamma\rho\tau\int_0^{\alpha} \sqrt{a^2 - x^2\sin^2\theta}\,d\theta,$$

where $\alpha = \sin^{-1}\frac{a}{x}$. Let $x\sin\theta = a\sin\phi$, where ϕ is the angle between QP and QO; &c.]

4. Find a function, ϕ, of r only which satisfies the equation

$$(\nabla^2 + a^2)\phi = 0,$$

where a is independent of r.

Ans. $\phi = A\frac{e^{ar}}{r} + B\frac{e^{-ar}}{r}$.

The equation (γ), p. 281, becomes $\frac{d^2.r\phi}{dr^2} + a^2.r\phi = 0$.

5. Fig. 228, p. 1, represents a homogeneous solid rectangular block whose density is ρ grammes per cub. cm.; the sides are $AD = 2a$ cm., $BD = b$ cm., $DO' = h$ cm.; find the attraction-intensity at a point, P, which is on the perpendicular to AD at its middle point and lies in the plane of the face $AOBD$.

Ans. If p is the distance (in centimètres) of P from the side AD, and if X, Z are the components of the force-intensity in and perpendicular to the plane $AOBD$,

$$X = 2\gamma\rho \int_p^{p+b} \sin^{-1} \frac{ah}{\sqrt{(a^2+x^2)(h^2+x^2)}} \cdot dx; \quad \text{(dynes per gramme.)}$$

$$Z = 2\gamma\rho \int_p^{p+b} \log_e \frac{\sqrt{h^2+x^2}\,(a+\sqrt{a^2+x^2})}{x\,(a+\sqrt{h^2+a^2+x^2})} \cdot dx. \quad \text{(dynes per gramme.)}$$

6. Apply the preceding to calculate the deviation of the plumb-line caused by a large rectangular table-land in the following instance.

'A table-land 1610 feet high, commencing at a distance of 20 miles from Takal K'hera near the great arc of meridian in India, runs 80 miles north, and 60 miles to the east and 60 to the west.' (Pratt's *Attractions, Laplace's Functions, and the Figure of the Earth*, p. 48.)

Observe that h is here very small compared with the other linear dimensions.

Assume ρ to be 2.8, i.e. about half the mean density of the Earth, or the density of statuary marble; also assume 160933 centimètres in 1 mile. Then, since a gramme mass weighs at the surface of the Earth about 980 dynes, the circular measure of the deviation is $\frac{X}{980}$; and the deviation is found to be about $4''.8$—so considerable a disturbance that (it is stated) the place in question was abandoned as a principal station of the survey. We have neglected Z in this result, as is, of course, allowable.

7. When by the method of Inversion (Art. 334) a system of points (x', y', z') is deduced from a given system (x, y, z), show that if the operations $x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}$, or $r\frac{d}{dr}$, and $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$ are, respectively, denoted by δ and ∇^2, we have

$$\delta' = -\delta,$$

$$\nabla'^2 = \frac{r^4}{k^4}\nabla^2 - \frac{2}{k^2}\delta.$$

CHAPTER XVIII.

ANALYSIS OF STRAINS AND STRESSES.

356.] **Definitions of Strain and Stress.** When a natural solid (such as iron, wood, &c.), or any material medium, is not acted upon by any external forces, its particles assume certain determinate distances from each other, and the body is then said to be in its *natural state.* But when forces act on it either at its surface or throughout its mass, or when any disturbance is propagated through its interior, these natural distances between its particles suffer alteration, and the body is said to be in a state of *strain.* Thus a fluid exerting pressure, a medium propagating sound, and the luminiferous ether when it is propagating light are instances of a body in a state of strain.

The change of the natural distances between the particles is always attended by the production of internal forces, or, as they are called, *internal stresses*, or simply stresses; and these stresses will depend, as we shall see, both on the nature of the body and on the nature of the strain in any case.

SECTION I.

Analysis of Small Strains.

357.] **Displacements of a Rigid Body.** It has been already pointed out (Chap. XV) that the general motion of a *rigid* body consists of a motion of translation which is the same for all its particles, together with a rotation round an axis through an angle which is the same for all its particles. These displace-

ments do not alter the distance between any two particles of the body, and they are therefore unaccompanied by the development of stress in its interior. Stress results only from the *alteration* of distances between pairs of particles, and hence in treating of strains and stresses all displacements, whether of translation or of rotation, which are impressed, with common magnitude, upon all particles of the body, may be discarded; and again any such common displacement may be freely introduced if it is found convenient for analysis.

358.] **Changes in Relative Co-ordinates.** Let a system of rectangular axes, Ox, Oy, Oz, (Fig. 290) be fixed in space; through any point, P, in the natural solid under consideration let Px, Py, Pz be drawn parallel to the fixed axes. Let the particle at P be displaced to P', and suppose that the co-ordinates (x, y, z) of P referred to the axes through O are increased by small quantities, u, v, w, respectively. The co-ordinates of P' are therefore $x+u$, $y+v$, $z+w$. Now these displacements u, v, w depend on the position of the point P, i.e. they are functions of its co-ordinates depending on the law according to which the strain is produced. We have then, when the kind of strain is specified, some such equations as

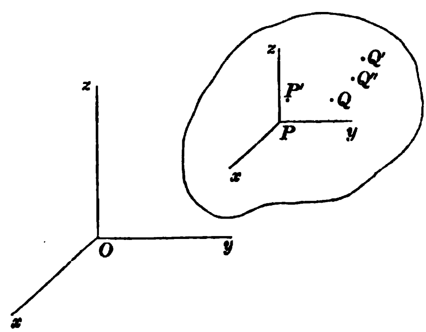

Fig. 290.

$$u = f_1(x, y, z), \quad v = f_2(x, y, z), \quad w = f_3(x, y, z),$$

where f_1, f_2, f_3 are symbols of functionality.

Let Q be a particle very near P, and let its co-ordinates with reference to the axes drawn through P be (ξ, η, ζ). Then the displacements of Q parallel to the axes are obviously

$$f_1(x+\xi,\ y+\eta,\ z+\zeta),$$
$$f_2(x+\xi,\ y+\eta,\ z+\zeta),$$
$$f_3(x+\xi,\ y+\eta,\ z+\zeta),$$

that is, by Taylor's Theorem,

$$u+\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz},\quad v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz},$$
$$w+\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}.$$

Suppose Q to come to Q' by displacement. Then in considering the nature of the strain in the neighbourhood of P, we may, by last Article, impress on every particle of the body a motion of translation represented in magnitude and sense by $P'P$, so that P' will be brought back to P without in any way interfering with the strain of the solid. By drawing $Q'Q''$ equal and parallel to $P'P$, the particle which was originally at Q may now be considered to be at Q''; and a similar process is to be repeated for all other particles. The part of the strain, therefore, due to the alteration of the distance between P and Q will depend on the co-ordinates of Q'' with reference to Px, Py, Pz. These co-ordinates are, of course, the excesses of those of Q' over those of P'; and therefore the relative co-ordinates of Q'' are

$$\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz},\quad \xi\frac{dv}{dx}+\eta\left(1+\frac{dv}{dy}\right)+\zeta\frac{dv}{dz},$$
$$\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\left(1+\frac{dw}{dz}\right);$$

in other words, the changes, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ in ξ, η, ζ, are

$$\Delta\xi=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz};\quad \Delta\eta=\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz};$$
$$\Delta\zeta=\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}. \qquad (a)$$

COR. 1. *All particles near P which in the natural state lie in one plane will after strain also lie in one plane.* For if the co-ordinates of Q'' are denoted by ξ', η', ζ', we have

$$\xi'=\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz},\ \eta'=\ldots,\ \zeta'=\ldots,$$

which equations, being linear, give ξ, η, ζ linearly in terms of ξ', η', ζ'. Remembering that $\frac{du}{dx}, \frac{dv}{dy}, \ldots$ are all small, these equations give $\xi=\xi'+$ small quantities of the order of $\frac{du}{dx}$, &c.; so that in any terms multiplied by $\frac{du}{dx}, \ldots$ ξ' may be put for ξ, η' for η, and ζ' for ζ.

Hence we have, to the order of accuracy adopted,

$$\left.\begin{aligned} \xi &= \xi'\left(1-\frac{du}{dx}\right)-\eta'\frac{du}{dy}-\zeta'\frac{du}{dz}, \\ \eta &= -\xi'\frac{dv}{dx}+\eta'\left(1-\frac{dv}{dy}\right)-\zeta'\frac{dv}{dz}, \\ \zeta &= -\xi'\frac{dw}{dx}-\eta'\frac{dw}{dy}+\zeta'\left(1-\frac{dw}{dz}\right). \end{aligned}\right\} \quad (1)$$

Therefore if all the points (ξ, η, ζ) lie in the plane

$$A\xi+B\eta+C\zeta+D=0,$$

all the points (ξ', η', ζ') will also lie in a plane. That is, every plane curve is strained into a plane curve in a different plane.

COR. 2. *All particles near P which in the natural state lie in one right line will after strain also lie in one right line.* For if we have

$$A\xi+B\eta+C\zeta+D=0 \text{ and } A'\xi+B'\eta+C'\zeta+D'=0,$$

we shall have ξ', η', ζ', also satisfying two linear equations.

COR. 3. *Two parallel right lines in the natural state are changed into two parallel right lines (with a different direction) in the strained state.*

For, one of the two lines being given by the equations

$$A\xi+B\eta+C\zeta+D=0, \quad A'\xi+B'\eta+C'\zeta+D'=0,$$

the other will be given by two equations in which the terms D and D' alone are altered. Substitute for ξ, η, ζ their values in terms of ξ', η', ζ', and observe that the values of D and D' do not influence the direction cosines of the line into which any one is converted by strain.

359.] **Elongation in any Direction.** Supposing P and Q to be, as before, two particles in the natural state of the body, the *elongation* in the direction PQ is defined as the ratio of the change produced by strain in the distance between these same particles to the original distance between them. Hence the elongation in the direction PQ is $\frac{PQ''-PQ}{PQ}$, or $\frac{\Delta\rho}{\rho}$, if ρ denotes PQ, and $\Delta\rho$ the change in ρ.

Now
$$\rho^2 = \xi^2+\eta^2+\zeta^2,$$
$$\therefore \quad \rho\Delta\rho = \xi\Delta\xi+\eta\Delta\eta+\zeta\Delta\zeta;$$

or if we substitute for $\Delta\xi$, $\Delta\eta$, and $\Delta\zeta$ their values from last Article,

$$\rho\,\Delta\rho = \xi^2\frac{du}{dx} + \eta^2\frac{dv}{dy} + \zeta^2\frac{dw}{dz} + \xi\eta\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \eta\zeta\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \zeta\xi\left(\frac{dw}{dx} + \frac{du}{dz}\right).$$

Let the cosines of the angles made by PQ with Px, Py, Pz be l, m, n, respectively, let

$$\frac{du}{dx} = a,\ \frac{dv}{dy} = b,\ \frac{dw}{dz} = c,$$

$$\frac{du}{dy} + \frac{dv}{dx} = 2s_3,\ \frac{dv}{dz} + \frac{dw}{dy} = 2s_1,\ \frac{dw}{dx} + \frac{du}{dz} = 2s_2,$$

and denote the elongation by ϵ; then the last equation gives

$$\epsilon = al^2 + bm^2 + cn^2 + 2\,lms_3 + 2\,mns_1 + 2\,nls_2. \qquad (1)$$

The elongation in any direction may be graphically represented as follows:

Construct at P the quadric surface whose equation referred to the spatially fixed axes Px, Py, Pz is

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = k^2 \qquad (2)$$

where k is any constant linear magnitude. If r is the length of the line PQ intercepted by this surface, we have

$$r^2(al^2 + bm^2 + cn^2 + 2\,lms_3 + 2\,mns_1 + 2\,nls_2) = k^2;$$

$$\therefore\quad \epsilon = \frac{k^2}{r^2}, \qquad (3)$$

or *the elongation in any direction varies inversely as the square of the radius vector of the Elongation Quadric in this direction*, if we agree to call the above surface the *Elongation Quadric.*

It is possible, however, that equation (2) may fail to represent the elongation in *all* directions. For there may be contraction (negative elongation) in some directions, and then (2) will represent a hyperbolic surface, the radii of which will give as in (3) the *elongations*, while the contractions must be given by constructing the surface

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = -k^2, \qquad (4)$$

which is the hyperboloid conjugate to that which gives the elongations.

Unless, then, all lines are contracted or all lines elongated, there will really be two quadrics required, one to represent elongations and the other to represent contractions.

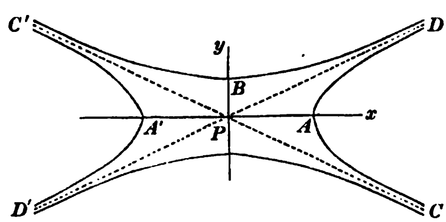

Fig. 291.

For example, consider the simple case in which the strain is made by drawing out all lines perpendicular to the plane yz in the same proportion, and contracting all lines perpendicular to the plane xz in the same proportion; so that

$$u = ax, \quad v = -by, \quad w = 0.$$

Then the elongation is given by the equation $\epsilon = al^2 - bm^2$. Now this expression is negative when $bm^2 > al^2$, and if we construct a surface whose equation is $a\xi^2 - b\eta^2 = 0$, i.e. two planes through the axis of z, this surface will form the boundary between lines which are elongated and lines which are contracted. The elongations are given by the radii of the surface $a\xi^2 - b\eta^2 = k^2$, a hyperbolic cylinder, the section of which by the plane xy is represented in Fig. 291 by the curve $(DAC, D'A'C')$; and the contractions by the conjugate surface $b\eta^2 - a\xi^2 = k^2$, which is represented by $(DBC', D'B'C)$; the planes of no elongation or contraction being the asymptotic planes, DD', CC', of these surfaces.

All lines through P along which the elongation is the same lie on a cone whose equation is easily found from (1). For, putting $\epsilon(l^2+m^2+n^2)$ for ϵ, we have

$$(a-\epsilon)l^2 + (b-\epsilon)m^2 + (c-\epsilon)n^2 + 2s_3 lm + 2s_1 mn + 2s_2 nl = 0;$$

and if ξ, η, ζ are the co-ordinates of any point on the line (l, m, n), we have $l:m:n = \xi:\eta:\zeta$; therefore this equation gives

$$(a-\epsilon)\xi^2 + (b-\epsilon)\eta^2 + (c-\epsilon)\zeta^2 + 2s_3\xi\eta + 2s_1\eta\zeta + 2s_2\zeta\xi = 0,$$

which, if ϵ is constant, denotes a cone whose vertex is P. This is called the *cone of equal elongation*. If ϵ is taken $= 0$, we get a cone of no elongation, and it is evidently (when real) the

asymptotic cone both of the Elongation Quadric and of the Compression Quadric.

Cor. 1. *The elongations in the directions of the axes of* x, y, z *are, respectively,* a, b, c, or $\frac{du}{dx}, \frac{dv}{dy}, \frac{dw}{dz}$.

Cor. 2. The elongation is the same along all parallel lines in the neighbourhood of P. For if R is any point very near P, the value of ϵ along a direction (l, m, n) at R is got by using the values of a, b, c, s_1, s_2, s_3 at R in equation (1). But these values at R differ from the values at P by infinitesimals of the second order. Therefore, &c.

Cor. 3. *Any small parallelogram or parallelopiped in the natural state in the neighbourhood of P is changed into another parallelogram or parallelopiped by the strain.*

For (Cor. 3, Art. 358) any two parallel lines are strained into two parallel lines, and (Cor. 2, Art. 359) they are equally elongated. Therefore, &c.

Cor. 4. *A small circle very near P in any plane is strained into an ellipse in a different plane.*

For, let AQB (Fig. 292) be a circle in the natural state; let OA and OB be any two rectangular diameters, Q any point on the circle, and QM and QN perpendiculars on OA and OB. Let the lines OA and OB become oa and ob (in a different plane) by the strain, and let Q become q. The circle will become a curve in the plane of oa and ob by Cor. 1, Art. 358. Also if qm and qn are drawn parallel to ob and oa, the lines QM and QN will become qm and qn; for M must become some point on oa (Cor. 2, Art. 358), and OB and QM must become parallel lines (Cor. 3, Art. 358).

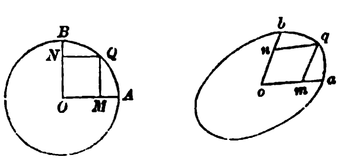

Fig. 292.

Again, if ϵ is the elongation along OA,

$$oa = (1+\epsilon)\,OA \text{ and } om = (1+\epsilon)\,OM;$$

$$\therefore \quad \frac{OM}{OA} = \frac{Om}{Oa},$$

similarly $$\frac{ON}{OB} = \frac{on}{ob}.$$

But $$\frac{OM^2}{OA^2}+\frac{ON^2}{OB^2}=1,$$

therefore $$\frac{om^2}{oa^2}+\frac{on^2}{ob^2}=1,$$

which shows that the curve on which q lies is an ellipse having the lines oa and ob for conjugate semi-diameters.

Hence every pair of rectangular radii of a circle is strained into a pair of semi-conjugate diameters of an ellipse; and since among these latter there is one rectangular pair (the axes of the ellipse), it follows that *some two rectangular diameters of the circle are strained into two rectangular lines.* Hence in every plane near P can always be found two rectangular lines which are strained into two rectangular lines.

COR. 5. *Any two small coplanar areas in the natural state are strained into two coplanar areas having the same ratio to each other as the unstrained areas.*

For let CAB and $C'A'B'$ be any two elementary rectangles in the same plane near P such that AB is parallel to $A'B'$ and AC parallel to $A'C'$. Then by Cor. 3 these will be strained into two parallelograms, cab and $c'a'b'$, such that ab is parallel to $a'b'$ and ac to $a'c'$.

Hence $$\frac{\text{area } cab}{\text{area } c'a'b'}=\frac{ac\times ab}{a'c'\times a'b'}.$$

Let ϵ be the elongation in the direction AB and ϵ' that in the direction AC; then

$$ab=(1+\epsilon)AB,\ a'b'=(1+\epsilon)A'B';$$

$$ac=(1+\epsilon')AC,\ a'c'=(1+\epsilon')A'C';$$

therefore $$\frac{\text{area } cab}{\text{area } c'a'b'}=\frac{AC\times AB}{A'C'\times A'B'}=\frac{\text{area } CAB}{\text{area } C'A'B'}.$$

Now, whatever be the two areas, they can each be broken up into an infinitely great number of small parallel rectangular strips, and the ratios of the strained areas of these strips being the same as those of the unstrained, the whole strained areas are to each other as the unstrained ones.

COR. 6. *Every small sphere in the natural state is strained into a small ellipsoid.* This is evident from Cor. 4, since the sphere, being a surface every section of which is a circle, must alter into

a surface every section of which is an ellipse. Nevertheless for clearness we may repeat the proof of that Cor. Let OA, OB, OC be any three rectangular semi-diameters of the sphere, Q any point on the sphere, QR a line parallel to OC terminated by the plane OAB, and RM, RN parallels to OB and OA. Let the lines OA, OB, OC be strained into oa, ob, oc, and Q to q. Then by Cor. 3, Art. 359, QR, RM, and RN will be strained into qr, rm, and rn which are parallels to oc, ob, and oa terminated by the planes oab, oac, and obc. Also by Cor. 2,

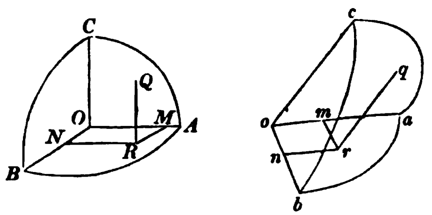

Fig. 293.

$$\frac{oa}{OA} = \frac{om}{OM}, \text{ i.e. } \frac{om}{oa} = \frac{OM}{OA};$$

similarly $$\frac{on}{ob} = \frac{ON}{OB}, \quad \frac{qr}{oc} = \frac{QR}{OC}.$$

But $$\frac{OM^2}{OA^2} + \frac{ON^2}{OB^2} + \frac{QR^2}{OC^2} = 1,$$

therefore $$\frac{om^2}{oa^2} + \frac{on^2}{ob^2} + \frac{qr^2}{oc^2} = 1,$$

which shows that the surface on which q lies is an ellipsoid having oa, ob, oc for a system of conjugate semi-diameters.

Hence *every rectangular set of radii of a sphere in the natural state is strained into a system of conjugate semi-diameters of the ellipsoid into which the sphere is changed;* and it follows that there is one rectangular set which is strained into a rectangular set and altered in directions, the latter being the axes of the ellipsoid into which the sphere is strained.

COR. 7. *Any two small volumes in the natural state are strained into two small volumes which bear the same ratio to each other as the unstrained volumes.* The proof of this proceeds exactly as in Cor. 5.

360.] **Lines of no Rotation.** Let us enquire whether, with the given strain, it is possible to find a particle Q, in the natural state of the body, such that its displaced position, Q'', shall be

on the line PQ. If this is so, all particles (near P) on the line PQ will retain the same direction with respect to P; i.e. the line PQ will not suffer rotation by the strain.

The direction cosines of PQ'' are $\dfrac{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}}{PQ''}, \ldots,$ and those of PQ are $\dfrac{\xi}{PQ}, \ldots.$ Hence if these are the same,

$$\frac{\xi\left(1+\frac{du}{dx}\right)+\eta\frac{du}{dy}+\zeta\frac{du}{dz}}{PQ''}=\frac{\xi}{PQ},$$

with two similar equations. Now $PQ''=(1+\epsilon)PQ$; hence

$$\left(\frac{du}{dx}-\epsilon\right)\xi+\frac{du}{dy}.\eta+\frac{du}{dz}.\zeta=0,$$

with two similar equations; or if l, m, n be the direction cosines of PQ, a line of no rotation,

$$\left.\begin{aligned} l\left(\frac{du}{dx}-\epsilon\right)+m\frac{du}{dy}+n\frac{du}{dz}&=0,\\ l\frac{dv}{dx}+m\left(\frac{dv}{dy}-\epsilon\right)+n\frac{dv}{dz}&=0,\\ l\frac{dw}{dx}+m\frac{dw}{dy}+n\left(\frac{dw}{dz}-\epsilon\right)&=0.\end{aligned}\right\} \tag{1}$$

By eliminating l, m, n from these equations, we obtain the cubic equation for ϵ

$$\begin{vmatrix} \frac{du}{dx}-\epsilon, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dv}{dx}, & \frac{dv}{dy}-\epsilon, & \frac{dv}{dz} \\ \frac{dw}{dx}, & \frac{dw}{dy}, & \frac{dw}{dz}-\epsilon \end{vmatrix}=0, \tag{2}$$

which gives necessarily one real value of ϵ and may give three real values.

Hence *in the small general strain of an elastic solid there is at every point at least one line of no rotation.*

361.] **Change of inclination of two lines.** In the unstrained state let there be two points, Q_1 and Q_2 very near P, and let ϕ be the angle between the lines PQ_1 and PQ_2. We propose to find the angle between the lines into which these are strained. Let $(\xi_1\,\eta_1\,\zeta_1)$ and $(\xi_2\,\eta_2\,\zeta_2)$ be the co-ordinates of Q_1 and Q_2 with

reference to Px, Py, Pz (Fig. 290); and supposing that the strained positions of Q_1 and Q_2 are Q_1'' and Q_2'', whose co-ordinates are $(\xi_1' \eta_1' \zeta_1')$ and $(\xi_2' \eta_2' \zeta_2')$, we have, by Art. 357,

$$\xi_1' = (1+a)\xi_1 + \eta_1 \frac{du}{dy} + \zeta_1 \frac{du}{dz}, \quad \eta_1' = \xi_1 \frac{dv}{dx} + (1+b)\eta_1 + \zeta_1 \frac{dv}{dz},$$

$$\zeta_1' = \xi_1 \frac{dw}{dx} + \eta_1 \frac{dw}{dy} + (1+c)\zeta_1,$$

$$\xi_2' = (1+a)\xi_2 + \eta_2 \frac{du}{dy} + \zeta_2 \frac{du}{dz}, \quad \eta_2' = \xi_2 \frac{dv}{dx} + (1+b)\eta_2 + \zeta_2 \frac{dv}{dz},$$

$$\zeta_2' = \xi_2 \frac{dw}{dx} + \eta_2 \frac{dw}{dy} + (1+c)\zeta_2.$$

Hence, neglecting squares and products of $a, b, \ldots, \frac{du}{dy}, \ldots$, we have

$$\xi_1'\xi_2' + \eta_1'\eta_2' + \zeta_1'\zeta_2' = \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2$$
$$+ 2(a\xi_1\xi_2 + b\eta_1\eta_2 + c\zeta_1\zeta_2) + (\xi_1\eta_2 + \xi_2\eta_1)\left(\frac{du}{dy} + \frac{dv}{dx}\right)$$
$$+ (\eta_1\zeta_2 + \eta_2\zeta_1)\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + (\zeta_1\xi_2 + \zeta_2\xi_1)\left(\frac{dw}{dx} + \frac{du}{dz}\right).$$

If ϕ' is the angle between PQ_1'' and PQ_2'',

$$\cos\phi' = \frac{\xi_1'\xi_2' + \eta_1'\eta_2' + \zeta_1'\zeta_2'}{PQ_1''.PQ_2''};$$

so that if ϵ_1 and ϵ_2 are the elongations in the directions PQ_1 and PQ_2, and $(l_1 m_1 n_1)$, $(l_2 m_2 n_2)$ the direction cosines of the lines PQ_1 and PQ_2, the above equation gives

$$(1+\epsilon_1)(1+\epsilon_2)\cos\phi' = \cos\phi + 2(al_1l_2 + bm_1m_2 + cn_1n_2)$$
$$+ 2s_3(l_1m_2 + l_2m_1) + 2s_1(m_1n_2 + m_2n_1) + 2s_2(n_1l_2 + n_2l_1);$$

or dividing out by $(1+\epsilon_1)(1+\epsilon_2)$,

$$\cos\phi' = \cos\phi(1 - \epsilon_1 - \epsilon_2) + 2(al_1l_2 + bm_1m_2 + cn_1n_2)$$
$$+ 2s_3(l_1m_2 + l_2m_1) + 2s_1(m_1n_2 + m_2n_1) + 2s_2(n_1l_2 + n_2l_1), \quad (1)$$

the products of the elongations and the small quantities $a, b, \ldots$ $s_3, \ldots$ being rejected. The change in the cosine of the angle between any two rectangular lines is got by putting $\phi = \frac{\pi}{2}$. Denoting this change by $2s$, we have

$$s = al_1l_2 + bm_1m_2 + cn_1n_2$$
$$+ s_3(l_1m_2 + l_2m_1) + s_1(m_1n_2 + m_2n_1) + s_2(n_1l_2 + n_2l_1). \quad (2)$$

COR. 1. *The quantities* $2s_3$, $2s_1$, $2s_2$ *are, respectively, the cosines of the angles between the strained positions of the axes of* (x, y), (y, z), (z, x).

Hence the lines through P which in the unstrained state of the body were parallel to the axes of reference are changed by strain into slightly oblique pairs of lines containing, respectively, angles equal to

$$\frac{\pi}{2}-2s_3, \quad \frac{\pi}{2}-2s_1, \quad \frac{\pi}{2}-2s_2.$$

COR. 2. The result at the end of Cor. 6, Art. 359, easily follows from the value of $\cos\phi'$ in (1). For if $\phi=\frac{\pi}{2}$ and also $\phi'=\frac{\pi}{2}$, the directions of the lines PQ_1 and PQ_2 are connected by the equation

$$al_1l_2+bm_1m_2+cn_1n_2+s_3(l_1m_2+l_2m_1)+s_1(m_1n_2+m_2n_1)$$
$$+s_2(n_1l_2+n_2l_1)=0,$$

or $$(al_1+s_3m_1+s_2n_1)l_2+(s_3l_1+bm_1+s_1n_1)m_2+$$
$$(s_2l_1+s_1m_1+cn_1)n_2=0,$$

which shows that PQ_1 and PQ_2 are conjugate diameters of the quadric

$$a\xi^2+b\eta^2+c\zeta^2+2s_3\xi\eta+2s_1\eta\zeta+2s_2\zeta\xi=k^2,$$

k being any constant.

COR. 3. *The quantities* $b+c$, $c+a$, $a+b$ *are, respectively, the areal dilatations, that is, the ratios of increase of small areas to their original values in the planes of* yz, zx, xy.

For, since all small areas near P in the plane yz are altered in the same ratio, to determine this ratio we may take a small rectangle with lengths m and n along Py and Pz. The sides of this become (Cor. 1, Art. 359) $(1+b)\,m$ and $(1+c)\,n$, and, the cosine of the angle between them becoming s_1, the sine of this angle is 1 to the order of accuracy adopted. Hence the new area is

$$(1+b)(1+c)\,mn, \quad \text{or} \quad mn+(b+c)\,mn;$$

or if A and A' are the unstrained and strained areas

$$\frac{A'-A}{A}=b+c=\text{areal dilatation}.$$

Similarly for dilatations in the other planes.

COR. 4. *The quantity $a+b+c$ is the cubical dilatation, that is, the ratio of the increase of any small volume at P to the unstrained magnitude of this volume.*

For, since all small volumes near P are increased in the same ratio, to determine this ratio we may take a small and rectangular parallelopiped with edges m, n, p along the axes Px, Py, Pz. These edges become $(1+a)\,m$, $(1+b)\,n$, $(1+c)\,p$, respectively, and the sines of the angles between them are each 1, to the order adopted. Hence the strained volume is

$$(1+a)(1+b)(1+c)\,mnp, \quad \text{or} \quad mnp+(a+b+c)\,mnp;$$

so that if V and V' are the unstrained and strained volumes,

$$\frac{V'-V}{V}=a+b+c.$$

COR. 5. We conclude at once that, *whatever system of rectangular lines is drawn through P, the sum, $a+b+c$, of the elongations along them is constant.*

For, the ratio in which any volume is increased cannot depend on any particular set of axes of reference. This also follows from the value of ϵ given in Art. 359.

362.] **Transformation of Strain.** *Given the components of a strain with reference to one set of rectangular axes, to find the components of the same strain with reference to any other set of rectangular axes.*

The components with reference to a set of axes, Px, Py, Pz, being $a, b, c, 2s_3, 2s_1, 2s_2$ $\left(\text{or } \frac{du}{dx}, \ldots, \frac{du}{dy}+\frac{dv}{dx}, \ldots\right)$, we wish to find them with reference to a set, Px', Py', Pz', whose direction cosines are (l, m, n), (l', m', n'), (l'', m'', n''), respectively.

The value of $\frac{du'}{dx'}$ is simply the elongation in the direction (l, m, n). Hence

$$a'=al^2+bm^2+cn^2+2s_3\,lm+2s_1\,mn+2s_2\,nl,$$

with exactly similar values of b' and c'.

Again, $\frac{du'}{dy'}+\frac{dv'}{dx'}$ is simply the cosine of the angle between the strained positions of the two lines Px', Py'; hence, by (2) of last Art.,

$$s_3'=all'+bmm'+cnn'+s_3(lm'+l'm)+s_1(mn'+m'n)+s_2(nl'+n'l),$$

with exactly similar values of s_1' and s_2'.

Two strains having reference to two distinct sets of axes are equivalent when each produces the other; and either may be substituted for the other.

In particular, given the strain components, e_1, e_2, e_3, referred to the principal axes of the strain, we have for the components with reference to any system of rectangular axes the values

$$\begin{aligned} a &= e_1 l^2 + e_2 m^2 + e_3 n^2; & s_1 &= e_1 l' l'' + e_2 m' m'' + e_3 n' n'', \\ b &= e_1 l'^2 + e_2 m'^2 + e_3 n'^2; & s_2 &= e_1 l l'' + e_2 m m'' + e_3 n n'', \\ c &= e_1 l''^2 + e_2 m''^2 + e_3 n''^2; & s_3 &= e_1 l l' + e_2 m m' + e_3 n n'. \end{aligned}$$

363.] **The Strain Ellipsoid.** It has been already proved (Cor. 6, Art. 359) that a small sphere in the unstrained state of the body is converted by the strain into an ellipsoid. This latter surface is called the *Strain Ellipsoid* of the given strain. We here exhibit its deduction analytically.

Let the point Q (Fig. 290) be any point on a sphere of radius r and centre P. Then, Px, Py, Pz being axes of co-ordinates,

$$\xi^2 + \eta^2 + \zeta^2 = r^2.$$

It is required to find the surface traced out by Q'', the strained position of Q, as the latter varies on the surface of the sphere. The co-ordinates of Q'' being, as in Art. 358, ξ', η', ζ', we have by squaring and adding

$$\xi^2 + \eta^2 + \zeta^2 = \left(1 - 2\frac{du}{dx}\right)\xi'^2 + \left(1 - 2\frac{dv}{dy}\right)\eta'^2 + \left(1 - 2\frac{dw}{dz}\right)\zeta'^2$$
$$- 4 s_3 \xi'\eta' - 4 s_1 \eta'\zeta' - 4 s_2 \zeta'\xi',$$

or $$\left(\tfrac{1}{2} - a\right)\xi'^2 + \left(\tfrac{1}{2} - b\right)\eta'^2 + \left(\tfrac{1}{2} - c\right)\zeta'^2 - 2 s_3 \xi'\eta' - 2 s_1 \eta'\zeta'$$
$$- 2 s_2 \zeta'\xi' - \frac{r^2}{2} = 0, \quad (1)$$

which is a quadric, and necessarily an ellipsoid since a sphere must be strained into a closed surface. As we have been using ξ, η, ζ to denote running co-ordinates, we may without confusion write the equation of the strain ellipsoid

$$\left(\tfrac{1}{2} - a\right)\xi^2 + \left(\tfrac{1}{2} - b\right)\eta^2 + \left(\tfrac{1}{2} - c\right)\zeta^2 - 2 s_3 \xi\eta - 2 s_1 \eta\zeta - 2 s_2 \zeta\xi$$
$$- \tfrac{1}{2} r^2 = 0. \quad (2)$$

364.] **Principal Axes and Principal Elongation of a Strain.** *The principal axes of a strain at any point P are those three rectangular lines* (Cor. 6, Art. 359) *which become by the strain the axes of the strain ellipsoid;* and since in general the direction of a

line is altered by the strain, the principal axes of the strain are, in general, rotated by the strain about the point P.

The principal elongations of a strain at any point P are the elongations along the principal axes. We shall denote these by e_1, e_2, e_3.

COR. If the axes of co-ordinates at P are taken in the directions of the axes of the strain ellipsoid, the quantities s_1, s_2, and s_3 are all zero, as is evident from (2) of last Art., and the equation of this ellipsoid will be

$$(\tfrac{1}{2}-e_1)\,\xi^2+(\tfrac{1}{2}-e_2)\,\eta^2+(\tfrac{1}{2}-e_3)\,\zeta^2-\tfrac{1}{2}\,r^2=0. \qquad (a)$$

365.] **Pure Strain.** A strain is said to be *pure* when the lines at P which become the axes of the strain ellipsoid are unaltered in their directions by the strain.

366.] **Conditions for a Pure Strain.** If ϵ is the elongation in the direction of any radius vector of the strain ellipsoid, we have

$$\rho = r(1+\epsilon),$$

where ρ is the length of this radius vector and r the radius of the sphere which becomes by strain the strain ellipsoid.

Hence if the axes of this ellipsoid are a, β, γ, we have

$$a = r(1+e_1),$$
$$\beta = r(1+e_2),$$
$$\gamma = r(1+e_3)$$

Now if l, m, n are the direction-cosines of any axis, we have (see Salmon's *Geometry of Three Dimensions*, or Frost's *Solid Geometry*)

$$\left.\begin{aligned}(\tfrac{1}{2}-a)\,l-s_3.m-s_2.n &= \lambda l,\\ -s_3.l+(\tfrac{1}{2}-b)\,m-s_1.n &= \lambda m,\\ -s_2.l-s_1.m+(\tfrac{1}{2}-c)\,n &= \lambda n,\end{aligned}\right\} \qquad (1)$$

the three values $(\lambda_1, \lambda_2, \lambda_3)$ of λ obtained from these equations being such that the equation of the ellipsoid referred to its own axes would be

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2-\frac{r^2}{2}=0.$$

Hence $\quad \lambda_1 = \dfrac{r^2}{2a^2} = \tfrac{1}{2}-e_1;\ \ \lambda_2 = \tfrac{1}{2}-e_2;\ \ \lambda_3 = \tfrac{1}{2}-e_3.$

Therefore if ϵ stands for any one of the principal elongations, e_1, e_2, e_3, the equations (1) become, for the direction of any *axis*,

$$\left.\begin{aligned}(a-\epsilon)\,l+s_3 m+s_2 n &= 0,\\ s_3 l+(b-\epsilon)\,m+s_1 n &= 0,\\ s_2 l+s_1 m+(c-\epsilon)\,n &= 0.\end{aligned}\right\} \qquad (2)$$

Now if there are three unrotated lines, they are given by equations (1), Art. 360; and if the same lines are determined by (2), we must have

$$\frac{du}{dy} = \frac{dv}{dx} = s_3; \quad \frac{du}{dz} = \frac{dw}{dx} = s_2; \quad \frac{dv}{dz} = \frac{dw}{dy} = s_1;$$

and the conditions for pure strain are that the displacements u, v, w satisfy the equations

$$\frac{du}{dy} - \frac{dv}{dx} = 0, \quad \frac{du}{dz} - \frac{dw}{dx} = 0, \quad \frac{dv}{dz} - \frac{dw}{dy} = 0. \tag{3}$$

These are the well-known conditions that the expression

$$u\,dx + v\,dy + w\,dz,$$

in which u, v, w are functions of x, y, z, should be the perfect differential of a single function, $\phi(x, y, z)$. When this function exists, i.e. when the strain is pure, it is called the *Displacement Potential* of the strain.

Hence the components, $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, of the strain (given in Art. 358) become when the strain is pure

$$\left.\begin{aligned} \Delta\xi &= a\xi + s_3\eta + s_2\zeta, \\ \Delta\eta &= s_3\xi + b\eta + s_1\zeta, \\ \Delta\zeta &= s_2\xi + s_1\eta + c\zeta, \end{aligned}\right\} \tag{4}$$

i.e. the coefficient of η in $\Delta\xi$ is the same as the coefficient of ξ in $\Delta\eta$, &c.; and this is the distinguishing character of a pure strain. A pure strain is also called an *irrotational* strain.

The values of the principal elongations of a strain are the roots of the cubic equation

$$\begin{vmatrix} a-\epsilon, & s_3, & s_2 \\ s_3, & b-\epsilon, & s_1 \\ s_2, & s_1, & c-\epsilon \end{vmatrix} = 0,$$

or $$\epsilon^3 - (a+b+c)\,\epsilon^2 + (ab+bc+ca-s_1^2-s_2^2-s_3^2)\,\epsilon + as_1^2 + bs_2^2 + cs_3^2 - abc - 2s_1s_2s_3 = 0.$$

367.] Rotation and Strain proper. *Every strain can be resolved into a pure strain and a rotation.* By a *rotation* here is meant such a displacement as a *rigid* body undergoes in turning round an axis, and we propose to show that the general small displacement at any point P of a body, may be produced by two operations, viz. first holding fixed in directions the principal axes of the strain and straining the body to a certain extent, and then rotating it as a rigid body about a certain axis.

It has been shown (p. 104) that if a rigid body receives *small* angular displacements, ω_1, ω_2, ω_3, round three fixed rectangular axes, the displacements of the co-ordinates, ξ, η, ζ of any point in it are

$$\omega_2\zeta-\omega_3\eta, \quad \omega_3\xi-\omega_1\zeta, \quad \omega_1\eta-\omega_2\xi. \tag{1}$$

(Such a displacement has, of course, no displacement potential; for if these displacements are denoted by u, v, w, we have $\frac{du}{d\eta}-\frac{dv}{d\xi}$ equal to $-2\omega_3$ and not equal to zero.)

Now the component, $\Delta\xi$, of the displacement of Q along the axis Px is (Art. 358)

$$a\xi+\eta\frac{du}{dy}+\zeta\frac{du}{dz};$$

and this $= a\xi+\frac{1}{2}\left(\frac{du}{dy}+\frac{dv}{dx}\right)\eta+\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right)\zeta+\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right)\zeta$

$$-\frac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\eta.$$

Hence, with the same values of s_1, s_2, s_3 as before, we have

$$\Delta\xi = a\xi+s_3\eta+s_2\zeta+\left\{\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right)\zeta-\frac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\eta\right\},$$

$$\Delta\eta = s_3\xi+b\eta+s_1\zeta+\left\{\frac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)\xi-\frac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\zeta\right\},$$

$$\Delta\zeta = s_2\xi+s_1\eta+c\zeta+\left\{\frac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right)\eta-\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right)\xi\right\}.$$

A comparison with (1) shows that the portions in brackets in these expressions denote rotations, as of a rigid body, about the axes through the small angles

$$\omega_1=\frac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right), \quad \omega_2=\frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right), \quad \omega_3=\frac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right), \tag{2}$$

which are equivalent to a rotation through $\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}$ about one line (p. 103); while the portions of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ outside the brackets denote a pure strain by Art. 366.

If the axes of reference, Px, Py, Pz, are chosen in the directions of the principal axes, the strain portion of the displacement will be expressed by

$$\Delta\xi = e_1\xi, \quad \Delta\eta = e_2\eta, \quad \Delta\zeta = e_3\zeta,$$

i.e. the strain is produced simply by multiplying the co-ordinates of every particle by the numbers $1+e_1$, $1+e_2$, $1+e_3$. A *simple*

elongation of a body in a direction perpendicular to any plane means the drawing out from the plane of every particle through a distance proportional to the perpendicular from the particle on the plane, so that those particles farthest from the plane in the natural state are most drawn away, but all in the same proportion to their original distances from it.

By this Article we see that *every small displacement at a point P can be produced by three successive simple elongations followed by a rotation, as of a rigid body, about an axis through P.*

The rotation part of the displacement in the neighbourhood of a point does not belong to the *strain* proper of the substance at the point. This rotation will frequently be called in the sequel the *vortical rotation.*

368.] **Significations of** s_1, s_2, s_3. Let the axes Px and Py become by strain Px'' and Py'', Fig. 294. (Of course it is supposed, as in Art. 358, that P is brought back to its original position after the strain.) All particles in the plane of Px and Py originally are in the (different) plane of Px'' and Py'' after the strain; and if A is a particle on the axis of y and AB a line parallel to Px, the line of particles AB will become (Cor. 3, Art. 358) a line of particles $A''B''$ parallel to Px''. Let fall a perpendicular, $A''p$, from A'' on Px''. Then the particle (A'') which was at A has advanced in front of P parallel to the line Px'' through the distance Pp. Now $Pp = PA'' \cos x''Py'' = 2PA''.s_3$ (Cor. 1, Art. 361); and $PA'' = (1+b)PA$; therefore $Pp = 2(1+b)s_3.PA$; or, neglecting the product bs_3,

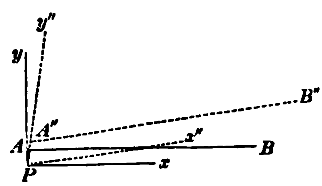

Fig. 294.

$$\frac{Pp}{PA} = 2s_3.$$

Hence the quantity $2s_3$ is the rate (per unit of distance between the two lines) at which particles on any line AB parallel to Px have slid beyond the corresponding particles on Px. Evidently it is also the rate at which sliding has taken place between particles on Py and lines parallel to Py.

Or again, imagine a little parallelopiped at P having its edges along the lines Px, Py, Pz. Then $2s_3$ is the rate at which the

face parallel to that in the plane xz has slid in front of the latter; or the rate at which the face parallel to the plane yz has slid in front of the face in the plane yz.

Similarly for the values of s_1 and s_2.

The effects of the separate components of strain in producing deformation may be usefully studied by finding the displaced positions of the various corners of the cube represented in Fig. 228, p. 1, as given by the equations (4), p. 391. Thus, if a alone exists, every point will be simply displaced parallel to Ox. If s_1 alone exists, the edge OA will not be moved; the points C and H will move through equal small distances along CF and HO'; while B and D will be moved along BF and DO'. This cube is thus changed into an oblique parallelopiped.

DEF. When a plane is held fixed in a body and all planes in the body parallel to it are slid in the same direction and sense parallel to the fixed plane, each through a distance proportional to its distance from the fixed plane, the strain so produced is called a *shearing strain*.

The ratio of the distance through which any plane has slid to its distance from the fixed plane is called the *amount* of the shear. Hence *the quantities* $2s_1$, $2s_2$, $2s_3$ *are the small shears of the axes of* (y, z), (z, x), (x, y) *respectively, at the point* P.

From Fig. 294 it is clear that the change in the cosine of the angle between any two lines at right angles in the natural state is the shear in their plane of lines parallel to either.

369.] **Shearing Strain.** The two kinds of strain with which we are most concerned in the theory of Elasticity are *Cubical Dilatation* and *Shearing Strain.* We propose, therefore, to consider this latter more particularly here.

Treating it first analytically, and confining our attention to a shear, $2s_3$, of the two rectangular lines Ox and Oy, the elongation quadric would be

$$2s_3\,\xi\eta = k^2,$$

the axes of co-ordinates being the lines Ox and Oy.

But this equation denotes a hyperbola in the plane xy referred to its asymptotes; and if we alter the axes of co-ordinates to the axes of the curve, the equation referred to them will be

$$s_3(\xi^2-\eta^2) = k^2.$$

A comparison with the general equation of the elongation quadric shows that this equation denotes an elongation s_3 (half

the shear) of the body along one axis of the curve, accompanied by an elongation $-s_3$ (i.e. an equal *compression*) of the substance along the other axis.

Hence *the shearing strain of a body can be produced by a simple elongation* (*equal to half the shear*) *along one line and a simple compression of equal amount along a perpendicular line.*

The same result is easily found also by considering the displaced positions of the various corners of the cube in Fig. 295, due to the shearing strain $2s_1$ alone (shear of the axes y, z), and given by equations (4), p. 391.

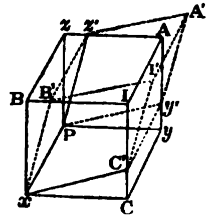

Fig. 295.

These equations give for the displacements of any point

$$\Delta\xi = 0,\ \Delta\eta = s_1\zeta,\ \Delta\zeta = s_1\eta;$$

so that the points y, z will be displaced to y', z', through equal distances; no point on Px will be displaced; the points B, C will come to B', C', such that $BB' = zz'$, $CC' = yy'$; finally, the points A, I will be displaced to A', I' along the diagonals PA and xI, respectively, the lines $z'A'$ and $y'A'$ being parallel to Py' and Pz', so that the squares $zPyA$ and $BxCI$ become lozenges by the strain, and the cube is altered into an oblique parallelopiped.

The angle $z'Py' = \frac{\pi}{2} - 2s_1$; the diagonal zy is shortened into $z'y'$, while the diagonal PA is elongated into PA'. If $l =$ the length of an edge of the cube,

$$z'y' = l(1-s_1)\sqrt{2}, \text{ and } PA' = l(1+s_1)\sqrt{2}.$$

The squares zPx, AyC, and all sections parallel to them remain squares, and are therefore quite undistorted; and the length of every line in these planes remains unaltered. The same is true of all planes parallel to xPy. Hence the shearing strain could be replaced by a strain in which every line near P parallel to the diagonal zy is compressed, and every line near P parallel to xI is elongated, the amount of the compression and of the elongation being s_1.

The axes of the elongation quadric, $2s_1(y^2 - z^2) = k^2$, belonging to this strain are the internal and external bisectors of the angle zPy, together with Px.

We have been considering small displacements; but let us now consider an elongation of any amount along a line Ox, and an equal compression along a perpendicular Oy (Fig. 296). Suppose that all lines in the body parallel to Ox are increased in the ratio $a : 1$, and that all lines parallel to Oy are diminished in the ratio $1 : a$; and consider displacements in the plane xy. There will, of course, be similar displacements in all planes parallel to xy. The displacement of the point O may be impressed in reversed direction on all points, so that O may be considered as at rest.

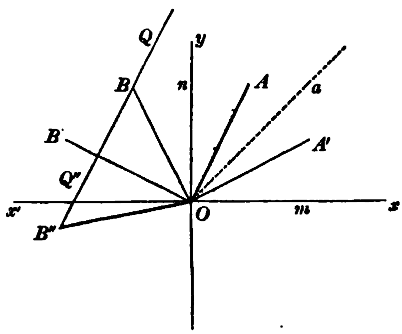

Fig. 296.

Draw OA, of any length, making the angle $AOx = \tan^{-1}a$. From A let fall An perpendicular to Oy. Then An becomes elongated by the strain parallel to Ox into $a \,.\, An$; but $a \,.\, An = nO$; therefore by this strain A is drawn out to a, Aa being parallel to Ox, and a a point on the bisector, Oa, of the angle xOy. From a draw am perpendicular to Ox. Then by the strain parallel to Oy, am becomes shortened into $\dfrac{am}{a}$. Now if we draw OA' making with Ox an angle equal to AOy, this line will meet am in a point, A', such that $A'm = \dfrac{am}{a}$. Hence after the two strains A will come to A'; and we see that *OA' is equal in length to OA, and that they are both equally inclined to the bisector of the angle xOy.*

In the same way if OB be drawn making $\angle BOx' = \tan^{-1}a$, the length of OB will be unaltered, the point B will come to B', and the lines OB and OB' are equally inclined to the bisector of the angle $x'Oy$. Also OA is perpendicular to OB'. Hence since parallel lines are all altered in the same ratio, all lines parallel to OA are unaltered in length, and all lines parallel to OB are unaltered in length.

Imagine a plane through OA perpendicular to the plane of the

paper, and let any curve whatever be traced out in this plane. The curve will remain perfectly undistorted after the strain. For, all lines perpendicular to the plane of the paper obviously remain so and are unaltered in length, and all lines parallel to the plane of the paper remain parallel to this plane, while of these latter those which are parallel to OA remain unaltered in length. Hence ordinates and abscissæ of the above-named curve parallel to OA and to a normal to the plane of the paper remain perpendicular to each other and unaltered in length. The curve, therefore, as regards magnitude and shape remains exactly as it was; its plane only is altered (to the plane through OA' perpendicular to the paper).

It follows, of course, *that all lines, whatever be their directions, in the plane through OA perpendicular to the paper remain unaltered in length.*

Similarly all lines in the plane through OB and the normal to the paper remain unaltered in magnitude; and all figures in this plane also remain undistorted.

The planes through the normal to the paper and the lines OA and OB are called *the planes of no distortion.*

Suppose that we impress on the body a common motion of rotation about the normal to the paper at O so as to bring OA' into coincidence with OA. This motion will, of course, be unaccompanied by any strain. Then OB' will come to OB'', and BB'' is perpendicular to OB' and parallel to OA, as is very easily seen.

Draw BQ parallel to OA. Then since the length of BQ remains unaltered, Q will come to Q'', a point such that $B''Q'' = BQ$. Hence all particles in the line BQ are slid parallel to AO through a space BB''. Now if p is the length of the perpendicular from B on OA,

$$\frac{BB''}{p} = a - \frac{1}{a},$$

as is easily found.

Consequently in this strain if *the undistorted plane OA is held fixed, every plane, BQ, parallel to it is slid parallel to it through a distance proportional to the perpendicular distance between BQ and OA*; and this is the usual way of representing a shearing strain.

Of course the strain may otherwise be produced (neglecting the effect of mere rotation common to all points) by holding fast

distance from the axis. That is, at each point there will be uniform elongation and uniform contraction. This will be proved in the last section of this chapter. Hence if the axis of the bar is taken as that of z, and the axes of x and y are in the plane of the fixed base,

$$u = -ax, \quad v = -ay, \quad w = cz$$

will express the displacements of any point, the quantities a and c being constant throughout the bar. This is the case of *Traction*. Suppose that, the base being still held fixed, the free extremity is twisted round through any angle (measured by the angle through which any diameter of the section revolves); then every other normal section of the bar will turn through an angle proportional to the distance, z, of this section from the fixed base.

If l = length of bar, a = angle through which its free end is twisted, every point in the section considered will be twisted through an angle $a\frac{z}{l}$. Hence the displacements of a point x, y in this section are (the twisting taking place from axis of x towards axis of y)

$$y = -\frac{azy}{l}, \quad v = \frac{azx}{l}, \quad w = 0.$$

This strain is called *Torsion*.

372.] **Lines of Flow and Vortex Lines.** Just as a Line of Force has been defined (p. 316) as a curve at every point of which the resultant force of attraction of a system is directed along the tangent, so a *Line of Flow* is defined to be a curve at every point of which the resultant displacement of the particle existing there is directed along the tangent.

Again, we have seen that the displacement at any point can be produced by a pure strain together with a rotation round an axis through the point. A curve such that at every point of it the rotation corresponding to that point takes place round the tangent is called a *Vortex Line*.

In analogy with a Tube of Force, we have a *Tube of Flow*. If through points constituting the contour of any area we draw Lines of Flow, these lines form a surface called a Tube of Flow. Similarly, if through the points constituting the contour of any area we draw Vortex Lines, these lines will make a surface which may be called a *Vortex Tube*.

When the normal section of the Vortex Tube is everywhere very small, it is called a *Vortex Filament.* Such a filament, AB, is represented in Fig. 297.

373.] **Equipotential Surfaces.** When the strain at every point is irrotational, the quantity $udx + vdy + wdz$ is a perfect differential of a function $\phi(x, y, z)$. Describe in the body a series of surfaces the equation of any one of which is

$$\phi(x, y, z) = C. \tag{1}$$

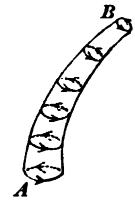

Fig. 297.

Then by giving C a series of different values we shall have a series of surfaces, exactly analogous to the equipotential surfaces of an attracting mass; and these equipotential surfaces of strain will be related to the lines of flow exactly as the equipotential surfaces of attraction are to the lines of force; that is, at every point the line of flow is perpendicular to the equipotential surface. For, the direction-cosines of the normal to the surface (1) at any point (x, y, z) are proportional to $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$; i.e. to u, v, w. But u, v, w, being the components of the displacement, are of course proportional to the direction-cosines of the line of flow. Therefore, &c.

The potential function of any small strain being ϕ, we see that $\frac{d\phi}{dx}$ is the displacement parallel to the axis of x; and since the axis of x may be in any direction, *the displacement in any direction is the rate of variation, per unit of length, of potential in this direction.*

It follows that the resultant displacement (which is perpendicular to the surface $\phi = C$) is $\frac{d\phi}{dn}$, where n denotes length measured along the normal to the surface, and the displacement is measured in the same sense as n.

Let two very close equipotential surfaces, $\phi = C_1$, $\phi = C_2$, be described. Denote these by ϕ_1 and ϕ_2. Then at all points on ϕ_1 the resultant displacement is inversely proportional to the normal distance at this point between the surfaces ϕ_1 and ϕ_2.

For, if at any point on the surface ϕ_1 the normal distance between it and ϕ_2 is Δn, the displacement is $\frac{\Delta\phi}{\Delta n}$ or $\frac{\phi_2 - \phi_1}{\Delta n}$.

But for all points considered $\phi_2 - \phi_1 = C_2 - C_1 =$ a constant; therefore the displacement varies inversely as Δn.

374.] **Circulation.** Suppose any curve, AB, to be traced out in the body, and let the displacement of each particle, P, on the curve between A and B be resolved along the tangent to the curve at P (the resolution taking place between A and B in a sense opposite to that of watch-hand rotation); then the sum obtained between A and B by multiplying this resolved part of displacement by the element, ds, of the curve at P and adding all such products is called the *circulation* between A and B. Hence, by definition, the circulation from B to A is equal and opposite to the circulation from A to B.

The components of the displacement parallel to the axes being, as before, u, v, w, and the direction-cosines of the tangent to the curve at P being $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, the circulation is

$$\int\left(u\frac{dx}{ds} + v\frac{dy}{ds} + w\frac{dz}{ds}\right) ds, \text{ or } \int(udx + vdy + wdz),$$

the integral being taken from A to B.

Supposing that there is no rotation, or, in other words, that there is a displacement potential which has a value ϕ_1 at A and ϕ_2 at B, the circulation from A to B is $\phi_2 - \phi_1$; it therefore depends merely on the co-ordinates of A and B and not at all on the curve between them along which it is taken.

If the curve is closed, B coincides with A, and the circulation is zero, it being still supposed that the strain is irrotational. If A and B are any two points on an equipotential surface, the circulation along any path from one to the other is zero.

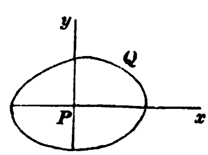

Fig. 298.

We now proceed to consider the case in which rotation exists, and to prove the following fundamental theorem :—

The circulation round any small plane curve described round any point, P, in the body is equal to twice the product of the area of the curve and the component of rotation at P perpendicular to the plane of the curve.

Let Q (Fig. 298) be any point on the small curve whose plane is taken as that of xy; denote the components of the displacement of P by u, v, w; and the co-ordinates of Q with reference

to P by ξ, η. Then the displacements of Q parallel to the axes are

$$u+\xi\frac{du}{dx}+\eta\frac{du}{dy},\quad v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy},\quad w+\xi\frac{dw}{dx}+\eta\frac{dw}{dy};$$

and the component of these along the tangent at Q is

$$\left(u+\xi\frac{du}{dx}+\eta\frac{du}{dy}\right)\frac{d\xi}{ds}+\left(v+\xi\frac{dv}{dx}+\eta\frac{dv}{dy}\right)\frac{d\eta}{ds}.$$

When this is multiplied by ds and integrated, we shall have (since u, v, $\frac{du}{dx}$, ... are constant for all points on the curve)

$$u\int d\xi+v\int d\eta+\frac{du}{dx}\int\xi d\xi+\frac{dv}{dy}\int\eta d\eta+\frac{dv}{dx}\int\xi d\eta+\frac{du}{dy}\int\eta d\xi,$$

of which all the integrals except the last two vanish, since the curve is closed. Now $\int\xi d\eta$ = area of curve = A; and $\int\eta d\xi = -A$, since the two integrations are carried round at the same time from x to y. Hence the circulation $= A\left(\frac{dv}{dx}-\frac{du}{dy}\right)$

$$= 2A\,.\,\omega_3,$$

(p. 392) ω_3 being the rotation round axis of z at P, i.e. perpendicular to the plane of the curve.

Suppose that any surface, plane or curved, bounded by any curve, $ABCD$ (Fig. 299), is traced out in the body and that at each point on this surface we take the component of rotation round the normal to the surface, multiply this component by the element of superficial area at the point, and take the sum of all such products. This sum is called the *surface-integral of normal rotation*. The normal must be supposed to be drawn away from the same side of the surface at every point, and the rotation is supposed to take place opposite to that of the hands of a watch held so that the normal passes up through its face.

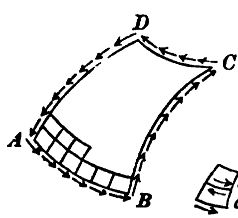

Fig 299.

It is very easy to prove that this *surface-integral of rotation is equal to one half the circulation round the edge, ABCD, of the surface.* For, let the surface be broken up into an indefinitely great number of little plane areas. Then the sum of the circu-

lations round these areas is twice the surface integral of rotation (by what has just been proved). But the circulations in the common portions of every two contiguous areas are directly opposed, and therefore mutually destructive, as is seen by drawing any two such little areas, a and b, apart; hence the circulation exists only along lines which do not form common parts of contiguous areas, i.e. along the edge which bounds the surface.

If the surface has no bounding edge, i.e. if it is a closed surface, the surface-integral of rotation over it is zero.

If the surface, without being closed, is such that at every point of it the rotation takes place about a tangent line to the surface, the circulation round its bounding edge is zero. Such a surface is that of a vortex filament (Fig. 297); or that represented in Fig. 300, which consists of a vortex tube whose ends are any two irregular curves whatever. The sum of the circulations round the terminal sections D and E of this tube, *estimated in the cyclical order indicated round the contour in* Fig. 299, is zero, i.e. *the circulations round any two sections whatever of a vortex tube are equal*; or, in other words, the circulation round any section, normal or oblique, plane or tortuous, of a vortex tube is constant.

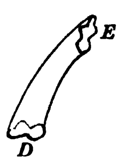

Fig. 300.

EXAMPLES.

1. Prove analytically that the shear of any two rectangular lines intersecting at any point is equal to the difference between the elongations along the internal and external bisectors of the angle between them.

Let the axes of co-ordinates be the principal axes of the strain at the point. Then the value of s given in equation (2), Art. 361, becomes

$$s = e_1 ll' + e_2 mm' + e_3 nn', \qquad (a)$$

the direction-cosines of the lines being (l, m, n) and (l', m', n'), and the shear $2s$. Now the direction-cosines of one bisector are $l-l'$, $m-m'$, $n-n'$, each divided by the square root of the sum of the squares of these quantities, i.e. by $\sqrt{2}$, since the lines are rectangular; and the direction-cosines of the other bisector are $l+l'$, $m+m'$, $n+n'$ each divided by $\sqrt{2}$. Let ϵ and ϵ' be the elongations along these bisectors. Then, by Art. 359,

$$2\epsilon = e_1(l-l')^2 + e_2(m-m')^2 + e_3(n-n')^2,$$
$$2\epsilon' = e_1(l+l')^2 + e_2(m+m')^2 + e_3(n+n')^2;$$

therefore $\epsilon' - \epsilon = 2(e_1 ll' + e_2 mm' + e_3 nn')$,

or $\epsilon' - \epsilon = 2s$,

which proves the proposition.

2. Find the pair of rectangular lines in a given plane for which the shear is greatest.

In any plane the elongation is greatest along one axis of the conic in which this plane cuts the Elongation Quadric, and least along the other. Therefore the *difference* of elongation along two rectangular lines is greatest for this pair; and therefore, by last example, the shear of the two rectangular lines of whose angle these axes are the external and internal bisectors is greatest.

Hence the shear in a given plane is greatest for two lines making angles of $\frac{\pi}{4}$ with the axes of the conic in which the given plane cuts the Elongation Quadric.

The magnitude of the shear for any two rectangular lines in the plane is easily found and represented by a curve.

Let the axes of x and y be taken in the given plane and coincident with the axes of the section of the Elongation Quadric in the plane. Then s_3 must $= 0$ for these axes. Also let one of two lines along which we wish to find the shear make an angle θ with the axis of x. Then in the expression for s (Art. 361) we have $l_1 = \cos\theta$, $m_1 = \sin\theta$, $l_2 = -\sin\theta$, $m_1 = \cos\theta$, $n_1 = n_2 = 0$; therefore

$$s = \tfrac{1}{2}(b-a)\sin 2\theta,$$

or

$$2s = (b-a)\sin 2\theta = \text{shear},$$

which of course shows that the shear is a maximum along lines bisecting the angles between the axes of the section. The curve whose polar equation is $r = (b-a)\sin 2\theta$ consists of four loops, one in each quadrant, and its radius-vector gives the shear for any directions, denoted by θ and $\frac{\pi}{2} + \theta$.

It follows that the two rectangular lines whose shear is absolutely the greatest at a point in the body are those in the plane of the greatest and least axes of the Elongation Quadric (or of the Strain Ellipsoid) and making angles of $\frac{\pi}{4}$ with them, and that their shear is $e_1 - e_3$, if we assume e_1, e_2, e_3 to be in descending order of magnitude.

3. Represent the shears of all pairs of rectangular lines obtained by taking a given line and all those at right angles to it at a given point.

Let the point O (Fig. 240, p. 37) be the point in the strained body through which the pairs of rectangular lines are drawn; OP, OA, OZ represent the principal axes of the strain at O; let OL be the given invariable line with which any variable line, OM (not drawn in the figure) at right angles OL is to be associated; draw the arc ZL,

and produce it to cut PA in N; let LN be produced to R so that $LR = \frac{\pi}{2}$; at R draw a great circle, RM, perpendicular to LR; then the variable line OM is always somewhere in the plane of this great circle. Let the position of the given line OL be defined by the angles ZL $(=\theta)$ and PN $(=\phi)$, and let the variable angle RM be χ. Then using the value of s given in (a) example 1, and substituting for the direction-cosines l, l', &c., of the lines OL and OM their values in terms of θ, ϕ, χ, we easily find

$$2s = (e_1 \cos^2\phi + e_2 \sin^2\phi - e_3) \sin 2\theta . \cos\chi \\ -(e_1 - e_2) \sin 2\phi \sin\theta . \sin\chi,$$

which can be written in the form

$$2s = K \cos(\chi + a). \tag{1}$$

If $e_1 > e_2 > e_3$, and we put $d_1 = e_2 - e_3$, $d_2 = e_1 - e_3$, $d_3 = e_1 - e_2$, we have $K = \sin\theta\sqrt{d_3^2 \sin^2 2\phi + (d_1 + d_2 + d_3 \cos 2\phi)^2 \cos^2\theta}$,

$$\tan a = \frac{d_3 \sin 2\phi}{(d_1 + d_2 + d_3 \cos 2\phi)\cos\theta}.$$

Equation (1) gives at once a graphic method of representing the shear by means of the lengths of chords of a circle whose diameter $= K$.

4. If λ, μ, ν are the direction-cosines of any plane, and $2s$ the maximum shear of lines in the plane, show that

$$4s^2 = (\overline{e_2 + e_3}\lambda^2 + \overline{e_3 + e_1}\mu^2 + \overline{e_1 + e_2}\nu^2)^2 - 4(e_2 e_3 \lambda^2 + e_3 e_1 \mu^2 + e_1 e_2 \nu^2).$$

5. Prove that a simple elongation in any direction is equivalent to a uniform cubical dilatation together with two shears, each having the given direction for one axis, the other axes being at right angles to it and to each other.

Consider a cube whose three edges at the point O are Ox, Oy, Oz, and suppose the given simple elongation, ϵ, to take place along Ox. We may consider this as $\frac{1}{3}\epsilon + \frac{1}{3}e + \frac{1}{3}\epsilon$ along Ox, and we may suppose an elongation $\frac{1}{3}\epsilon$ along Oy together with an elongation $-\frac{1}{3}\epsilon$ (or a contraction) in the sense of yO; and similarly $\frac{1}{3}\epsilon$ and $-\frac{1}{3}\epsilon$ in Oz. Now $\frac{1}{3}\epsilon$ along Ox, Oy, and Oz (and of course along all lines parallel to these) constitutes (p. 388) a cubical dilatation ϵ; while $\frac{1}{3}\epsilon$ along Ox and $-\frac{1}{3}e$ along Oy constitute (Art. 369) a shear, whose amount is $\frac{2}{3}\epsilon$. Therefore, &c.

6. Resolve a simple elongation ϵ in a given direction into its components with reference to three rectangular axes.

Ans. If the direction-cosines of the direction of elongation with reference to the three axes are l, m, n, the elongations and shears to which ϵ is equivalent are

$$\epsilon l^2,\ \epsilon m^2,\ \epsilon n^2,\ 2\epsilon lm,\ 2\epsilon mn,\ 2\epsilon nl.$$

For, if ξ, η, ζ be the co-ordinates of any point before strain, the length of the perpendicular from this point on the plane through the

origin perpendicular to the direction (l, m, n) is $l\xi+m\eta+n\zeta$; and the point (ξ, η, ζ) is drawn out along this perpendicular through a distance $\epsilon(l\xi+m\eta+n\zeta)$. The projection of this distance along the axis of x is $\epsilon l\,(l\xi+m\eta+n\zeta)$; hence the strained co-ordinates (ξ', η', ζ'), are

$$\xi'=\xi+\epsilon l(l\xi+m\eta+n\zeta),\quad \eta'=\eta+\epsilon m(l\xi+m\eta+n\zeta),$$
$$\zeta'=\zeta+\epsilon n(l\xi+m\eta+n\zeta).$$

Comparing these values of ξ', η', ζ' with those given at p. 392, we see that

$$\epsilon l^2=a,\ \epsilon m^2=b,\ \epsilon n^2=c,\ \epsilon lm=s_3,\ \epsilon mn=s_1,\ \epsilon nl=s_2,$$

which are the required components of the elongation with reference to the axes.

7. Find the condition that, in the general small strain, there should be two planes of no elongation.

Ans. $\begin{vmatrix} a, & s_3, & s_2 \\ s_3, & b, & s_1 \\ s_2, & s_1, & c \end{vmatrix}=0.$ Hence one of the principal elongations must be zero (see p. 391).

8. Given two small strains,

$$(a,\ b,\ c,\ 2s_1,\ 2s_2,\ 2s_3),\ (a',\ b',\ c',\ 2s_1',\ 2s_2',\ 2s_3'),$$

find the resulting elongation quadric and strain ellipsoid.

Ans. In the previous equations of these surfaces put $a+a'$ for a, &c., s_3+s_3' for s_3, &c.

9. Resolve a shear, $2s$, of two given rectangular lines into its components along three given rectangular axes.

Ans. If the direction-cosines of the two given lines with reference to the given axes are (l, m, n), (l', m', n'), the components are

$$2sll',\ 2smm',\ 2snn',\ 2s(lm'+l'm),\ 2s(mn'+m'n),\ 2s(nl'+n'l).$$

10. Find the conditions that a strain whose components with reference to three given rectangular axes are given should be equivalent to a shear.

Ans. $\begin{vmatrix} a, & s_3, & s_2 \\ s_3, & b, & s_1 \\ s_2, & s_1, & c \end{vmatrix}=0$ and $a+b+c=0.$

The first of these expresses that the product of the three principal elongations is zero, and the second that their sum (the cubical dilatation) is zero. Hence the principal elongations are of the forms $e, -e, 0$.

11. Find the Vortex Lines in the case of Torsion.

Ans. The rotations at any point are

$$\omega_1=-\frac{ax}{2l},\ \omega_2=-\frac{ay}{2l},\ \omega_3=\frac{az}{l}.$$

Hence the differential equations of the Vortex Lines are

$$\frac{dx}{x}=\frac{dy}{y}=-\frac{dz}{2z}.$$

The Vortex Lines are therefore the intersections of $\frac{y}{x}=c_1$ and $x^2z=c_2$. The vortex line at any point lies in the plane through this point and the axis about which the torsion takes place.

12. When the small strain $(a, b, c, 2s_1, 2s_2, 2s_3)$ is equivalent to a shear, find the magnitude of the shear.

Ans. If $2s$ is the shear, $s=\sqrt{s_1^2+s_2^2+s_3^2+\frac{1}{2}(a^2+b^2+c^2)}$. To get this equate the components in example 9 to $a, b, c, 2s_1, \ldots$. Squaring and adding the last three, we have

$$s^2(1-l^2l'^2-\ldots+2ll'mm'+\ldots)=s_1^2+s_2^2+s_3^2;$$

or $$s^2[1-2(l^2l'^2+m^2m'^2+n^2n'^2)]=s_1^2+s_2^2+s_3^2;$$

therefore the rest follows from the first three.

13. Prove that torsion is equivalent to shear at each point, and find its amount.

Ans. Let P be the point considered, PO the perpendicular (of length r) from P on the axis of torsion, and let the strain be expressed as in Art. 371; then the amount of the shear is $\frac{ar}{l}$, and the strain is a shear of the line drawn through P parallel to the axis of torsion and a line perpendicular to this one and to PO.

14. Find the areal dilatation on a plane the direction-cosines of whose normal are l, m, n.

Ans. $a+b+c-(al^2+bm^2+cn^2+2lms_3+2mns_1+2nls_2)$.

Section II.

Analysis of Stresses.

375.] **Intensity of a Stress.** If a force whose magnitude is P acts over an area S in such a way that there is all over the area the same force on the same amount of area, the force is said to be *uniformly distributed* over the area; and the *intensity* of force on the area is $\frac{P}{S}$, i.e. the rate at which the force is distributed per unit of area. Thus the atmospheric pressure on any area at the surface of the earth is roughly 15 lbs. weight on

every square inch, and if the unit of force is a pound weight and the unit of length an inch, the intensity of atmospheric pressure is represented by the number 15.

If force acts over an area in such a way that there is not the same amount exerted on the same area everywhere, the distribution is not uniform; and in this case we can speak only of the intensity of force *at each particular point*. If about any point we describe a very small area, dS, on which we may assume the distribution of force to be constant, and if dF is the amount of force on it, the intensity of force at the point selected is $\frac{dF}{dS}$. An instance of this occurs when the area pressed is any non-horizontal area in a heavy liquid. The intensity of pressure at points in the upper part of the area is less than the intensity at points in the lower part.

376.] **Stress at a Point.** At any point, P, of the body consider a small plane surface of area dS and any position. This may be regarded as separating the part (A) of the body at one side of it from the part (B) at the other side. Then the particles in this element plane, when the body is strained in any manner, are subject to certain forces proceeding from the particles at the side (A) and resulting from the elongation or contraction of the natural distances. The resultant of these forces is called the stress on the side (A) of the element plane.

The particles in the element plane are also subject to forces proceeding from particles at the side (B) of the plane; and the resultant of these latter is, of course, a stress equal and opposite to the first-mentioned stress.

The resultant stress (on either side of the element plane) divided by the area, dS, is the intensity of stress on the plane; and the resultant stress may be either normal to the plane, oblique to it, or in it.

If at the same point P we consider a small plane surface of the same area as before, but of different position, the resultant stress on it will, generally speaking, be different both in magnitude and in direction from the previous stress. In the case of a perfect fluid body the magnitude of the stress is constant and its direction is normal to the element plane, whatever be the position of the latter at the point P.

Hence in the case of a strained body the term 'stress at a

point' has no definite meaning until we specify the element plane on which the stress acts.

377.] **Equilibrium of an elementary Parallelopiped.** At any point, P (Fig. 295, p. 395), whose co-ordinates with reference to three fixed rectangular axes are (x, y, z) let a very small rectangular parallelopiped, $PxyzIC$, of the substance be separated in imagination from the rest of the body by means of element planes perpendicular to the fixed axes. We may then, if we actually produce on the faces of this element the stresses which are produced on them by the neighbouring portions of the body, consider the equilibrium of the element apart from the remainder*.

The resultant stress on any face, zPy, may be considered as acting at the middle point of the face. Let this stress per unit area be resolved into three components, p_{xx}, p_{xy}, p_{xz}, parallel to the axes, and in the *negative* senses of these axes. In the suffixes the first letter indicates the axis to which the face is perpendicular, and the second the axis parallel to which the component acts. Let the intensities of the stress components on the face zPx be p_{yx}, p_{yy}, p_{yz}, of which the second is a normal tension, these components being also in the negative senses of the axes. Let the components of stress intensity for the face xPy be similarly p_{zx}, p_{zy}, p_{zz}, in the negative senses of the axes.

At the middle points of the opposite faces of the parallelopiped the stress components will, of course, be in the positive senses of the axes. Thus the normal stresses on the faces are all *tractions*, so that if in any case they are really *pressures* (as in perfect fluids) they are to be considered negative. The oblique components, p_{xy}, &c., are *shearing stresses*.

Let the lengths of the edges of the parallelopiped be dx, dy, dz. Then these stress components are all functions of the position of P, i.e. each of them is some function of (x, y, z). And the co-ordinates of the point x in the figure are $(x+dx, y, z)$; so that if $p_{xx}=f(x, y, z)$, the p_{xx} for the face $BxCI$ is $f(x+dx, y, z)$, i.e.

* In considering the equilibrium of an element of a fluid body it is customary to say that we consider it as *solidified* and acted on by the stresses (pressures) which the fluid exerts on its surface. This solidification is, however, wholly unnecessary and misleading—if, indeed, it is not actually wrong. The element while regarded as forming part of the body is not solidified, but is kept in its condition by the very forces which, by supposition, are produced on it by other means. If these forces were by themselves sufficient in the one case, they must be so in the other, without the aid of solidification.

it is $p_{xx}+\frac{dp_{xx}}{dx}dx$, neglecting $(dx)^2$, &c. This component is, as said, directed in the sense Px. Hence the components of intensity of stress on $BxCI$ are

$$p_{xx}+\frac{dp_{xx}}{dx}.dx;\quad p_{xy}+\frac{dp_{xy}}{dx}.dx;\quad p_{xz}+\frac{dp_{xz}}{dx}.dx.$$

Similarly for the components of intensity of stress on the other faces. To get the whole amount of stress in any direction on any face, the intensity in this direction must, of course, be multiplied by the area of the face. Let us calculate the whole amount of stress parallel to Px exerted on the parallelopiped. The face zPy will contribute $p_{xx}\times dydz$, in the negative direction, while the opposite face, $BxCI$, will contribute

$$\left(p_{xx}+\frac{dp_{xx}}{dx}.dx\right)\times dydz;$$

and the sum of these is $\frac{dp_{xx}}{dx}\times dxdydz$. The face zPx will give a stress $p_{yx}\times dzdx$ parallel to Px, and the opposite face will give $\left(p_{yx}+\frac{dp_{yx}}{dy}dy\right)dzdx$; and the sum of these is $\frac{dp_{yx}}{dy}.dxdydz$; similarly, the faces xPy and BzA will give $\frac{dp_{zx}}{dz}.dxdydz$. Hence the whole stress force acting on the element in the direction Px is $\left(\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\right)dxdydz$. Some external force (gravity, or other) may also act on each element of the body. Such a force will always be proportional to the quantity of matter in the element. Suppose ρ to be density of the body at P; then, approximately, the quantity of matter in the parallelopiped is $\rho dxdydz$. Let the components of the external force which is felt at P along the axes of x, y, z be X, Y, Z, per unit of mass. Then the component of the external force along Px exerted on the element is $\rho Xdxdydz$. Equating to zero the sum of the components along Px of all forces exerted on the element, we have

$$\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}+\rho X=0.$$

Similarly,

$$\left.\begin{aligned}\frac{dp_{xy}}{dx}+\frac{dp_{yy}}{dy}+\frac{dp_{zy}}{dz}+\rho Y=0,\\ \frac{dp_{xz}}{dx}+\frac{dp_{yz}}{dy}+\frac{dp_{zz}}{dz}+\rho Z=0,\end{aligned}\right\}\qquad(1)$$

the last two equations being obtained by resolving forces along the axes of y and z.

In a perfect fluid the shearing intensities are zero, and, by a fundamental result for the stress of every strained body, which will be presently given, the normal intensities are of equal amount for all planes at P; so that $p_{xx} = p_{yy} = p_{zz}$, each being a pressure, equal to $-p$.

For any kind of body we obtain another important set of equations by expressing the equilibrium of the *moments* of the forces acting on the parallelopiped. For example, take moments about the line joining the middle points of the opposite faces zPy and BxC. The external force* acting on the parallelopiped may be considered to act at its middle point; it will therefore contribute nothing to the moments about the axis chosen. Neither will the stresses on these faces themselves, since these stresses act at the middle points of the faces. Of the stresses on the faces zPx and AyC the components $p_{yz} \times dx\,dz$ and

$$\left(p_{yz} + \frac{dp_{yz}}{dy} . dy\right) dx\,dz,$$

which are parallel to Pz, will alone contribute moments. The moment of the first is $p_{yz} \times dx\,dz \times \frac{dy}{2}$, or $\frac{1}{2} p_{yz} dx\,dy\,dz$, and the moment (in the same sense) of the second is

$$\left(p_{yz} + \frac{dp_{yz}}{dy} dy\right) dx\,dz \times \frac{dy}{2},$$

or $\frac{1}{2} p_{yz} dx\,dy\,dz$, neglecting the term $dx\,(dy)^2\,dz$. The sum of these moments is $p_{yz} dx\,dy\,dz$.

Again, of the stresses on the faces xPy and BzA the components, $p_{zy} \times dx\,dy$ and $\left(p_{zy} + \frac{dp_{zy}}{dz} . dz\right) dx\,dy$ will alone con-

* It is important for the student to distinguish two species of external force acting on any body. There may be external forces which act only at *particular points on its surface*—as, for example, when a beam rests against the ground and against a wall, the reactions of the ground and wall—and there may be external forces which affect *every element* inside the body—as, in the same case, the attraction of the earth which produces a force (the weight) on each element of the beam. The latter are called *continuous, or bodily, forces*. Thus a strained body may be affected by both—the above beam, if slightly flexible, will be bent. The forces (per unit of mass), X, Y, Z, in equations (1) belong exclusively to the second kind. Forces of the first kind do not enter into these equations; they are like the terminal tensions of a string, and are required for determining the values of constants which occur in the integrals of the differential equations (1) of equilibrium.

tribute; and the sum of their moments is $p_{zy}\,dx\,dy\,dz$, which is obviously in the sense opposite to that of the previous moment.

Hence equating the sum of these moments to zero,

$$p_{yz} = p_{zy}.$$

Similarly,
$$\left.\begin{aligned} p_{zx} &= p_{xz}, \\ p_{xy} &= p_{yx}, \end{aligned}\right\} \qquad (2)$$

which are obtained by taking moments about the lines joining the middle points of the other pairs of opposite faces.

The stress (per unit of area) on the face zPy can be resolved into two, viz. one normal to the face and the other in the face. The first is p_{xx}, and the second (which is the resultant shearing force intensity on the face) is $\sqrt{p^2_{xy}+p^2_{xz}}$. Equations (2) obviously assert that if we take any two element planes at right angles to each other at any point of the body, *the component along the normal to the second of the stress per unit area on the first is equal to the component along the normal to the first of the stress per unit area on the second.* We shall now see that this very important result is true for two element planes inclined at any angle to each other.

To save a multiplicity of symbols, we shall (with Lamé) use the following notation:—

$$p_{xx} = N_1; \;\; p_{yy} = N_2; \;\; p_{zz} = N_3,$$
$$p_{zy} = p_{yz} = T_1; \;\; p_{zx} = p_{xz} = T_2; \;\; p_{xy} = p_{yx} = T_3.$$

Fig. 301 represents these component intensities of stress, in the senses in which they are assumed to act in all our subsequent equations, at points indefinitely close to P in the three co-ordinate planes.

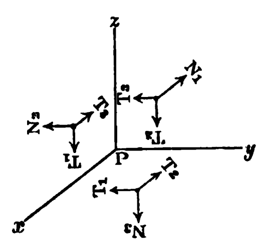

Fig. 301.

378.] Equilibrium of an elementary Tetrahedron. Consider now the equilibrium of the indefinitely small tetrahedron whose vertex is P (Fig. 301) and whose base is the triangle formed by the points marked x, y, z—the plane of these points being any plane whatever in the neighbourhood of P. Let l, m, n be the direction-cosines of the perpendicular from P on this plane. Let P, Q, R be the components, along the axes Px, Py, Pz, of the intensity of

stress exerted on this plane. It is required to express these components in terms of N_1, N_2, N_3, T_1, T_2, T_3.

Let A be the area of the triangular face xyz; then the areas of the triangular faces zPy, zPx, xPy are lA, mA, nA, respectively.

Now for the equilibrium of the tetrahedron resolve the forces acting on it along Px. The face zPy will contribute the term $-N_1 \times lA$; the face zPx will contribute $-T_3 \times mA$; the face xPy the term $-T_2 \times nA$; the face xyz the term $P \times A$; while the component of the external force is $\rho X \times$ the volume of the tetrahedron, or $\frac{1}{3}\rho X \times hA$, where h is the length of the perpendicular from P on the plane xyz.

Hence the equation of equilibrium is

$$P - lN_1 - mT_3 - nT_2 + \tfrac{1}{3}\rho hX = 0,$$

or, in the limit,

Similarly,
$$\left.\begin{aligned} P &= lN_1 + mT_3 + nT_2, \\ Q &= lT_3 + mN_2 + nT_1, \\ R &= lT_2 + mT_1 + nN_3, \end{aligned}\right\} \qquad (3)$$

the terms depending on the external bodily force disappearing because they are infinitesimals of the third order (being proportional to the *volume* of the tetrahedron) while the stresses are of the second order being proportional to the areas of the faces of the tetrahedron. These equations give the intensity of stress in magnitude and direction on any assigned element plane when the stresses on *three* rectangular element planes are known; they are, in fact, *the composition and resolution of stress*.

Any one of these equations (3) suffices for the proof of the important general theorem of projection already referred to. For P is the projection, along the normal to the element plane zPy, of the intensity of stress on the element plane xyz, and

$$lN_1 + mT_3 + nT_2$$

is the projection, along the normal to the latter plane, of the intensity of stress on the former. This theorem is true therefore for any two element planes at a point.

Remark. The components of stress on an element plane at the bounding surface of the body are to be equated to the components of the external force applied to the surface at the element.

COR. 1. It follows immediately from this theorem of the projections of two stresses that if there is at a point in the body

any plane on which the stress is zero, the lines of action of the stresses on all other planes at this point lie in this plane of zero stress.

COR. 2. *If the stress on every element plane at a point in a body is normal to the plane, the intensity of the stress is constant for all element planes at the point.*

For, let p and q be the intensities of stress on two planes, each stress being normal to the corresponding plane; and let ϕ be the angle between the two normals. Then by the theorem of projection

$$p \cos \phi = q \cos \phi,$$

$$\therefore p = q,$$

i.e. the intensity of the stress is constant on all planes at the given point.

Thus in a perfect fluid the stress on every element plane at a point is a normal pressure; hence its intensity is constant in all directions round the point—a result which is one of the elementary principles of Hydrostatics.

A perfect fluid may, therefore, be completely defined as *a body such that, however it may be strained, the stress on every element-plane at every point is a normal pressure*—the equality following from the normality.

When the stress on an element plane, ϖ, exerted by the part, A, of the body on one side of it consists of a force whose component normal to ϖ is directed from this plane towards the part A, the stress on ϖ is called *tension*; and when the normal component is directed from A to ϖ, it is called *pressure*. All perfect fluid stress is, as just said, pressure. In general at every point inside a strained body there will be some planes on which the stress is pressure, and others on which the stress is tension.

It may assist the student to understand the nature of the action of stress on an element plane if we draw a figure representing the equilibrium of these stresses on an element of the body. Thus, if we take the elementary parallelopiped PI (Fig. 295, p. 395) to be a cube, and also take (as we may) the stress on any face as acting at its middle point, the forces in the plane of xy may be represented as in Fig. 302, which is that of a section of the cube through its centre and parallel to the plane of xy. If there were no stresses on planes parallel to xy, this figure would completely represent the equilibrium of the cubical ele-

ment. (Since the faces have been all taken as equal in area, the *intensities* of stresses are proportional to the stresses acting on them).

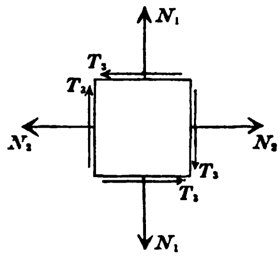

Fig. 302.

It is evident, of course, that when the stresses on *any three* planes at a point (rectangular or not) are known, the stress on every plane at this point can be found both in magnitude and in line of action. For we may consider the equilibrium of the tetrahedral element contained by the assumed plane and the three given ones, and the required force will be equal and opposite to the resultant of three given forces.

Let it, for example, be given that the stress at any point P is a shearing stress in each of two rectangular planes, there being no stress on planes perpendicular to both of them. Suppose that all planes in the neighbourhood of P which are perpendicular to the plane of the paper and parallel to CD (Fig. 303) are subject to a shearing stress, and that all planes parallel to AD and perpendicular to the paper are also subject to shearing stress, and that planes parallel to the paper are not subject to stress. The intensities of these shearing stresses are obviously equal (either by what precedes, or by considering the equilibrium of a small prism whose base is the square $ABCD$ and whose edges are perpendicular to the paper. The equality of moments round an axis through P perpendicular to the figure gives the equality of the intensities of these shears); let their common intensity be S, and suppose them represented by the arrows.

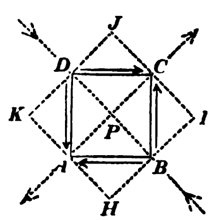

Fig. 303.

Draw the plane AC, and consider the equilibrium of the portion ADC of the body (or rather of a little right prism whose base is ADC). It is kept in equilibrium by the forces S acting in the lines DC and DA and by the stress on the face AC. This last must (since it may be supposed to act at the middle point of AC) act in the line PD from P to D. If h is the height of the

prism, the areas of its faces are $h \times AC$, $h \times CD$, $h \times DA$; so that the forces acting in DC and DA are $S \times h \times DC$ and $S \times h \times DA$; and their resultant, F, which is equal and opposite to the stress on AC, is given by the equation

$$F = \sqrt{S^2 \times h^2 \times DC^2 + S^2 \times h^2 \times DA^2} = S \times h \times AC;$$

$$\therefore \quad \frac{F}{h \times AC} = S,$$

i.e. the intensity of stress on the face AC is equal to the intensity of the shearing stress on each of the other two faces; moreover, the stress on AC is normal to AC. This stress is the action of the portion of the body at the right hand side of AC on the particles in the plane AC, and since it acts in the sense PD, it is a *pressure*. Hence if the portion of the body at the right hand side of AC, or of any plane parallel to it and near it be removed, a pressure of intensity S must be applied to the plane in the sense PD. The action of the part of the body at the left hand side of AC, or of any parallel to and near it, consists, of course, of a *pressure* in the opposite sense; so that if we draw two element planes HI and JK parallel to AC and consider the portions of the body at the right of the first and at the left of the second as removed, two pressures (indicated by the arrows pointing to B and D) must be applied to the portion of the body contained between these planes.

Similarly, by drawing BD and considering the equilibrium of the prism standing on the base BCD, we see that the action of the portion of the body at the lower side of BD on the particles in this face consists of a normal stress of intensity S directed in the sense CP, i.e. *towards* the parts considered as removed; in other words, this stress is a *tension*. Consequently if we isolate in imagination a small prism of the body standing on the square $HIJK$, we regard it as acted on by two *pressures* on its faces HI and JK, and by two *tensions* on its faces IJ and KH.

The state of stress of the body at P may just as well be produced by applying normal stress (pressure), of the same intensity as the shearing stress, to all planes parallel to AC and near it, and normal stress (tension), of same intensity, to all planes parallel to BD and near it; in other words, *we may substitute this state of stress for the shearing stress.*

Hence *a shearing stress on two rectangular planes at any point*

produces equal normal stresses of opposite signs (pressure and tension) and of intensities equal to that of the shearing stress on the two planes which bisect the angles between them.*

This result follows, of course, from equations (3) by taking the lines from P perpendicular to CD and BC as axes of x and y, and putting $N_1 = 0$, $N_2 = 0$, $N_3 = 0$, $T_1 = 0$, $T_2 = 0$, $T_3 = S$, $l = m = \frac{1}{\sqrt{2}}$, $n = 0$. From these equations also we deduce the magnitude and line of action of the stress on any plane near P.

The student will do well, however, to deduce from the figure the stress on any plane through (or near) P perpendicular to the figure.

379.] **Transformation of Stress.** *Given the conditions of stress of a body at any point in it with reference to one set of rectangular planes, to find the condition of stress at the same point with reference to any other set of rectangular planes.*

Let the given stresses at a point P, on three rectangular planes of xy, yz, zx, be N_1, N_2, N_3, T_1, T_2, T_3, as in last Article. Then the components along the axes of x, y, z of the stress per unit area on an element plane at the point the direction-cosines of whose normal are l, m, n are given by equations (3) of last Article. The resolved part, T, of this stress along any line whose direction-cosines are λ, μ, ν is $\lambda P + \mu Q + \nu R$; i.e.

$$T = l\lambda N_1 + m\mu N_2 + n\nu N_3 + (l\mu + m\lambda)T_3 + (m\nu + n\mu)T_1 + (n\lambda + l\nu)T_2. \quad (1)$$

If the line along which the stress is resolved is the normal to the element plane itself, the component, N, is $lP + mQ + nR$;

i.e. $$N = l^2 N_1 + m^2 N_2 + n^2 N_3 + 2lmT_3 + 2mnT_1 + 2nlT_2. \quad (2)$$

Let it be required to find the intensities of stress on three other rectangular element planes at P whose normals are Px', Py', Pz'; and let the direction-cosines of these normals with respect to Px, Py, Pz be (l, m, n), (l', m', n'), (l'', m'', n''), respectively. Denote the components of the intensity of stress on the plane $y'z'$ by N'_1 along Px', T'_3 along Py', and T'_2 along Pz'; the components of the intensity of stress on the plane $z'x'$ by T'_3 along Px', N'_2 along Py', and T'_1 along Pz'; and those of the

* Compare with the corresponding result in the case of shearing *strain*. The shearing strain may be replaced by two simple elongations, the magnitude of each being *half* that of the shear. See (p. 395.)

intensity of stress on the plane $x'y'$ by T'_2 along Px', T'_1 along Py', and N'_3 along Pz'.

Then N'_1 is given by (2); N'_2 is obtained by using $(l', m' n')$ for (l, m, n) in (2); N'_3 by using (l'', m'', n'') for (l, m, n) in (2); T'_3 by using (l', m', n') for (λ, μ, ν) in (1); T'_2 by using (l'', m'', n'') for (λ, μ, ν) in (1); and T'_1 by using $(l' m', n')$ for (l, m, n), and (l'', m'', n'') for (λ, μ, ν) in (1).

It will be seen from this that in transforming from one set of rectangular axes to another, the quantities $N_1, N_2, N_3, T_3, T_1, T_2$, transform like $x^2, y^2, z^2, xy, yz, zx$.

The system of stress, thus calculated, on the new planes may be substituted for the original system of stress—the two systems are, in other words, perfectly equivalent, and either will produce the other.

In particular, if A, B, C are the principal intensities of stress at a point, the components of stress intensity on any system of rectangular planes at the point are—

$$N_1 = Al^2 + Bm^2 + Cn^2; \quad T_1 = Al'l'' + Bm'm'' + Cn'n'',$$
$$N_2 = Al'^2 + Bm'^2 + Cn'^2; \quad T_2 = All'' + Bmm'' + Cnn'',$$
$$N_3 = Al''^2 + Bm''^2 + Cn''^2; \quad T_3 = All' + Bmm' + Cnn'.$$

380.] **Cone of Shearing Stress.** The expression (2) for the normal component of intensity of stress on a plane may for all values of l, m, n (i.e. for all element planes at the point considered) retain a positive value. In this case the normal component of stress is a *tension* on all planes. Or the expression may be negative for all planes, and then the normal stress will be *pressure* all round. Or, finally, it may be positive for some directions and negative for others. It will then be zero for some directions; i.e. there will be planes on which the stress is entirely tangential. The directions of the normals to these planes are given by the equation

$$N_1 l^2 + N_2 m^2 + N_3 n^2 + 2T_3 lm + 2T_1 mn + 2T_2 nl = 0,$$

and therefore the normals trace out the cone

$$N_1 x^2 + N_2 y^2 + N_3 z^2 + 2T_3 xy + 2T_1 yz + 2T_2 zx = 0, \qquad (1)$$

the planes themselves tracing out the cone whose generators are perpendicular to the generators of this cone. This latter cone, when it exists, is called the *Cone of Shearing Stress.*

381.] **Principal Planes of a Stress.** The angle between the direction of stress and the plane on which it acts depends on the

plane chosen. Let us try whether, with any given stress, it is possible to find a plane on which the stress is normal.

If F is the resultant stress on a plane the direction-cosines of whose normal are (l, m, n), and if F acts in the normal, $P = lF$, $Q = mF$, $R = nF$, and equations (3) of Art. 378 become

$$\left.\begin{aligned} lN_1 + mT_3 + nT_2 &= lF, \\ lT_3 + mN_2 + nT_1 &= mF, \\ lT_2 + mT_1 + nN_3 &= nF; \end{aligned}\right\} \tag{1}$$

and these give, by elimination of the direction-cosines, the cubic for F

$$\begin{vmatrix} N_1 - F, & T_3, & T_2 \\ T_3, & N_2 - F, & T_1 \\ T_2, & T_1, & N_3 - F \end{vmatrix} = 0,$$

or $$F^3 - (N_1 + N_2 + N_3)F^2 + (N_1N_2 + N_2N_3 + N_3N_1 - T_1^2 - T_2^2 - T_3^2)F - (N_1N_2N_3 - N_1T_1^2 - N_2T_2^2 - N_3T_3^2 + 2T_1T_2T_3) = 0.$$

This equation, as is well known, gives three real values of F, and equations (1) will give the direction-cosines of the planes subject to these normal stresses. The coefficients of this equation have, as is also well known, the same values no matter what three rectangular planes are taken as those of reference.

All theorems, therefore, concerning stress may be simplified by supposing that we have selected as planes of reference the three on which the stresses are normal. These are called the *principal planes* of the stress at the point considered. Let the stresses on them (per unit area, of course) be denoted by A, B, C.

The equations (1) which determine the planes and magnitudes of the principal stresses show that these planes are the principal planes of the quadric

$$N_1x^2 + N_2y^2 + N_3z^2 + 2T_3xy + 2T_1yz + 2T_2zx = f, \tag{2}$$

f being any constant force magnitude.

The equation of the tangent plane to this quadric at the point x', y', z' is

$$(N_1x' + T_3y' + T_2z')x + (T_3x' + N_2y' + T_1z')y + (T_2x' + T_1y' + N_3z')z = f.$$

Let a normal be drawn to any element plane at the point, P, considered, and let r be the length of this normal from P to the surface of this quadric. Then by putting lr, mr, nr for x', y', z',

the tangent plane at the extremity of this normal is (by the values of P, Q, R in p. 414)

$$Px + Qy + Rz = \frac{f}{r}. \tag{3}$$

The direction-cosines of the perpendicular from P on this plane are $\frac{P}{F}, \frac{Q}{F}, \frac{R}{F}$, where F is the resultant stress (per unit area) on the element plane; and these show that the resultant stress acts in this perpendicular. Again, if p is the length of the perpendicular from P on the plane (3), we have

$$F = \frac{f}{pr}, \tag{4}$$

the value of the resultant stress.

If the axes of the quadric (2) are taken as those of co-ordinates, we have

$$N_1 = A,\ N_2 = B,\ N_3 = C,\ T_1 = T_2 = T_3 = 0;$$

and the quadric has for equation

$$Ax^2 + By^2 + Cz^2 = f.$$

The cone traced out by the normals to the planes of shearing stress is obviously the asymptotic cone of the quadric (2); and if this cone is real, its reciprocal cone (the cone of shearing stress) will separate the planes on which the stress is pressure from those on which it is tension. When the cone is imaginary, all planes at the point P will be subject to stress of one kind—either pressure or tension.

When the cone is real, the quadric (2) must be accompanied by another whose equation is obtained by merely changing f to $-f$, as has been explained in the analogous case of strain (p. 381).

Another graphic mode of connecting the stress on a plane with the position of the plane is this. Let the principal planes be taken as the co-ordinate planes; then the components of the intensity of stress on any plane (l, m, n) are by equations (3), p. 414,

$$\left.\begin{aligned} P &= lA, \\ Q &= mB, \\ R &= nC. \end{aligned}\right\} \tag{5}$$

Hence $\frac{P^2}{A^2} + \frac{Q^2}{B^2} + \frac{R^2}{C^2} = 1$. Of course P, Q, R are the co-

ordinates of the extremity of the line representing the intensity of stress on the plane (l, m, n). Hence the extremities of lines representing in magnitude and direction the intensities of stresses on all planes at O lie on the ellipsoid

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1, \tag{6}$$

whose semi-axes are in magnitudes and directions the principal intensities of stress at P.

If a tangent plane be drawn to this ellipsoid parallel to the plane whose stress is considered, the length of the perpendicular from the centre on the tangent plane represents the magnitude of the intensity of stress, as is obvious by squaring and adding the sides of equations (5).

The ellipsoid (6) may for shortness be called the *Stress Ellipsoid.*

In proving general properties of stress simplicity is, of course, gained by taking the principal axes of the stresses as those of reference. Thus, with these axes, the cone of shearing stress is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0,$$

and that traced out by the normals to planes of shearing stress is $Ax^2 + By^2 + Cz^2 = 0$; so that for the reality of these cones (i.e. *for the existence of planes subject wholly to shearing stress*) the principal stresses must consist either of one tension and two pressures, or two tensions and one pressure. With any system of axes the equation of the cone of shearing stress is

$$\begin{vmatrix} N_1 & T_3 & T_2 & x \\ T_3 & N_2 & T_1 & y \\ T_2 & T_1 & N_3 & z \\ x & y & z & 0 \end{vmatrix} = 0.$$

382.] **Work done in Strain.** We propose to investigate the work done in the strain of any small volume of the body. About the point P (Fig. 290, p. 377) let any small closed surface be drawn in the natural state of the body. Let dS be any element of this surface, and let the direction-cosines of the normal to this element, measured outwards, be l, m, n. Then the components of *intensity* of stress (resulting from strain) on the element plane dS being P, Q, R, and the final displacements

of the mean point of the element being $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$, the work done in the displacement of the element will be (see p. 339, vol. i)

$$\tfrac{1}{2}(P\Delta\xi + Q\Delta\eta + R\Delta\zeta)\,dS.$$

Hence the work done in the strain of the volume contained in the whole surface is

$$\tfrac{1}{2}\int(P\Delta\xi + Q\Delta\eta + R\Delta\zeta)\,dS.$$

Substituting for P its value (p. 414), the term PdS becomes $(lN_1 + mT_3 + nT_2)\,dS$.

But if $d\sigma_1$, $d\sigma_2$, $d\sigma_3$ are the projections of dS on the planes of yz, zx, and xy, respectively, $ldS = d\sigma_1$, $mdS = d\sigma_2$, $ndS = d\sigma_3$; so that the work done becomes

$$\tfrac{1}{2}\int(N_1\Delta\xi + T_3\Delta\eta + T_2\Delta\zeta)\,d\sigma_1 + \tfrac{1}{2}\int(T_3\Delta\xi + N_2\Delta\eta + T_1\Delta\zeta)\,d\sigma_2 + \tfrac{1}{2}\int(T_2\Delta\xi + T_1\Delta\eta + N_3\Delta\zeta)\,d\sigma_3.$$

The intensities of stress N_1, N_2,... may be considered as constant over the surface and taken outside the integral signs. Also substituting for $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ their values (Art. 358), we have

$$\int\Delta\xi\,d\sigma_1 = \int\left(\xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz}\right)d\sigma_1 = \frac{du}{dx}\int\xi\,d\sigma_1 + \frac{du}{dy}\int\eta\,d\sigma_1 + \frac{du}{dz}\int\zeta\,d\sigma_1.$$

Now, the surface being closed, $\int\xi\,d\sigma_1 = d\Omega$ = volume enclosed by surface; and $\int\eta\,d\sigma_1 = \int\zeta\,d\sigma_1 = 0$, since, the normal being always drawn outwards, the elementary projections $d\sigma_1$ on one side of the plane yz must be given a sign opposite to the sign of those on the other side.

In this way we have also

$$\int\eta\,d\sigma_2 = \int\zeta\,d\sigma_3 = d\Omega;\ \int\xi\,d\sigma_2 = \int\xi\,d\sigma_3 = \ldots = 0.$$

Hence the work of straining the element of volume considered is

$$\tfrac{1}{2}(N_1 a + N_2 b + N_3 c + 2T_1 s_1 + 2T_2 s_2 + 2T_3 s_3)\,d\Omega, \tag{α}$$

where a, b, c, $2s_1$, $2s_2$, $2s_3$ are, as usual, the simple elongations and shears of the strain. If we use the principal elongations and stresses, the work is

$$\tfrac{1}{2}(Ae_1 + Be_2 + Ce_3)\,d\Omega. \tag{β}$$

EXAMPLES.

1. To resolve a shearing stress of intensity S, which is exerted on two given rectangular planes at any point into its components with reference to any three rectangular planes at the point.

Let P (Fig. 295, p. 395) be the point, and suppose that the stress on all planes parallel to zPy is a shearing stress of intensity S, and that the stress on all planes parallel to zPx is also a shearing stress of (necessarily) the same intensity (see p. 413), while there is no stress on planes parallel to xPy.

Let the direction-cosines of the normals, Px, Py, Pz to these planes with reference to any three rectangular axes $P\xi$, $P\eta$, $P\zeta$, be (l, m, n), (l', m', n'), (l'', m'', n''). Then for the system of planes on which the stresses are given we have $N_1' = N_2' = N_3' = 0$, and also $T_1' = T_2' = 0$, since there is no stress on xPy (see Fig, 301). Therefore if P', Q', R' are the components along Px, Py, Pz of the intensity of stress on a plane whose direction-cosines with respect to these lines are λ, μ, ν, we have

$$P' = \mu S, \quad Q' = \lambda S, \quad R' = 0.$$

Hence the components along Px, Py, Pz of the intensity of stress on the plane $\eta\zeta$ are
$$P' = l'S, \quad Q' = lS, \quad R' = 0;$$

and N_1 is the sum of the components of these along the axis of ξ;

therefore $\quad N_1 = lP' + l'Q' + l''R' = 2ll'S.$

Also $\quad T_3 = mP' + m'Q' + m''R' = (lm' + l'm)S,$

$$T_2 = nP' + n'Q' + n''R = (ln' + l'n)S;$$

and hence the components of the given shearing stress are

$$2ll'S, \; 2mm'S, \; 2nn'S, \; (lm' + l'm)S, \; (ln' + l'n)S, \; (mn' + m'n)S.$$

(Compare with the resolution of a shearing strain, p. 407.)

2. Two normal stresses on two rectangular planes are combined with two shearing stresses on the same planes; find the principal planes and intensities of the resultant stress.

Let Fig. 301, p. 413, represent the normal stresses N_1 and N_2 acting on planes at right angles to each other. Since there is no stress on any plane parallel to the plane of the paper, $N_3 = T_1 = T_2 = 0$, and the stress on *every* plane lies in the plane of the paper (p. 414). Also $T_3 = S$, and the principal planes are obviously perpendicular to the plane xPy. Let the normal to any plane passing through the line Pz make an angle θ with the direction of N_1. Then the components of stress on this plane are

$$P = N_1 \cos\theta + S\sin\theta,$$
$$Q = S\cos\theta + N_2 \sin\theta.$$

For a principal plane $P = F.\cos\theta$, $Q = F.\sin\theta$, where F is a principal stress. Hence

$$(N_1 - F).\cos\theta + S.\sin\theta = 0,$$
$$S.\cos\theta + (N_2 - F).\sin\theta = 0.$$

From these equations we find the two principal intensities of stress to be
$$\tfrac{1}{2}[N_1+N_2\pm\sqrt{(N_1-N_2)^2+4S^2}],$$
and the directions of the principal planes are given by the equation
$$\tan 2\theta = \frac{2S}{N_1-N_2}\cdot$$

2. If there is no normal stress on a certain plane, and no normal stress on any plane perpendicular to this plane, prove that the system of stress can be defined with reference to two planes (i. e. the state of stress on two planes will serve to determine the stress on every plane at the point considered).

Let the plane xPy (Fig. 301) be that on which there is no normal stress, there being also no normal stress on any plane through Pz. Draw any plane through Pz, and let the normal to it make an angle ϕ with Py. Then if P, Q, R are the components of stress intensity on this plane, since $N_1 = N_2 = N_3 = 0$, we have
$$P = T_3 \cos\phi\,;\;\; Q = T_3 \sin\phi\,;\;\; R = T_2 \sin\phi + T_1 \cos\phi.$$

Also the component of this stress along the normal to the plane is $Q\cos\phi + P\sin\phi$; therefore $T_3 = 0$, and the system contains only the two intensities T_1, T_2. The stress on every plane through Pz is a shearing stress parallel to Pz, and its amount varies from $\sqrt{T_1^2+T_2^2}$ to zero. If $\tan\phi = \frac{T_2}{T_1}$ we get a plane on which the stress has the first of these values, and for the plane at right angles to this through Pz the stress $= 0$. Let a cube of the substance be determined by these planes and the plane xPy. Then this cube experiences equal shearing stresses, each $= \sqrt{T_1^2+T_2^2}$, on two pairs of opposite faces, and no stress whatever on the remaining pair of opposite faces (the resultant stress intensity on the face xPy being $\sqrt{T_1^2+T_2^2}$).

3. Find the element-plane at any point on which the shearing stress is greatest.

Let A, B, C be the principal stress intensities at the point, and let l, m, n be the direction-cosines of the normal to any plane. Then, since $P = lA$, $Q = mB$, $R = nC$, if S is the component of the stress in the plane, we have
$$S^2 = l^2A^2 + m^2B^2 + n^2C^2 - (l^2A + m^2B + n^2C)^2.$$

Let l, m, n be expressed in terms of the colatitude and longitude determining the normal; that is,
$$l = \sin\theta\cos\phi,\;\; m = \sin\theta\sin\phi,\;\; n = \cos\theta.$$

Then we find
$$\frac{S^2}{\sin^2\theta} = \beta^2\cos^2\phi + a^2\sin^2\phi - (\beta\cos^2\phi + a\sin^2\phi)^2\sin^2\theta, \qquad (1)$$
where $\beta = A - C$, $a = B - C$.

Supposing the principal stresses to be all of the same sign, and $A > B > C$, we see that S will be a maximum with respect to ϕ when

$\phi = 0$, i.e. the normal to the required plane must be in the plane of A, C. The value of θ which will then make S a maximum is $\frac{\pi}{4}$; that is, the normal bisects the angle between the axes of greatest and least stress, and then

$$S = \tfrac{1}{2}(A - C).$$

If the principal stresses are not all of the same sign, it is easily found that S is a maximum when the normal lies in the plane of the axes of *algebraically* greatest and *algebraically* least stress, and bisects the angle between them, its value being half the algebraic difference of these stress intensities.

4. If the stress on any plane is wholly a shearing stress, prove that its line of action is the line of contact of the plane with the cone of shearing stress, and find its magnitude.

5. If at a point the principal stresses consist of two tensions, A and $B\,(A > B)$ and a pressure C, find the plane whose stress is wholly shearing and of maximum intensity.

$$Ans.\ l = \left(\frac{C}{A+C}\right)^{\frac{1}{2}};\ m = 0;\ n = \left(\frac{A}{A+C}\right)^{\frac{1}{2}},$$

and the intensity $= \sqrt{AC}$.

6. Find the conditions that the stress $(N_1, N_2, N_3, T_1, T_2, T_3)$ shall produce shearing stress on two planes only, and these rectangular.

$$Ans.\ \begin{vmatrix} N_1, & T_3, & T_2 \\ T_3, & N_2, & T_1 \\ T_2, & T_1, & N_3 \end{vmatrix} = 0, \text{ and } N_1 + N_2 + N_3 = 0.$$

Hence the product of the three principal intensities $= 0$, and sum $= 0$; therefore one of them $= 0$, and the other two are a tension and an equal pressure.

7. From the equations of equilibrium of an elementary parallelopiped, p. 411, deduce the six equations which are sufficient for the equilibrium of a rigid body.

Multiply the first by $dx\,dy\,dz$, and integrate by parts exactly as in Art. 346. Thus we get

$$\iiint \rho X\,dx\,dy\,dz + \int (lN_1 + mT_3 + nT_2)\,dS = 0,$$

where dS is an element of the bounding surface of the body.

But $lN_1 + mT_3 + nT_2$ is the x-component of the stress at a point on the surface, i.e. the x-component of the external force (if any) applied at the point. If this is denoted by X_0, we have (denoting the element of volume by $d\Omega$)

$$\int \rho X\,d\Omega + X_0 = 0,$$

with two similar equations, which are exactly the equations of translation in Art. 240.

Similarly, multiplying the second by $x\,dx\,dy\,dz$, and the first by $y\,dx\,dy\,dz$, subtracting the first from the second, and integrating throughout the body, we have

$$\int\rho(Yx-Xy)\,d\Omega+\int(Y_0x-X_0y)\,dS=0,$$

with two similar equations, which are the equations of moments of Art. 240.

8. From the equations of equilibrium of an elementary parallelopiped deduce the equations of equilibrium of a perfectly flexible string.

383.] **Virtual Work of Strain.** The body under strain having assumed its state of equilibrium, let any *further* very small increments be imagined to be produced in the strain components, so that the displacements u, v, w of a point P become further increased by $\delta u, \delta v, \delta w$; and let us consider the amount of work done in this further displacement by the stresses exerted on the faces of a small parallelopiped, $dx\,dy\,dz$, at P. The total x-stress on the parallelopiped is $\left(\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}\right)dx\,dy\,dz$, and the work done by this component in the virtual displacement is the product of the component and δu. Hence the virtual work of the stress on all elements of volume is

$$\iiint\Big[\left(\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}\right)\delta u+\left(\frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}\right)\delta v$$
$$+\left(\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}\right)\delta w\Big]\,dx\,dy\,dz. \qquad (a)$$

By integration by parts, we obtain another form of this expression. Thus, by exactly the same process as that employed in Art. 346, we have

$$\int\Big[(lN_1+mT_3+nT_2)\,\delta u+(lT_3+mN_2+nT_1)\,\delta v$$
$$+(lT_2+mT_1+nN_3)\,\delta w\Big]\,dS$$
$$-\iiint\Big[N_1\frac{d\delta u}{dx}+N_2\frac{d\delta v}{dy}+N_3\frac{d\delta w}{dz}$$
$$+T_1\left(\frac{d\delta v}{dz}+\frac{d\delta w}{dy}\right)+\dots\Big]\,dx\,dy\,dz, \qquad (\beta)$$

where dS is an element of the bounding surface of the body. The first integral is simply the virtual work of the surface stress on the body, and this surface stress consists merely of external force applied to the body. The triple integral in (β) is therefore

properly the virtual work of the stress throughout the body, which is therefore

$$-\int (N_1 \delta a + N_2 \delta b + N_3 \delta c + 2 T_1 \delta s_1 + 2 T_2 \delta s_2 + 2 T_3 \delta s_3)\, d\Omega, \quad (\gamma)$$

where $d\Omega$ is an element of volume of the body.

It is easy to see directly that this is the expression for the virtual work of the stresses. For, let PI (Fig. 295) represent a small parallelopiped in a state of stress. Whether we suppose its edges to be $(1+a)\,dx$, $(1+b)\,dy$, $(1+c)\,dz$, or simply dx, dy, dz is indifferent. Let them be the former, and let $N_1 + \epsilon$ represent the mean intensity of tension on the planes between $zPyA$ and $BxCI$. Then ϵ is evidently a small quantity of the order of magnitude of dx, and we have a tension

$$(N_1 + \epsilon)(1 + b + c)\, dy\, dz$$

in the substance, parallel to Px. If the length $(1+a)\,dx$ is slightly increased, or imagined to be increased, so as to become $(1 + a + \delta a)\, dx$, the work of this tension is

$$-(N_1 + \epsilon)(1 + b + c)\, dy\, dz \,.\, \delta a \,.\, dx,$$

or

$$-N_1 \delta a \,.\, dx\, dy\, dz,$$

rejecting infinitesimals of the second order.

Similarly for the other normal stresses, N_2, N_3. To find the virtual work of T_1, observe that it is not produced by any of the elongations a, b, c; it is due to the sliding of the face $BzAI$, the face $xPyC$ being imagined as fixed, and also to the sliding of the face $IAyC$, the face $BzBx$ being imagined fixed.

The first of these shears gives

$$-T_1 (1 + a + b)\, dx\, dy \,.\, \delta s_1 \,.\, dz,$$

since the relative displacement of the faces by shear is $s_1 . dz$ in the strain, and the further displacement imagined is $\delta s_1 . dz$. This virtual work is $-T_1 \delta s_1 . dx\, dy\, dz$, neglecting infinitesimals of higher order. Similarly the other sliding contributes

$$-T_1 \delta s_1 \,.\, dx\, dy\, dz\,;$$

and T_1 therefore does work of the amount

$$-2 T_1 \delta s_1 \,.\, dx\, dy\, dz.$$

Hence (γ) follows.

384.] **Stress Potential.** It is necessary that the expression

$$-(N_1 \delta a + N_2 \delta b + N_3 \delta c + 2 T_1 \delta s_1 + 2 T_2 \delta s_2 + 2 T_3 \delta s_3) \qquad (1)$$

which when multiplied by $d\Omega$, the element of volume, is the virtual work of the stress of this element, should be an exact

differential of some function of the strain components, and therefore of the form

$$-\delta\phi\,(a, b, c, s_1, s_2, s_3). \tag{2}$$

For, if the imagined further displacement is actually made, and the element $d\Omega$ is made to pass through a series of states of strain—say from the state in which the strain components are $(a', b', c', s_1', s_2', s_3')$ to that in which they are $(a'', b'', \ldots)$, the work actually done by the stress is

$$-d\Omega \int (N_1 da + N_2 db + N_3 dc + 2\,T_1 ds_1 + 2\,T_2 ds_2 + 2\,T_3 ds_3),$$

the integration extending from the first state to the second.

Now, unless the quantity under the integral sign is an exact differential, $d\phi$, the work done in passing from the first to the last state will depend on the intermediate states—or on what we may call the 'path of the strain'—so that on the return from the second to the first state by a different 'strain path' the work given back by the stresses would not be the same as that required to produce the original change of state. There is thus either a loss or gain of work done on the element, and the excess or defect of work must shew itself by a gain or loss of kinetic energy in the element. Such energy might be the molecular energy called Heat. But if we assume that the states of strain are produced very slowly, so that no appreciable velocity, molecular or other, is ever generated, no energy of any appreciable amount is ever generated or destroyed in the element. Hence the work done by the stress in the passage from any one state of strain along any 'strain path' to another state of strain must be independent of the path, and this can be so only if the expression (1) is of the form (2).

Consider, for example, what would happen if the element of work were equal to $d\Omega(bda - adb)$.

Representing values of a and b by abscissæ and ordinates with reference to two fixed rectangular axes through an origin O, the work of the stress from the state represented by the point A, whose co-ordinates are (a', b'), to the point B, whose co-ordinates are (a'', b''), would be represented by double the area included between the lines OA and OB, and any arbitrary curve whatever drawn between A and B, so that by perpetually making the element reach B by a strain path represented by a curve S, and return to A by a strain path represented by a curve S', there would be in each cycle of changes a gain (or loss) of work represented by double the area enclosed by these curves.

The work done by the stresses, therefore, in the strain of the body from its natural state to the state in which $(a, b, \dots 2s_1, \dots)$ are the strain components at any point is

$$-\int \phi \, d\Omega,$$

where ϕ is the Potential of the strain and is used, for simplicity, instead of $\phi(a, b, c, s_1, s_2, s_3)$.

The work done by the stresses is equal and opposite to that done by the forces externally applied to the body, if the strain is produced without appreciable velocity.

Section III.

Stress in Terms of Strain.

385.] **Isotropic Body.** A body is said to be *isotropic*, if its structure in the neighbourhood of any point is the same in all directions round the point. More definitely, let P be any point in the body, and Q a point distant l from P, in any direction; let a little cylinder having PQ for its axis and having a very small transverse section, σ, be imagined to be cut out of the body; then if to stretch this cylinder—one end being fixed and the other pulled—by a constant amount, δl, requires the same force no matter what the direction of PQ is, the substance is isotropic. In other words, if Young's modulus is the same for slender cylinders cut out in all directions, the substance is isotropic. As examples of approximately isotropic solids, we may cite glass and steel.

If this modulus is not constant for all such cylinders, the body is anisotropic, or as M. de Saint-Venant calls it, *heterotropic.* The term isotropic is due to Cauchy.

Heterotropy may exist in all degrees; that is, a heterotropic solid may have certain planes with respect to which its structure is symmetrical—as, for instance, forged metallic pieces, woods, and slates—without possessing structural symmetry with regard to any other planes.

A crystalline body is, of course, an example of heterotropy.

386.] **Extension and Lateral Contraction.** Confining our attention for the present to the case of an isotropic solid, suppose that we take a cube of the substance, $zPxyI$ (Fig. 295), and

apply tension to the two opposite faces $BIAZ$ and $xCyP$, these tensions being uniformly distributed over the faces, with intensity p; then, in virtue of the isotropy of the substance, the other pairs of parallel faces will be drawn towards each other, through the same distance—in other words, there will be uniform lateral contraction of the prism.

As before, let c be the elongation of the edges parallel to Pz, and let η be the ratio of the contractions (parallel to Px and Py) to the elongation, so that

$$a = -\eta c, \; b = -\eta c.$$

Then, θ being the cubical dilatation,

$$\theta = (1-2\eta)c.$$

Also, E being Young's modulus for the substance,

$$p = E.c.$$

A vigorous controversy exists with regard to the coefficient η, M. de Saint-Venant maintaining, on the one side, both as a mathematical and as an experimental result that for all hard fine-grained isotropic solids η is constant and equal to $\frac{1}{4}$, while M. Lamé (and with him English writers generally) denies this constancy and leaves its value indeterminate.

Subsequently we shall give Saint-Venant's argument; but no inconvenience will arise from leaving indeterminate the value of η. Those cases in which experiment finds for η values different from $\frac{1}{4}$ are disposed of by Saint-Venant by saying that either the bodies to which they refer are not solid isotropic bodies, or the displacements produced in them are not small. In this way he disposes of cork and indiarubber (which is really a cellular substance the pores of which are filled with a liquid), and also of jellies, in which the displacements are far greater than are contemplated in the theory of small strains. (See Saint-Venant's annotated translation of Clebsch's *Theorie der Elasticität Fester Körper*, p. 67.)

Solids for which $\eta = \frac{1}{4}$ are sometimes called 'perfect solids.' According to Saint-Venant, all hard fine-grained solids are perfect solids.

387.] **Moduli of Extension and Shear.** Young's modulus is the modulus of extension, which we may formally define as follows: *If the ends of a cylinder are pulled by two equal and opposite forces, acting along its axis, and distributed uniformly over the ends, no lateral or other forces being applied, the ratio of the*

force-intensity on the ends to the (fractional) elongation of the cylinder is the modulus of extension of the substance.

If P is the magnitude of each force, $\sigma =$ area of section, $l =$ length of unstrained cylinder, $\Delta l =$ increase of length,

$$\frac{P}{\sigma} = E . \frac{\Delta l}{l} .$$

In such case there is, as just said, always lateral contraction; and it is to be carefully noted that it is only when forces act on the ends and no forces act on the sides that the intensity of the tension $= E \times$ the elongation. Thus we must not expect to find that $N_1 = E . a$, for example, where N_1 and a are, respectively, the normal intensity of tension and the elongation along the axis of x at any point of a strained solid. For, the faces of the cube (Fig. 295) are all acted upon by forces, and not merely the faces zPy and BxC.

Modulus of Shear, or Sliding. If one face of a cube, or any prism, of a substance is held fixed, while to the opposite face is applied uniformly distributed force in the plane of this face, the ratio of the force-intensity on the face to the shear produced is the modulus of shear.

Let P be the whole force applied to the face, $\sigma =$ area of the face, $l =$ length of prism, $\Delta\tau$ the sliding displacement of any point in the face, and μ the modulus of shear; then

$$\frac{P}{\sigma} = \mu . \frac{\Delta\tau}{l} .$$

The expression $\frac{\Delta\tau}{l}$ is what we have previously denoted by $2s$, the shear. (See Art. 369.)

Saint-Venant uses G for this modulus of shear or sliding (*glissement*), while Lamé uses μ.

The elastic quality of every isotropic solid is completely expressed by these two moduli, E and μ. For perfect fluids (liquids or gases) $\mu = 0$.

The modulus of shear is also the *modulus of torsion*. For (example 13, p. 408) torsion is equivalent to shear at every point; and if, as in Art. 371, the axis of torsion is taken as that of z, the stress intensities, T_1 and T_2, on the transverse section of the cylinder at P are given by the equations

$$T_1 = 2\mu s_1 = \mu a \frac{x}{l}; \ T_2 = -\mu a \frac{y}{l},$$

so that if T is the resultant stress, $T = \mu a \frac{r}{l}$, which shows that μ is also the resistance to torsion.

To express the modulus of shear in terms of Young's modulus and the coefficient of lateral contraction. In Fig. 295, suppose the faces $BIAz$ and $xCyP$ to have tension uniformly distributed over them, with intensity p, there being no other forces applied to the cube. Let $l =$ the length of each edge of the cube, and separate it, in imagination, into two wedges by the diagonal plane $xIAP$. Considering the equilibrium of the upper wedge, we see that the stress produced on its face $xIAP$ by the lower wedge must be a tension at its middle point equal and opposite to pl^2; and as the area of this face $= l^2\sqrt{2}$, the intensity of this stress $= \frac{p}{\sqrt{2}}$. Resolving this into a normal and tangential stress intensity, the latter $= \frac{p}{2}$. Hence if $2s$ is the fractional sliding, or shear, of planes parallel to $xIAP$, we have

$$\frac{p}{2} = 2\mu s. \tag{α}$$

Now this shear is the change in the cosine of the angle between the diagonals xI and BC; and by the applied traction the edge CI and all parallel to it are lengthened by Δl, while by lateral contraction the edge BI and all parallel to it become $l - \eta\Delta l$. Hence the square $xCIB$ becomes a rectangle whose sides are $l + \Delta l$ and $l - \eta\Delta l$. Also (Art. 361) the angle between the diagonals of this rectangle is $\frac{\pi}{2} - 2s$.

Hence $$\tan\left(\frac{\pi}{4} - s\right) = \frac{l - \eta\Delta l}{l + \Delta l} = \frac{1 - s}{1 + s};$$

therefore $$s = (1 + \eta)\frac{\Delta l}{2l}.$$

Since, then, $p = E\frac{\Delta l}{l}$, we have from (α)

$$\mu = \frac{E}{2(1 + \eta)}. \tag{β}$$

For 'perfect solids' we have

$$\mu = \frac{2}{5}E. \tag{γ}$$

The modulus of compressibility, or resistance to compression, can be easily expressed in terms of μ and E.

Generally a modulus of rigidity, or resistance to a strain, of any kind is the force intensity (per unit area) producing the strain divided by the fractional measure of the strain; so that every rigidity is a force per unit area. Supposing that the whole surface of a cube is subject to uniform intensity of pressure, or tension, p, if θ is the cubical compression or dilatation, the resistance, k, to compression is given by the equation

$$k = \frac{p}{\theta}.$$

Now let the cube in question be that in Fig. 295; and observe that the elongation a will be produced by three distinct and superposed causes:—1°, the elongation which would be produced if only the faces $zPyA$ and $BxCI$ were pulled, the amount of this being $\frac{p}{E}$; 2°, the lateral contraction which would be produced if only the faces $zPxB$ and $AyCI$ were pulled, the amount of this being $\eta\frac{p}{E}$; and 3°, the lateral contraction which would be produced if only the faces $PxCy$ and $zBIA$ were pulled, the amount of which is also $\eta\frac{p}{E}$.

Hence
$$a = (1-2\eta)\frac{p}{E},$$

$$\therefore \quad \theta = 3(1-2\eta)\frac{p}{E},$$

$$\therefore \quad k = \frac{E}{3(1-2\eta)}. \qquad (\delta)$$

For Young's modulus in terms of the moduli of compression (k) and distortion (μ), we have

$$E = \frac{9k\mu}{3k+\mu}.$$

We have here used a principle which is largely employed in the theory of small strains, viz. *the principle of the independence and superposition of strains due severally to given superposed systems of stress.*

388.] **Stress Components in terms of Strain Components.** It is required to express the stress components (N_1, N_2, N_3,

T_1, T_2, T_3) in terms of the strain components ($a, b, c, 2s_1, 2s_2, 2s_3$) in the case of a strained isotropic body.

Take the cube in Fig. 295. Then, as just explained, the elongation a is due to the separate actions of the normal tensions N_1, N_2, N_3, the first producing a stretch equal to $\frac{N_1}{E}$, and the latter two producing lateral contractions equal to $\eta\frac{N_2}{E}$ and $\eta\frac{N_3}{E}$.

Hence $$Ea = N_1 - \eta(N_2 + N_3). \tag{1}$$

Similarly $$Eb = N_2 - \eta(N_3 + N_1), \tag{2}$$

$$Ec = N_3 - \eta(N_1 + N_2). \tag{3}$$

By definition we have also

$$T_1 = 2\mu s_1,\ T_2 = 2\mu s_2,\ T_3 = 2\mu s_3. \tag{4}$$

We have also by solving for N_1,

$$N_1 = \frac{\eta E}{(1+\eta)(1-2\eta)}\cdot\theta + \frac{E}{1-\eta}\cdot a, \tag{5}$$

with similar values of N_2, N_3. Let

$$\lambda = \frac{\eta E}{(1+\eta)(1-2\eta)}. \tag{6}$$

Then we have

$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu a;\ T_1 = 2\mu s_1, \\ N_2 &= \lambda\theta + 2\mu b;\ T_2 = 2\mu s_2, \\ N_3 &= \lambda\theta + 2\mu c;\ T_3 = 2\mu s_3, \end{aligned}\right\} \tag{7}$$

in the simple notation of Lamé.

It will be observed that for perfect solids $\lambda = \mu$.

This remarkably simple method of expressing the stress components in terms of those of strain is due to Clebsch. (Saint-Venant's edition of Clebsch, p. 14.)

389.] **Method of Cauchy.** This method consists in assuming that at every point in a strained isotropic body the principal axes of the strain coincide with the principal axes of the stress. Here then we have

$$s_1 = s_2 = s_3 = 0,\quad T_1 = T_2 = T_3 = 0.$$

Also we can assume

$$A = (\lambda + 2\mu)e_1 + \lambda e_2 + \lambda e_3,$$

where λ and μ are constants; for e_2 and e_3 must evidently have the same coefficient in the value of A, since the body is elastically symmetric with regard to the axes of y and z (and, of course,

with regard to *all* axes) and the plane on which N_1 acts is also symmetrically placed with respect to them. Thus

$$\left.\begin{aligned} A &= \lambda\theta + 2\mu e_1, \\ B &= \lambda\theta + 2\mu e_2, \\ C &= \lambda\theta + 2\mu e_3, \end{aligned}\right\} \qquad (1)$$

where $\theta = e_1 + e_2 + e_3 =$ the cubical dilatation, and e_1, e_2, e_3 are the principal elongations.

It is required to express the components, $N_1, N_2, N_3, T_1, T_2, T_3$, of the stress at the point considered in the body with reference to three rectangular axes at the point and the corresponding components of the strain. Let (l, m, n), &c., be the direction-cosines of the new axes with reference to the principal axes of strain and stress. Then by multiplying both sides of equations (1) by l^2, m^2, n^2, respectively, and adding, we have, by Art. 379, precisely the value of N_1 obtained in last Article.

Similarly, by multiplying the sides of equations (1) by $l' l''$, $m' m''$, $n' n''$, and adding, we have the value of T_1 before obtained.

390.] **Method of Thomson.** Denote, as before, the resistances to compression and distortion (or shear) by k and μ, respectively.

Then, to find the stresses called into play by a simple elongation, a, along the axis of x, resolve this elongation exactly as in example 5, p. 406, into a cubical dilatation, a, together with two shears. Now, by our above definition, the dilatation will cause a normal intensity of stress equal to ka on each face of a cubical element whose edges coincide with Ox, Oy, and Oz at the point, O.

Consider the elongation $\frac{1}{3}a$ along Ox and the accompanying contraction $\frac{1}{3}a$ along Oz. These give shears each equal to $\frac{2}{3}a$ on the planes $OCHD$ inclined at angles of 45° to Ox and Oz; and these shears will, by the above definition, give rise to shearing stresses each of intensity $\frac{2}{3}\mu a$ on these planes. Again, by p. 416, these shearing stresses will give rise to normal stresses each of intensity $\frac{2}{3}\mu a$ on planes parallel to OH and CD; and it is obvious that the normal stress on the plane OH (or rather the plane through OH perpendicular to the paper) produced by the

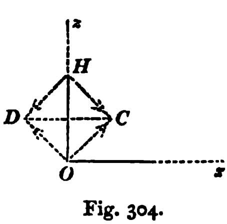

Fig. 304.

portion of the body to the right of OH will be *tension*, i.e., it will be in the sense Ox; while on the plane CD or (Ox) the normal stress produced by the portion of the body at the upper side of the figure will be *pressure*, i.e. it will be in the sense zO.

Similarly, by considering the other shear (that which consists of elongation $\frac{1}{3}a$ along Ox and contraction $\frac{1}{3}a$ along Oy), we have a further normal tension equal to $\frac{2}{3}\mu a$ on the plane perpendicular to Oz; and normal pressure $\frac{2}{3}\mu a$ on the plane perpendicular to Oy. Hence the elongation a gives normal stresses

$$(k+\tfrac{4}{3}\mu)\,a,\quad (k-\tfrac{2}{3}\mu)\,a,\quad (k-\tfrac{2}{3}\mu)\,a,$$

on the planes perpendicular to Ox, Oy, Oz, respectively.

Similarly the elongation b (which is along Oy) gives normal stresses

$$(k-\tfrac{2}{3}\mu)\,b,\quad (k+\tfrac{4}{3}\mu)\,b,\quad (k-\tfrac{2}{3}\mu)\,b$$

in the same directions; and the remaining elongation, c, gives

$$(k-\tfrac{2}{3}\mu)\,c,\quad (k-\tfrac{2}{3}\mu)\,c,\quad (k+\tfrac{4}{3}\mu)\,c.$$

Hence we have

$$N_1 = (k+\tfrac{4}{3}\mu)\,a+(k-\tfrac{2}{3}\mu)\,b+(k-\tfrac{2}{3}\mu)\,c;$$

or

$$N_1 = (k-\tfrac{2}{3}\mu)\,\theta+2\mu a,$$

as before. The values of T_1, T_2, T_3, are obvious from the definition of a modulus of shear.

291.] **Case of a Liquid.** A perfect fluid has a zero modulus of shear, i.e. $\mu = 0$. If it is a liquid, it is voluminally incompressible, therefore $\lambda = \infty$. Also $\theta = 0$, but $\lambda\theta$ is N_1, N_2, or N_3, which are all equal to $-p$, the pressure intensity at any point.

392.] **Strain Potential.** For an isotropic body the Strain Potential is easily found from the values of the stress components just given. If ϕ is the Strain Potential (Art. 384), we have

$$N_1da+N_2db+N_3dc+2\,T_1ds_1+2\,T_2ds_2+2\,T_3ds_3 \equiv d\phi.$$

Hence we easily find

$$2\phi = \lambda\,(a+b+c)^2+2\,\mu\,(a^2+b^2+c^2+2\,s_1^{\,2}+2\,s_2^{\,2}+2\,s_3^{\,2})\ldots. \quad (a)$$

Of course $N_1 = \dfrac{d\phi}{da}, \ldots 2\,T_1 = \dfrac{d\phi}{ds_1}, \ldots.$

If in (a) we substitute the values of the strain components in terms of those of stress, we have

$$2\phi = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\,\sigma^2-\frac{\sigma'}{\mu},$$

where $\sigma \equiv N_1+N_2+N_3$, and

$$\sigma' \equiv N_1N_2+N_2N_3+N_3N_1-T_1^2-T_2^2-T_3^2.$$

But by (6), Art. 388, we have $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ = Young's modulus; hence

$$2\phi = \frac{\sigma^2}{E} - \frac{\sigma'}{\mu}. \qquad (\beta)$$

393.] **Clapeyron's Theorem.** Since (Art. 384) $-\int \phi \, d\Omega$ is the work done by the stress in the passage of the body from the unstrained state to that in which ϕ is the Potential of strain of the element $d\Omega$, we have this work equal to

$$-\tfrac{1}{2}\int\left(\frac{\sigma^2}{E} - \frac{\sigma'}{\mu}\right) d\Omega, \qquad (\gamma)$$

which, with altered sign, is also the work of the external forces if the strain is very slowly produced. This expression for the work constitutes Clapeyron's Theorem. Observe, then, that in this expression (γ) the stresses involved in σ, σ' are those belonging to the *final* state of strain.

The work done in the passage to a certain final state of strain may, therefore, be expressed as an integral, through the body, either of a Potential of the stress or of a Potential of the strain, belonging to the final state—that is, either of the function ϕ given by (a) or of that given by (β).

To take the simplest illustration, consider the work done in the extension of a cylindrical bar of uniform section σ. The work diagram is a triangle, and if l_0 = natural length of bar, Δl = whole amount of extension, the *final* tension $T = E\sigma \frac{\Delta l}{l_0}$; and the work of the tension $= -\tfrac{1}{2} T . \Delta l$; so that the work is

$$-\frac{l_0 T^2}{2E\sigma}, \quad \text{or} \quad -\tfrac{1}{2} l_0 E\sigma . a^2,$$

where $a = \frac{\Delta l}{l_0}$ = the elongation.

EXAMPLES.

1. Find the work done in a gradual uniform compression of a body.

If V_0 is the unstrained volume, V the compressed volume, p the *final* intensity of pressure on the surface, the work is

$$\tfrac{1}{2} p (V_0 - V). \qquad (a)$$

In a uniform compression we have at each point

$$u = -ax, \quad v = -ay, \quad w = -az;$$

therefore $\theta = -3a$; $N_1 = N_2 = N_3 = -(3\lambda + 2\mu)a = -3ka$, where k is the modulus of compression; $k = \frac{p}{\theta}$, therefore $N_1 = -p$, so that the intensity of stress is constant throughout the body.

The work may also be expressed as

$$\frac{p^2 V_0}{2k}.$$

Observe that the element of work done by the pressure of a gas in changing its volume by dv is pdv, while our expression (a) gives the work of changing the volume of an elastic solid as equal to $\frac{1}{2}pdv$. There is an apparent contradiction; but observe that in all cases of equilibrium of strain, we assume the externally applied forces to be applied with *very gradually increasing magnitudes*, so that neither they nor the stresses reach at once their final values; the stresses grow from zero to certain final values, and our expressions for work done all involve the final values of the stresses and not intermediate values which they have in intermediate states of strain. Now as the components of stress are linear functions of those of strain, we obtain the elementary result which we know to hold for the expansion of a bar according to Hooke's Law, viz. that the work is *one half* the product of the whole extention and the *final* tension of the bar.

2. Find the work done in the distortion of a body (without compression or dilatation at any point).

Ans. $\mu\int(a^2+b^2+c^2+2s_1^2+2s_2^2+2s_3^2)\,d\Omega.$

For example, take the case of the torsion of a circular cylinder (Art. 371). Then $2s_1 = \frac{ax}{l}$, $2s_2 = -\frac{ay}{l}$; so that the work

$$= \frac{\mu a^2}{2l^2}\int r^2 d\Omega,$$

where r is the distance of the element $d\Omega$ from the axis of the cylinder. This work is

$$\mu \cdot \frac{\pi R^4 a^2}{4l},$$

where R = radius of cylinder.

If the torsion is produced by opposite couples applied at the ends, or by holding one end fixed and applying a couple of moment G to the other, this work is

$$\tfrac{1}{2} a R^2 G,$$

as will presently be shown.

3. Two uniform bars, CB, BA (Fig. 184, p, 224, vol. I) are freely jointed to each other at B, and have their ends C and A fixed by smooth pins; if a weight, W, so great that, in comparison, the weights of the bars may be neglected, is suspended from the joint B, find the vertical distance through which B will descend.

The weight W is supposed to have been put on gradually, so that at no moment is there any vibration produced. Also if Δz is the

vertical descent of B, the work done by this gradually accumulated weight is $\frac{1}{2} W\Delta z$. If T and T' are the tension and pressure in the bars BC and BA, respectively, and Δl, $\Delta l'$ the amounts by which they are elongated and shortened, we have the work done by the stresses $= -\frac{1}{2}T.\Delta l - \frac{1}{2}T'\Delta l'$; or if $BC = a$, $BA = c$, and the areas of their sections are σ, σ', this work is

$$-\tfrac{1}{2}\left(\frac{aT^2}{E\sigma} + \frac{cT'^2}{E\sigma'}\right).$$

Hence, since no appreciable kinetic energy is generated,

$$W.\Delta z = \frac{aT^2}{E\sigma} + \frac{cT'^2}{E\sigma'}.$$

But if BC and BA make a and γ with the vertical,

$$T = W\frac{\sin\gamma}{\sin\beta}; \quad T' = W\frac{\sin a}{\sin\beta},$$

where β is the angle between the bars at B. Hence

$$\Delta z = \frac{W}{E\sin^2\beta}\left(\frac{a\sin^2\gamma}{\sigma} + \frac{c\sin^2 a}{\sigma'}\right).$$

If the points C and A are in the same vertical line, and $AC = b$,

$$\Delta z = \frac{Wac}{Eb^2}\left(\frac{c}{\sigma} + \frac{a}{\sigma'}\right).$$

4. Three uniform bars AB, BC, CA, forming a triangle and freely jointed to each other rest in a vertical plane, the bar AC being horizontal and the ends A, C being supported on two fixed vertical pillars; a weight W is suspended from the joint B (by gradual accumulation); find the vertical descent of B.

Ans. If the normal sections of the bars opposite A, B, C are σ, σ', σ'', their lengths a, b, c, and the angles a, β, γ,

$$\Delta z = \frac{W}{E\sin^2\gamma}\left(\frac{a}{\sigma}\cos^2 a + \frac{b}{\sigma'}\cos^2\beta + \frac{c}{\sigma''}\cos^2 a\cos^2\beta\right).$$

5. One end of a bar of isotropic material is held fixed, and the bar hangs vertically; find its elongation caused by its weight.

Let AB be the bar in its natural state, P a point in AB at a distance z from A; let $A'B'$ represent the elongated bar, and let P' be the displaced position of P.

Then the intensity of stress on a normal section at $P' = E\dfrac{dw}{dz}$, where E is Young's modulus. But if σ is the area of the section at P', the intensity of stress $= \dfrac{\text{weight of length } PB}{\sigma} = \dfrac{W}{\sigma}\dfrac{l-z}{l}$, where W and l are the weight and length of the bar.

Hence
$$E\frac{dw}{dz} = \frac{W}{\sigma}\frac{l-z}{l}.$$

$$\therefore\ w = \frac{W}{E\sigma l}(lz - \tfrac{1}{2}z^2) + C,$$

where C is a constant. Now the value of w for the fixed end is zero, therefore $C=0$; and the value of w for the free end, B, is the amount of elongation. Hence, putting $z=l$,

$$\text{amount of elongation} = \frac{Wl}{2E\sigma}.$$

It is immaterial whether σ means the section of the bar $A'B'$ or the section of AB, since these areas differ by a small quantity of the first order.

6. To find the stresses produced at any point in a circular cylinder which undergoes torsion round its axis.

With the notation of p. 400, we have by Art. 388.

$$N_1=0,\quad N_2=0,\quad N_3=0,$$

$$T_1=\frac{\mu a}{l}x,\quad T_2=-\frac{\mu a}{l}y,\quad T_3=0.$$

The torsion may be produced either by fixing one end of the cylinder and applying a couple to the other end, or by applying two equal and opposite couples to the ends, each of which is free. By considering the equilibrium of a portion of the cylinder between one end and a section made at any point O (Fig. 305), on the axis perpendicularly to the axis, we see that the stress system exerted over this section by the remaining portion of the cylinder must be a couple equal in amount to the applied couple (F, F).

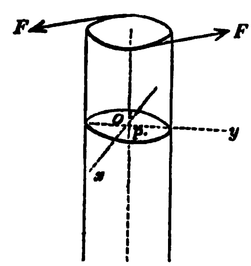

Fig. 305.

Let the fixed axes of x and y at O be Ox and Oy, and let P be a point in the section whose co-ordinates are x and y. Then the above values of the intensities of stress show that on the element area dS at P the two components of stress on the *lower* side of dS are $\frac{\mu a}{l}ydS$ in the direction Ox, and $\frac{\mu a}{l}xdS$ in the direction yO. The sum of their moments about Oz is $\frac{\mu a}{l}(x^2+y^2)dS$ in a sense opposite to that of the applied couple. Hence if the amount of this couple is denoted by G,

$$\frac{\mu a}{l}\int r^2 dS = G,$$

where $r=OP$, and the integration is extended over the whole area of the section at O. Now $\int r^2 dS$ is the moment of inertia, I, of the section about Oz. Therefore

$$\frac{\mu a}{l}I=G.$$

Let $\frac{a}{l}$, which is the rate of twist per unit length of the cylinder, be denoted by τ, and we have

$$\mu\tau I = G. \qquad (a)$$

Defining the torsional rigidity as the moment of the applied twisting couple divided by the rate of twist produced, we see that, for a solid circular cylinder,

$$\text{torsional rigidity} = \tfrac{1}{2}\mu \,.\, \pi R^4,$$

where R is the radius of the cylinder (since $I = \frac{1}{2}\pi R^4$).

The result in equation (a) is known as Coulomb's Law.

7. To show that Coulomb's Law cannot apply to a non-circular cylinder when it is acted on only by twisting couples at its extremities.

In order that the law of torsion strain expressed by the equations

$$u = -\tau yz, \quad v = \tau xz, \quad w = 0$$

may hold, we shall show that force must be applied over the bounding surface of the cylinder parallel to its axis.

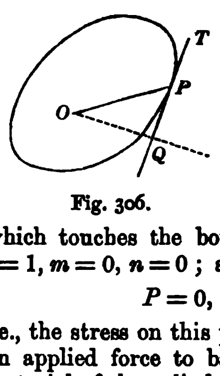

Fig. 306.

Let Fig. 306 represent a section of the cylinder perpendicular to its axis, the axis passing through O; let P be a point on the bounding surface, PT the tangent to the section, and OQ a perpendicular to PT. Let OQ be taken as axis of x, the axis of z being the axis of the cylinder; and let us calculate the stress on an element plane which touches the bounding surface at P. We have for this plane $l = 1, m = 0, n = 0$; and equations (3), p. 414, give (by last example)

$$P = 0, \quad Q = 0, \quad R = -\mu\tau y = -\mu\tau \,.\, PQ;$$

i.e., the stress on this plane is proportional to PQ, and there must be an applied force to balance this stress, since there is none of the material of the cylinder at the right-hand side of the plane.

8. Let there be a straight fibrous body or beam subject to a slight bending strain such that the fibres (*mean fibres*) which lie in a certain plane, although bent, are not elongated, and that the elongation (positive or negative) along every other fibre is proportional to its (positive or negative) distance from this plane, the bending of all fibres taking place parallel to a single plane which cuts the normal section of the bar perpendicularly. It is required to find for any normal section the sum of the moments, round the line in which it intersects the plane of the mean fibres, of the stresses which are exerted at the section by the strained fibres.

Suppose that after the bending any one section, AHB (Fig. 307), is brought by a motion as of a rigid body back to its old position, and let a neighbouring section then occupy the position $A'H'B'$. Let HH', cc' be two of the mean fibres which reach across from one of the sections to the other. Then the original distance between

the sections is HH' or cc'. Let this be denoted by ds. If PP' is any other fibre reaching across, Pn and $P'n'$ the perpendiculars from P and P' on the right lines cH and $c'H'$, the elongation along PP' (i.e. $\frac{PP'-ds}{ds}$) is proportional to Pn. Let the planes of the sections AHB and $A'H'B'$ intersect in a line OL, let ρ denote the length of the radius of curvature (cO) of the bent mean fibres (cc' or nn'), and $Pn = y$. Then evidently

$$\frac{PP'}{nn'} = \frac{\rho + y}{\rho},$$

$$\therefore \quad \frac{PP' - nn'}{nn'} = \frac{y}{\rho},$$

which is the elongation along PP'. For fibres at the lower side of cH, there is contraction, or negative elongation, and for these y is reckoned as negative.

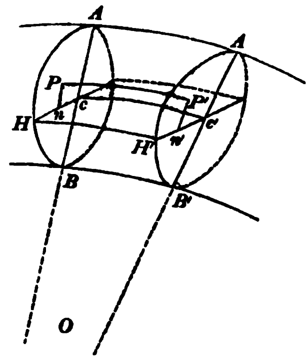

Fig. 307.

Now, by Hooke's Law, if we consider a small prism who sides are the fibres emanating from points on a very small area, $d\sigma$, at the point P, the stress of this prism (*assumed wholly longitudinal*) is

$$\frac{Ey}{\rho} d\sigma.$$

The moment of this force about cH is $\frac{Ey^2}{\rho} d\sigma$; therefore the sum of these moments all over the section AHB is $\frac{E}{\rho} \int y^2 d\sigma$, or

$$\frac{EI}{\rho}, \qquad (a)$$

where I is the Moment of Inertia of the section AHB about the line cH. [See the second paragraph of p. 432.]

Remark. If the end of a beam merely rest against a fixed surface, there will be no Bending Moment at this end, and $\rho = \infty$ at it. But if the end is tangentially fixed, there will be a Bending Moment at it, and its curvature will not be zero.

9. A uniform slightly elastic beam rests, in non-limiting equilibrium, with one end on the ground and the other against a vertical wall, the vertical plane through the beam being at right angles to the wall; find the form of the mean fibre of the beam. Let AB (Fig. 308) be the beam; GN the vertical through its centre of gravity, G; R and S the reactions of the ground and wall; ϕ the angle made by R with the vertical; a the angle which the tangent to the beam at A makes with the horizon; h and k the distances, Ax and Bx, of the extremities from the line of intersection of the ground and wall.

Let P be any point in the beam, at which we shall calculate the Bending Moment, i.e. the sum of the moments of all the forces acting on the beam between P and A; let the horizontal and vertical lines through A be taken as axes of x and y; let Q be any point between P and A; let the co-ordinates of P and Q be (x, y) and (x', y'), respectively; let the original length of the beam be l, and its weight W.

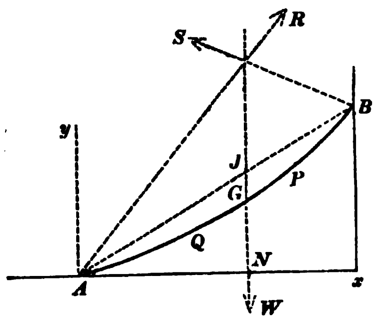

Fig. 308.

Then the weight of an element of length, ds', at Q is $\frac{W}{l}ds'$, and the moment of this force tending to produce curvature at P round a line (such as cH in Fig. 307) perpendicular to the plane of the figure is

$$-\frac{W}{l}(x-x')ds'.$$

Also the moment of R about this axis is

$$R(x\cos\phi-y\sin\phi).$$

Hence if ρ is the radius of curvature of the mean fibre at P, we have

$$\frac{EI}{\rho}=R(x\cos\phi-y\sin\phi)-\frac{W}{l}\int(x-x')ds', \tag{1}$$

the integration being performed from A to P.

If P is taken very close to A, the Bending Moment on the right side of (1) is zero, therefore ρ at $A=\infty$, i.e. A is a point of inflexion; and B is also a point of inflexion for a similar reason.

Assume
$$y=x\tan a+a_3x^3+a_4x^4+a_5x^5\ldots, \tag{2}$$
where $a_3, a_4, a_5, \ldots$ are all very small quantities; there being no term in x^2 since $\frac{d^2y}{dx^2}=0$ $(\rho=\infty)$ at A.

From (2), we find

$$\frac{ds}{dx}=\sec a+\sin a(3a_3x^2+4a_4x^3+5a_5x^4+\ldots)$$

$$\frac{d^2y}{dx^2}=6a_3x+12a_4x^2+20a_5x^3+\ldots.$$

Now $\frac{1}{\rho}=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$; and if we neglect products of $a_3, a_4, \ldots$, we shall have $\frac{1}{\rho}=\cos^3 a(6a_3x+12a_4x^2+20a_5x^3+\ldots)$.

Also

$$\int_0^x (x-x')\frac{ds'}{dx'}\cdot dx' = \tfrac{1}{2}x^2\sec a + \tfrac{1}{4}a_3\sin a\,.\,x^4 + \tfrac{1}{5}a_4\sin a\,.\,x^5 + \ldots.$$

Making these substitutions in (1), and equating to zero the coefficient of every power of x, we have

$$a_3 = \frac{R\sin\phi\,(\cot\phi - \tan a)}{6\,EI\cos^3 a},$$

$$a_4 = -\frac{W}{24\,lEI\cos^4 a},$$

while $a_5, a_6, \ldots$ are of the order $\frac{1}{(EI)^2}$ and may be neglected.

Also at the extremity B, $\frac{d^2y}{dx^2}$ must be zero; therefore

$$a_3 + 2\,ha_4 = 0;$$

and the equation of the mean fibre is

$$y = x\tan a + \frac{W}{24\,lEI}(2\,hx^3 - x^4)\sec^4 a.$$

By putting k and h for y and x, this equation gives

$$\tan a = \frac{k}{h} - \frac{Wh^3}{24hlEI}\sec^4 a.$$

Putting $\sec a = \frac{\sqrt{k^2+h^2}}{h}$ in the small term, we get

$$\tan a = \frac{k}{h} - \frac{Wl'^4}{24hlEI},$$

where l' is used for $\sqrt{k^2+h^2}$.

Substituting this value of a in the equation of the mean fibre, we have

$$y = \frac{k}{h}x - \frac{Wl'^4}{24\,lh^4EI}(h^3x - 2\,hx^3 + x^4),$$

which is the equation of the mean fibre, to the first power of $\frac{1}{EI}$.

It will be easily found that AN, the abscissa of the centre of gravity of the beam, is

$$\frac{h}{2}\left(1 + \frac{Wkl'^2}{60\,lEI}\right).$$

10. A rigid bar is supported nearly horizontally on three given vertical props which are slightly elastic; to determine the pressures on these props.

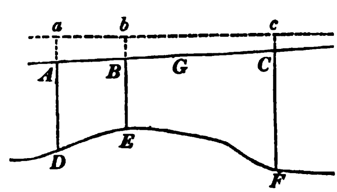

Fig. 309.

Suppose that the props are fixed in the ground at D, E, and F (Fig. 309), and their extremities were originally a, b, c, which are in a horizontal line; but that when the shrinking has taken place, their extremities, A, B, C, lie in a line slightly inclined to the horizon.

Let their original lengths be p, q, r, so that $Aa=\delta p$, $Bb=\delta q$, $Cc=\delta r$; let the pressures on them at A, B, and C be P, Q, and R; let G be the centre of gravity of the bar and W its weight.

Then we have

$$P+Q+R=W, \text{ and } P.GA+Q.GB-R.GC=0, \tag{1}$$

the second being obtained by moments about G.

Now if the areas of the normal sections of the props are α, β, γ, we have (Art. 387)

$$\frac{P}{\alpha}=E\frac{\delta p}{p}, \quad \frac{Q}{\beta}=E\frac{\delta q}{q}, \quad \frac{R}{\gamma}=E\frac{\delta r}{r}, \tag{2}$$

supposing that Young's modulus is the same for all.

Again, we must express the fact that ABC is a right line. Drawing through C a parallel to abs, we have

$$\frac{\delta p-\delta r}{\delta q-\delta r}=\frac{AC}{BC},$$

$$\therefore \quad BC.\delta p-AC.\delta q+AB.\delta r=0, \tag{3}$$

or, by (2),

$$\frac{p.BC}{\alpha}P-\frac{q.AC}{\beta}Q+\frac{r.AB}{\gamma}R=0. \tag{4}$$

The three equations (1) and (4) determine P, Q, R.

11. A heavy rigid slab is supported nearly horizontally on four given vertical props; to determine the pressures on these props.

Let A, B, C, D be the upper ends of the props when the shrinking has taken place; let the original lengths of the props be p, q, r, s; let the perpendiculars from A and C on the diagonal BD be p' and r'; let those from B and D on AC be q' and s'; let the perpendiculars from G, the centre of gravity of the slab, on AC and BD be x and y; let P, Q, R, S be the pressures on the props, whose sections are α, β, γ, δ, respectively; and let $W=$ weight of slab. Then we have obviously the statical equations

$$P+Q+R+S=W, \; Pp'-Rr'-Wx=0, \; Qq'-Ss'+Wy=0, \tag{1}$$

[G is supposed to lie within the area AOD] the two latter being equations of moments round BD and AC.

We must now express the fact that A, B, C, D lie in one plane.

To do this we shall calculate the vertical descent, $\delta\xi$, of the point O from the descents of A and C and also from those of B and D. Just as in last example, we have

$$\frac{\delta p-\delta r}{\delta\xi-\delta r}=\frac{AC}{OC}=\frac{p'+r'}{r'}, \quad \therefore \quad \delta\xi=\frac{r'\delta p+p'\delta r}{p'+r'}.$$

Similarly

$$\delta\xi=\frac{s'\delta q+q'\delta s}{q'+s'};$$

therefore

$$\frac{r'\delta p+p'\delta r}{p'+r'}=\frac{s'\delta q+q'\delta s}{q'+s'}. \tag{2}$$

Also, as before, $\frac{P}{a} = E\frac{\delta p}{p}$, &c., therefore (2) becomes

$$\frac{pr'}{a(p'+r')}P - \frac{qs'}{\beta(q'+s')}Q + \frac{rp'}{\gamma(p'+r')}R - \frac{sq'}{\delta(q'+s')}S = 0. \quad (3)$$

The four equations (1) and (3) determine the pressures.

12. A beam rests horizontally on any number of fixed right vertical props, and is loaded uniformly between each successive pair of props. Prove that if A_1, A_2, A_3 are any three successive props the bending stress couples of the beam at which are M_1, M_2, M_3, we shall have

$$8(a+b)M_2 + 4aM_1 + 4bM_3 = wa^3 + w'b^3, \quad (a)$$

where $a = A_1A_2$, $b = A_2A_3$, $w =$ load per unit length over A_1A_2, $w' =$ load per unit length over A_2A_3.

(This is known as the *Equation of Three Moments.*)

Let S_1' and S_1 be the shearing forces in normal sections just in front of A_1 (that is, towards A_2) and just behind it; S_2', S_2 the shearing forces in normal sections just in front of A_2 (i.e. towards A_3) and just behind it; S_3', S_3 the shearing forces just in front and just behind A_3.

Take A_2 as origin, A_2A_1 as axis of x, and the downward vertical line at A_2 as axis of y. Then taking any point, P, on the beam between A_2 and A_1, the bending stress couple at this point is $-EI\frac{d^2y}{dx^2}$; and equating to this the sum of the moments of the forces between P and A_2, we have

$$EI\frac{d^2y}{dx^2} = M_2 - S_2x + \tfrac{1}{2}wx^2;$$

$$\therefore \quad EIy = EI\tan a . x + \tfrac{1}{2}M_2 . x^2 - \tfrac{1}{6}S_2 . x^3 + \tfrac{1}{24}wx^4, \quad (1)$$

where a is the inclination of the tangent at A_2 to A_2A_1.

Similarly, taking a point, P', on the beam between A_2 and A_3, its co-ordinate with reference to A_2A_3 as axis of x being x',

$$EI\frac{d^2y'}{dx'^2} = M_2 - S_2'x' + \tfrac{1}{2}w'x'^2,$$

$$\therefore \quad EIy' = -EI\tan a . x' + \tfrac{1}{2}M_2 . x'^2 - \tfrac{1}{6}S_2' . x'^3 + \tfrac{1}{24}w'x'^4. \quad (2)$$

Now in (1) $y = 0$ when $x = a$; and in (2) $y' = 0$ when $x' = b$. Hence we have by addition of (1) and (2) with these substitutions

$$\tfrac{1}{2}(a+b)M_2 - \tfrac{1}{6}(a^2S_2 + b^2S_2') + \tfrac{1}{24}(a^3w + b^3w') = 0. \quad (3)$$

Again, for the equilibrium of the span A_1A_2, taking moments about A_1, we have

$$aS_2 = M_2 - M_1 + \tfrac{1}{2}wa^2; \quad (4)$$

and by moments about A_3 for the equilibrium of A_2A_3,

$$bS_2' = M_2 - M_3 + \tfrac{1}{2}w'b^2. \quad (5)$$

Eliminating S_2 and S_2' from (3), (4), (5) we have (a).

13. If the beam rests on three props only, two of which are at its extremities, find the pressures.

Put $M_1 = 0$, $M_3 = 0$; then (a) gives M_2, and (4), (5) give S_2, S_2'.

The pressure on A_1 is $wa - S_2$; that on A_2 is $S_2 + S_2'$; and that on A_3 is $w'b - S_2'$.

14. A weight is placed on an ordinary rectangular table which rests on the ground; calculate the pressures on the four legs, supposing that the legs may be treated as rigid in comparison with the ground.

Ans. If the adjacent sides at any corner A are b and a, and if x and y are the distances from these sides, respectively, of the point of application of the resultant of the sustained weight and the weight of the table, the pressure on the leg through A is

$$\frac{W}{2}\left(\frac{3}{2} - \frac{x}{a} - \frac{y}{b}\right),$$

where $W =$ sum of sustained weight and weight of table.

15. Prove that a circular cylinder can be subject to the strain

$$u = -\tau yz, \quad v = \tau zx, \quad w = cxy,$$

(its axis being axis of z) provided that surface stress parallel to the axis is supplied.

16. Determine the components of strain as quadratic functions of the co-ordinates so that at all points we shall have

$$N_1 = N_2 = T_1 = T_3 = 0\,;$$

and show that such strain will require the application of external force on the surface.

[Assume $u = px + qy + rz + \frac{1}{2}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)$, with similar values of v and w; then let the equations be satisfied at all points, i.e. equate to zero the coefficient of each variable.]

[The five following examples were communicated to the Author by the late Rev. Professor Townsend.]

17. A horizontal beam, supported at both ends, being loaded with any number of isolated weights, if the bending moments be equal at any pair of contiguous weights, P and Q, they are equal throughout the entire interval PQ.

18. A uniform load, PQ, is moved along a horizontal beam supported at both ends, A and B; prove that at a given point, O, in the beam the bending moment will be greatest when PQ occupies such a position that $\dfrac{OP}{OQ} = \dfrac{OA}{OB}$.

19. A uniform beam is *tangentially fixed* at both extremities A and B, D is its point of greatest deflection, C is the foot of the perpendicular from D on AB; X is any point in the line AB; a perpendicular to AB at X meets the bent beam in Y and the circular arc through A, D, B in Z.

Prove that
$$XY = \frac{XZ^2}{CD}.$$

20. A uniform beam is supported by four equidistant props, two of which are terminal; prove that the two points of inflection of its middle segment lie on the horizontal line of the props.

21. A uniform beam, AB, is supported horizontally at two points, C and D, in its length, C being adjacent to A and D to B. Prove that if two circles be described with C and D for centres and CA and DB for radii, respectively, the two points of inflexion of the beam are the two *limiting points* of the coaxal system determined by the circles.

394.] **Determinateness of Strain.** *If the external bodily and surface forces applied to a given elastic solid are given, the state of strain is completely determinate*—that is, there cannot be two different states of strain corresponding to these data.

For, if possible, let there be two different states of strain, $(a, b, c, 2s_1, 2s_2, 2s_3)$ and $(a', b', c', 2s_1', 2s_2', 2s_3')$ expressing the strain components at the same point P in these two different states. Reverse all the external forces and all the components of strain in the second state, and superpose this reversed state on the first. Thus we have the body acted upon by no external forces whatever, and yet strained, the typical components of the strain being $a-a', \ldots, 2(s_1-s_1'), \ldots$. Now the Potential Work of the stresses is equal to that of the external forces, which is zero; hence we have

$$\int \phi \, d\Omega = 0,$$

where 2ϕ has the value obtained by putting $a-a', \ldots, s_1-s_1'$, for $a, \ldots, s_1, \ldots$ in (a), Art. 392. But ϕ cannot possibly vanish except by the vanishing of all the components of strain individually, since it consists of the sum of a number of squares. We must therefore have $a' = a$, &c.; that is, the second state is identical with the first.

In any case, therefore, in which a given body is acted upon by given external forces and couples, if we find, by trial or otherwise, any *one* system of values of the displacements, u, v, w, satisfying the equations of equilibrium, we are assured that these constitute the only solution of the problem.

395.] **Differential Equations for Displacements.** If in the equations of equilibrium of an element (p. 411) we substitute the values of N_1, N_2, &c., given in equations (7), p. 435, we have

$$(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u = -\rho X, \qquad (1)$$

$$(\lambda+\mu)\frac{d\theta}{dy}+\mu\nabla^2 v = -\rho Y, \qquad (2)$$

$$(\lambda+\mu)\frac{d\theta}{dz}+\mu\nabla^2 w = -\rho Z, \qquad (3)$$

which are the differential equations from which the values of u, v, w must be found. Their integrals will involve arbitrary constants which must be found from the components of external force applied on the bounding surface of the body. Thus, if l, m, n are the direction-cosines of the normal to the surface at a point where the components of external force are P_0, Q_0, R_0, we must have

$$l(\lambda\theta + 2\mu a) + 2\mu(ms_3 + ns_2) = P_0,$$

with two similar equations.

Whenever the strain is pure, the general equations (1), (2), (3) can be expressed in a single equation. For in this case $\nabla^2 u \equiv \frac{d\theta}{dx}$, so that (1) becomes

$$(\lambda + 2\mu)\frac{d\theta}{dx} = -\rho X, \tag{4}$$

and the three equations of equilibrium are precisely the same in forms as those of a perfect fluid. They are, of course, all contained in the single equation

$$(\lambda + 2\mu)\,d\theta = -\rho(Xdx + Ydy + Zdz), \tag{5}$$

the direction in which the differentials are taken being any whatever.

When, in addition, the external forces have a Potential, V, the equation becomes simply

$$(\lambda + 2\mu)\,d\theta = -\rho\,dV, \tag{6}$$

so that if the body is homogeneous,

$$(\lambda + 2\mu)\,\theta + \rho V = \text{constant}. \tag{7}$$

Also in this case $\theta \equiv \nabla^2\phi$ where ϕ is the strain potential, so that ϕ is obtained from the equation

$$\nabla^2\phi + \frac{\rho}{\lambda + 2\mu}V = \text{const}. \tag{8}$$

If cylindrical co-ordinates are used, let the displacements of the co-ordinates z, ζ, ϕ (p. 282) be denoted by w, p, ϵ, respectively.

Then we have

$$\frac{d}{dx} = \cos\phi\frac{d}{d\zeta} - \frac{\sin\phi}{\zeta}\frac{d}{d\phi},$$

$$\frac{d}{dy} = \sin\phi\frac{d}{d\zeta} + \frac{\cos\phi}{\zeta}\frac{d}{d\phi},$$

$$\nabla^2 = \frac{d^2}{dz^2} + \frac{d^2}{d\zeta^2} + \frac{1}{\zeta}\frac{d}{d\zeta} + \frac{1}{\zeta^2}\frac{d^2}{d\phi^2}, \quad \text{(see p. 282)}$$

and since $u = p\cos\phi - \epsilon\zeta\sin\phi$, $v = p\sin\phi + \epsilon\zeta\cos\phi$, we have

$$\theta = \frac{dw}{dz} + \frac{dp}{d\zeta} + \frac{d\epsilon}{d\phi} + \frac{p}{\zeta},$$

and by substituting these new operators in (1), (2), (3) we obtain the differential equations in cylindrical co-ordinates. The cases to which cylindrical and spherical co-ordinates apply with special simplicity, are, however, much better treated by first using the Cartesian equations instead of the general equations for cylindrical and spherical co-ordinates (the latter being very complicated and unwieldy) and then using the cylindrical or spherical operations which are equivalent to ∇^2, &c.—a procedure which is illustrated in the solution of some of the following problems.

Examples.

1. To deduce the traction and torsion displacements of an elliptic cylinder.

For a cylinder or prism having any curve whatever for base, take the axis of the cylinder or prism as axis of z, and any two rectangular axes at the centre of the fixed base as axes of x and y. At any point on the lateral surface let the normal make an angle ψ with the axis of x. Then, since there is no stress whatever on the tangent plane at any point on the lateral surface, we have

$$N_1\cos\psi + T_3\sin\psi = 0, \tag{1}$$

$$T_3\cos\psi + N_2\sin\psi = 0, \tag{2}$$

$$T_2\cos\psi + T_1\sin\psi = 0. \tag{3}$$

These hold equally for torsion and traction. For both cases also we have internal differential equations obtained by putting

$$X = Y = Z = 0$$

in (1), (2), (3), p. 449.

Now, whatever be the shape of the base, the values

$$u = -ax, \quad v = -ay, \quad w = cz, \tag{4}$$

satisfy all these equations if the constants a and c are properly related. For these values given at once $T_1 = T_2 = T_3 = 0$, and to satisfy (1), (2), (3) completely we have only to make $N_1 = N_2 = 0$, i.e., to take

$$\frac{c}{a} = 2\frac{\lambda + \mu}{\mu}. \tag{5}$$

Also the values (4) satisfy the internal differential equations.

Finally, if a force F is applied to the second end of the cylinder

or prism, and if S is the area of this base, $\frac{F}{S}$ is the intensity of stress on this end, which must be equal to N_3, that is,

$$\lambda(c-2a)+2\mu c=\frac{F}{S}. \tag{6}$$

From (5), (6) we have

$$c=\frac{\lambda+\mu}{\lambda^2+2\lambda\mu+2\mu^2}\frac{F}{S};\quad a=\frac{\mu}{\lambda^2+2\lambda\mu+2\mu^2}\frac{F}{2S}.$$

For a 'perfect solid' $\lambda=\mu=\frac{2}{5}E$, and these become

$$c=\frac{F}{ES};\quad a=\frac{F}{4ES},$$

as they ought by Art. 386.

Consider now the torsion of the prism. Assume

$$u=-\tau yz,\quad v=\tau xz, \tag{A}$$

where τ is a constant (evidently the rate of twist), the value of w being undetermined. These give

$$\theta=\frac{dw}{dz},\ T_1=\mu\left(\frac{dw}{dy}+\tau x\right),\ T_2=\mu\left(\frac{dw}{dx}-\tau y\right),\ T_3=0.$$

Hence (1) and (2) require $N_1=N_2=0$, and therefore

$$\frac{dw}{dz}=0. \tag{7}$$

We have then from (3)

$$\left(\frac{dw}{dx}-\tau y\right)\cos\psi+\left(\frac{dw}{dy}+\tau x\right)\sin\psi=0, \tag{8}$$

while the equations of internal equilibrium reduce to $\nabla^2 w=0$, or

$$\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}=0. \tag{9}$$

The equation of the base (or transverse section) being $f(x, y)=0$, we may write (8) in the form

$$\left(\frac{dw}{dx}-\tau y\right)\frac{df}{dx}+\left(\frac{dw}{dy}+\tau x\right)\frac{df}{dy}=0. \tag{10}$$

The equations (9) and (10) hold for the torsion of any prism or cylinder whatever be the nature of its transverse section; and the problem simply reduces to determining w as a function of x and y so as to satisfy these two equations.

Take, in particular, the case of an elliptic cylinder, so that

$$f(x,y)\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1,$$

and (10) becomes

$$b^2\left(\frac{dw}{dx}-\tau y\right)x+a^2\left(\frac{dw}{dy}+\tau x\right)y=0,$$

which is obviously satisfied by $w=-\frac{\tau c^2}{a^2+b^2}\cdot xy$, which also satisfies (9).

Now let G be the moment of the twisting couple applied at the free end, the base being supposed fixed, or free and also acted upon by a couple $-G$. Consider the equilibrium of the portion of the cylinder included between the free end and any transverse section of the cylinder. Then the sum of the moments of the stresses on this section about the axis of the cylinder must be equal to G. Let x, y be the co-ordinates of any point on the section and dS the element of area at the point. Then the moment of the stress is

$$(xT_1 - yT_2)\,dS,$$

so that
$$\frac{2\tau}{a^2+b^2}\int(a^2y^2+b^2x^2)\,dS = \frac{G}{\mu},$$

or
$$\tau = \frac{a^2+b^2}{\pi a^3b^3}\frac{G}{\mu}.$$

Therefore the strain components in a twisted elliptic cylinder are

$$u = -\frac{a^2+b^2}{\pi a^3b^3}\frac{G}{\mu}\cdot yz;\quad v = \frac{a^2+b^2}{\pi a^3b^3}\frac{G}{\mu}\cdot xz;\quad w = -\frac{e^2}{\pi a^3b^3}\frac{G}{\mu}\cdot xy. \tag{11}$$

Hence in a twisted circular cylinder the transverse sections remain plane; but they do not remain so in an elliptic cylinder.

It will be easily found that the resultant shearing stress, $\sqrt{T_1^2+T_2^2}$, in the transverse section, at any point on the surface is

$$\frac{2G}{\pi ab}\cdot\frac{1}{p},$$

where p is the central perpendicular on the tangent line to the ellipse, so that this stress is greatest at the extremity of the minor axis of the section.

The torsional rigidity of a solid elliptic cylinder is

$$\frac{\pi a^3b^3}{a^2+b^2}\cdot\mu.$$

2. Determine the strains in a twisted rectangular prism.

Let the sides of the rectangle be $2a$, $2b$. Then the equations to be satisfied are

$$\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2} = 0, \quad \text{everywhere}; \tag{1}$$

$$\frac{dw}{dx}-\tau y = 0, \quad \text{when } x = \pm a; \tag{2}$$

$$\frac{dw}{dy}+\tau x = 0, \quad \text{when } y = \pm b; \tag{3}$$

the second and third being the equivalent of (10) of last example.

For simplicity, assume $w = \tau xy + w'$. Then $\dfrac{d^2w'}{dx^2}+\dfrac{d^2w'}{dy^2} = 0$, $\dfrac{dw'}{dx} = 0$ when $x = \pm a$, and $\dfrac{dw'}{dy} = -2\tau x$ when $y = \pm b$.

To satisfy the first, assume

$$w' = A\,(e^{my}-e^{-my})\sin mx.$$

Then the second equation requires $ma = (2n+1)\frac{\pi}{2}$, so that

$$w' = A_n\left\{e^{(2n+1)\frac{\pi y}{2a}} - e^{-(2n+1)\frac{\pi y}{2a}}\right\}\sin(2n+1)\frac{\pi x}{2a}, \qquad (4)$$

and the complete value of w' will be found by giving n all values from 0 to ∞, and adding all the terms together.

The last equation of condition for w' gives the following equation, which is to be an identity (as a Fourier development),

$$-2\tau x = \frac{\pi}{a}\Sigma(2n+1)A_n\cosh(2n+1)\frac{\pi b}{2a}\sin(2n+1)\frac{\pi x}{2a}, \qquad (5)$$

in which the hyperbolic cosine is used for shortness, as in the formulæ

$$\tfrac{1}{2}(e^{\phi}+e^{-\phi}) = \cosh\phi; \quad \tfrac{1}{2}(e^{\phi}-e^{-\phi}) = \sinh\phi.$$

To determine A_n, multiply both sides of (5) by $\sin(2n+1)\frac{\pi x}{2a}dx$, and integrate between $x=0$ and $x=a$, exactly as in the development of a Fourier series. Then we have all the terms on the right vanishing except A_n; and hence

$$A_n = \frac{16\tau a^2(-1)^{n+1}}{(2n+1)^3\pi^3\cosh(2n+1)\frac{\pi b}{2a}}. \qquad (6)$$

Substituting this in (4), we have

$$\frac{w}{\tau} = xy + \frac{32a^2}{\pi^3}\sum_0^\infty(-1)^{n+1}\frac{\sinh(2n+1)\frac{\pi y}{2a}\cdot\sin(2n+1)\frac{\pi x}{2a}}{(2n+1)^3\cosh(2n+1)\frac{\pi b}{2a}}, \qquad (7)$$

the suffixes indicating that n is to receive all values from 0 to ∞.

The value of τ is to be obtained from the magnitude of the externally applied couple, G, producing the torsion. Thus

$$\int(xT_1 - yT_2)\,dS = G, \qquad (8)$$

in which the integral expresses the sum of the moments of the stresses on any transverse section about the axis of the prism.

Now $T_1 = \mu(\frac{dw}{dy}+\tau x)$ and $T_2 = \mu(\frac{dw}{dx}-\tau y)$; and if for brevity we write the value of w in (7) in the form

$$\frac{w}{\tau} = xy + \Sigma B_n\sinh my\,.\,\sin mx,$$

we have

$$\frac{G}{\mu\tau} = 2\int x^2dS - \Sigma mB_n\int\int(y\sinh my\cos mx - x\sin mx\cosh my)\,dxdy,$$

the limits of x being $\pm a$, and those of y being $\pm b$.

The double integral is easily found to be

$$4(-1)^nB_n\left(\frac{b}{m}\cosh mb - \frac{2}{m^2}\sinh mb\right),$$

while $\int x^2 dS = \frac{4}{3}a^3 b$. Hence substituting for m and B_n, we have

$$\frac{G}{\mu\tau} = \frac{8}{3}a^3 b + \frac{256 a^3 b}{\pi^4}\sum_0^\infty \frac{1}{(2n+1)^4} - \frac{32^2 a^4}{\pi^5}\sum_0^\infty \frac{\tanh(2n+1)\frac{\pi b}{2a}}{(2n+1)^5}.$$

But it is a known trigonometrical result that

$$\sum_0^\infty \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

Hence finally,

$$\frac{1}{\tau} = \frac{16 a^3 b\mu}{G}\left\{\frac{1}{3} - \frac{64}{\pi^5}\frac{a}{b}\sum_0^\infty \frac{\tanh(2n+1)\frac{\pi b}{2a}}{(2n+1)^5}\right\}, \tag{9}$$

and we have, therefore, the components of strain, (A), p. 452, at every point of the prism.

The right-hand side of (9) multiplied by G is the torsional rigidity of a solid rectangular cylinder.

These results are, of course, due to Saint-Venant (see his edition of Clebsch, pp. 214, &c.) who notices, in particular, the case in which the transverse section is a very elongated rectangle ($\frac{a}{b}$ very small), and that in which it is a square.

For the first case, we get $\frac{G}{\tau} = 16a^3 b\mu(\frac{1}{3} - \cdot 21 \times \frac{a}{b})$, and for the square

$$\frac{G}{\tau} = \cdot 843462 \times I,$$

where $I = \frac{8}{3}\pi a^4 =$ the moment of inertia of the transverse section about the axis of the prism. Contrast these with the result (p. 442) for a circular cylinder.

3. To determine the strain and stress at any point of a spherical shell of any thickness, the outer and inner surfaces of which are each subject to uniformly distributed pressure.

The displacement at every point in the shell is necessarily radial, by symmetry; so that, if r is the central radius drawn to any point of its substance,

$$u = xf(r), \quad v = yf(r), \quad w = zf(r), \tag{α}$$

where $f(r)$ is some unknown function of r. Hence the strain is pure, and if ϕ is its potential, $\theta = \nabla^2\phi$. But by (6), p. 450,

$$\theta = A = \text{a constant,}$$

there being no external bodily forces. Also

$$d\phi = udx + vdy + wdz = rf(r)\,dr,$$

so that ϕ is a function of r only. Hence $\nabla^2\phi \equiv \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr})$, by p. 281; and we have

$$\frac{d}{dr}(r^2\frac{d\phi}{dr}) = Ar^2,$$

$$\therefore\ \frac{d\phi}{dr} = \frac{1}{3}Ar + \frac{B}{r^2}, \tag{β}$$

where B is another constant.

Now the normal stress, P, on a plane perpendicular to the radius vector r is given by the equation

$$P = \lambda\theta + 2\mu\frac{d^2\phi}{dr^2},$$

$$= A(\lambda + \tfrac{2}{3}\mu) - \frac{B}{r^3}; \qquad (\gamma)$$

and if p, p' are the intensities of pressure at the outer and inner surfaces of the shell, whose radii are a, a', respectively, the value of P in (γ) is $-p$ when $r = a$, and its value is $-p'$ when $r = a'$. From these A and B are then determined, and thence the stress on any assigned element plane.

We have $f(r) = \frac{1}{3}A + \frac{B}{r^3}$, from (β), and thence the values of u, v, w in (α).

For a plane occupying any position

$$N_1 = (\lambda + \tfrac{2}{3}\mu)A + \frac{2\mu B}{r^3}\left(1 - \frac{3x^2}{r^2}\right),$$

$$T_3 = -\frac{6\mu Bxy}{r^5},$$

with similar values of the other components.

To get the tearing stress at any point, take any point in the section made by the plane xy. The only stress on this plane is N_3, which is

$$(\lambda + \tfrac{2}{3}\mu)A + \frac{2\mu B}{r^3}.$$

In this expression, substituting the values of A and B as before determined, and denoting the intensity of the tearing stress by Q, we have

$$Q\left(\frac{1}{a'^3} - \frac{1}{a^3}\right) = \frac{p'}{a^3} - \frac{p}{a'^3} + \frac{p' - p}{2r^3},$$

which shows that if $p' > p$, the tendency to tear is a maximum at the inner surface.

This expression, being quite independent of the moduli λ and μ, is the same whatever be the isotropic substance of which the shell is made. The displacements and strains, however, do depend upon λ and μ. It is not difficult to give a common sense reason for this.

4. To calculate the strain and stress at any point of a sphere which when free of strain is homogeneous, the strain being produced by its self-attraction alone *.

Let ρ be the density of the sphere when homogeneous, and $-\theta$ the cubical compression at any point, P, of the heterogeneous sphere. The latter sphere will consist of spherical shells each of constant density $\rho(1-\theta)$. Let r' be the radius of any one of these shells

* For the sake of the exemplification of some points of general theory, I retain the solution of this question which I gave in the third edition. Professor Williamson has kindly sent me the simpler solution which follows the present one.

and θ' the value of θ on the shell. Then if X is the force per unit mass produced by the whole sphere at P, we have

$$X = -\frac{4\pi\gamma\rho x}{r^3}\int_0^r (1-\theta') r'^2 dr, \tag{1}$$

with similar values of Y and Z, where r is the distance of P from the centre, O, of the sphere. We may put this into the form

$$X = -\tfrac{4}{3}\pi\gamma\rho x\left(1-\frac{3}{r^3}\int_0^r \theta' r'^2 dr'\right). \tag{2}$$

In the differential equations for the components of strain (Art. 395) we have ρX, where ρ is the density at P, i.e. in this case $\rho(1-\theta)$. Hence (1) of Art. 395 becomes

$$(\lambda+\mu)\frac{d\theta}{dx} + \mu\nabla^2 u = \tfrac{4}{3}\pi\gamma\rho^2 x\left(1-\theta-\frac{3}{r^3}\int_0^r \theta' r'^2 dr'\right), \tag{3}$$

in which we have neglected the product of θ and the integral term, since θ is everywhere supposed small.

Differentiating (3) with respect to x, denoting, for simplicity, the definite integral by I, and observing that $\dfrac{d}{dx} = \dfrac{x}{r}\dfrac{d}{dr}$, we have

$$\begin{aligned}(\lambda+\mu)&\frac{d^2\theta}{dx^2} + \mu\frac{d}{dx}\nabla^2 u \\ &= \tfrac{4}{3}\pi\gamma\rho^2\left\{1-\left(1+\frac{3x^2}{r^2}\right)\theta - \frac{x^2}{r}\frac{d\theta}{dr} + \left(\frac{9x^2}{r^5}-\frac{3}{r^3}\right)I\right\}.\end{aligned} \tag{4}$$

Adding to this the similar results in v, w, and observing that

$$\frac{d}{dx}\nabla^2 u + \frac{d}{dy}\nabla^2 v + \frac{d}{dz}\nabla^2 w = \nabla^2\theta, \tag{5}$$

we have
$$(\lambda+2\mu)\nabla^2\theta = 4\pi\gamma\rho^2\left(1-2\theta-\tfrac{1}{3}r\frac{d\theta}{dr}\right). \tag{6}$$

Expressing the operator ∇^2 in terms of r (p. 281), and observing that θ is a function of r only, this equation becomes

$$c^2\frac{d^2(r\theta)}{dr^2} + r\frac{d(r\theta)}{dr} + 5r\theta - 3r = 0, \tag{7}$$

where $\dfrac{\lambda+2\mu}{4\pi\gamma\rho^2} = \dfrac{c^2}{3}$, the constant c being a linear magnitude, since λ and μ are each of the form $\dfrac{\text{force}}{(\text{length})^2}$, and $\gamma\rho^2$ is obviously of the same nature.

By assuming $r\theta = \phi + \dfrac{r}{2}$, equation (7) becomes

$$c^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + 5\phi = 0. \tag{8}$$

Now for any solid body c is an enormously great magnitude, so that if we can determine ϕ from (8) as a series proceeding by inverse powers of c, it will be sufficient to take the first two terms.

At the centre of the sphere the value of ϕ is zero, while $\frac{d\phi}{dr}$ at this point is unknown. Let $\left(\frac{d\phi}{dr}\right) = A$, and determine ϕ by the series

$$\phi = \phi_0 + r\left(\frac{d\phi}{dr}\right)_0 + \frac{r^2}{1.2}\left(\frac{d^2\phi}{dr^2}\right)_0 + \dots.$$

Then $\phi_0 = 0$; $\left(\frac{d\phi}{dr}\right)_0 = A$; $\left(\frac{d^2\phi}{dr^2}\right)_0 = 0$; $\left(\frac{d^3\phi}{dr^3}\right)_0 = -\frac{6A}{c^2}$; &c.

Hence, neglecting $\frac{1}{c^4}$ and higher powers of $\frac{1}{c}$, we have

$$\phi = A\left(r - \frac{r^3}{c^2}\right),$$

supposing, as we do, that no values of r within the limits of integration make $\frac{r}{c}$ other than a small fraction. Thus

$$\theta = \tfrac{1}{3} + A\left(1 - \frac{r^2}{c^2}\right). \tag{9}$$

Substitute this value of θ in (3), and we have, by putting

$$\tfrac{4}{3}\pi\gamma\rho^2 = \frac{\lambda + 2\mu}{c^2}$$

in the right-hand side and neglecting small quantities beyond $\frac{1}{c^2}$,

$$\nabla^2 u = -\frac{2A}{c^2}x, \tag{10}$$

which shows, by (β), p. 280, that u may be calculated by regarding it as the Potential at P produced by matter filling the sphere, the density of this matter at each point, P', being given by the equation

$$\rho' = \frac{Ax'}{2\pi\gamma c^2}. \tag{11}$$

To find the Potential of this at P, consider separately the part, u_1, due to the sphere of radius OP, and the part, u_2, due to the shell contained between this and the surface of the whole sphere, whose radius $= R$, suppose.

We have then, in the ordinary notation,

$$u_1 = \frac{A}{2\pi c^2}\int_0^r\int_0^\pi\int_0^{2\pi}\frac{x' r'^2 \sin\theta' dr' d\theta' d\phi'}{PP'}, \tag{12}$$

in which we develop $\frac{1}{PP'}$ in a series of Laplacians ascending by powers of $\frac{r'}{r}$, since $r' < r$. Putting $x' = r'\cos\theta' = r'\mu'$, and remembering that, by Art. 351, since μ' is a surface harmonic of

the first degree, the only term in the series that will not vanish in the double integration in μ' and ϕ', is that in L_1, we have

$$u_1 = \frac{A}{2\pi c^2}\int_0^r \int_{-1}^1 \int_0^{2\pi} \frac{L_1}{r^2} r'^4 dr'.\mu' d\mu' d\phi'$$

$$= \frac{Ar^3}{10\pi c^2}\int_{-1}^1 \int_0^{2\pi} \mu' L_1 d\mu' d\phi'$$

$$= \frac{2A}{15\pi c^2} r^2 x, \text{ by } (\eta), \text{ p. } 354. \tag{13}$$

To get u_2 we must develop $\frac{1}{PP'}$ in (12) in ascending powers of $\frac{r}{r'}$. As before, the only term of the Laplacian series that will not vanish is that in L_1; also the limits of r' are r and R. Hence we find

$$u_2 = \frac{A}{3c^2}(R^2 - r^2)x. \tag{14}$$

Adding (13) and (14), we have

$$u = \frac{Ax}{3c^2}(R^2 - \tfrac{3}{5}r^2), \tag{15}$$

with similar values of v and w.

Now with regard to the physical and analytical conditions to be satisfied by u, v, w, it is obvious, *à priori*, that their values at the centre of the sphere must vanish; that they must make the displacement at every point, P, radial (i.e. along PO); that they must give zero value to the stress at every point on the bounding surface, since we have supposed the sphere to be uninfluenced from without; and that they must satisfy the identity

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \equiv \theta. \tag{A}$$

Observe particularly that we have not in our investigation made use of (A) but merely of a more general relation (5), which will be equally well satisfied if any three functions, χ_1, χ_2, χ_3, each satisfying the equation $\nabla^2\phi = 0$, are added to u, v, w.

Now it will be found that (15) and the two analogous values of v and w satisfy (A), and all the other conditions just enunciated except that of making the components $lN_1 + mT_3 + nT_2$, &c., vanish at the surface.

Let us add an arbitrary term Bx to (15), and try whether all the required conditions are satisfied by the values

$$u = \frac{Ax}{3c^2}(R^2 - \tfrac{3}{5}r^2) + Bx, \tag{16}$$

$$v = \frac{Ay}{3c^2}(R^2 - \tfrac{3}{5}r^2) + By, \tag{17}$$

$$w = \frac{Az}{3c^2}(R^2 - \tfrac{3}{5}r^2) + Bz. \tag{18}$$

Satisfying (A), we have from (9)

$$\left(1-\frac{R^2}{c^2}\right)A-3B+\tfrac{1}{2}=0. \tag{19}$$

Again, $N_1=\lambda\left\{\tfrac{1}{2}+A\left(1-\frac{r^2}{c^2}\right)\right\}+2\mu\left\{B+\frac{A}{3c^2}\left(R^2-\tfrac{3}{5}r^2-\tfrac{6}{5}x^2\right)\right\}$,

and $$T_3=-\tfrac{4}{5}\frac{\mu A}{c^2}xy;\quad T_2=-\tfrac{4}{5}\frac{\mu A}{c^2}zx,$$

with similar values of the other stress components; and since the stress on any tangent plane is zero, we must have

$$N_1x+T_3y+T_2z=0 \tag{20}$$

when $r=R$, two similar equations also holding.

Now we find that x disappears from (20), which reduces to

$$\left\{\lambda\left(1-\frac{R^2}{c^2}\right)-\frac{8\mu}{15}\frac{R^2}{c^2}\right\}A+2\mu B+\tfrac{1}{2}\lambda=0, \tag{21}$$

the other two equations giving also this result.

Hence all the conditions of the problem can be satisfied by the values (16), &c., the constants A and B being determined from (19) and (21).

We easily find, to the order of approximation adopted,

$$A=-\tfrac{1}{2}\left\{1+\tfrac{2}{5}\cdot\frac{5\lambda+6\mu}{3\lambda+2\mu}\cdot\frac{R^2}{c^2}\right\}, \tag{22}$$

$$B=-\frac{15\lambda+14\mu}{15(3\lambda+2\mu)}\cdot\frac{R^2}{c^2}, \tag{23}$$

$$\theta=\frac{1}{2c^2}\left\{r^2-\tfrac{3}{5}\cdot\frac{5\lambda+6\mu}{3\lambda+2\mu}\cdot R^2\right\}. \tag{24}$$

As a verification, suppose that the sphere was a mass of incompressible fluid. Then $\lambda=\infty$, $\theta=0$, but $\lambda\theta=-p$, where p is the intensity of pressure at any point (Art. 391); and (24) gives

$$p=\tfrac{2}{3}\pi\gamma\rho^2(R^2-r^2),$$

which can be deduced at once from the fundamental equations of Hydrostatics.

Professor Williamson solves this problem by the obvious principle that the strain at every point is pure, and by applying at once equation (6), p. 450. Thus we have

$$(\lambda+2\mu)\frac{d\theta}{dr}+\rho(1-\theta)\frac{dV}{dr}=0.$$

But $$\frac{dV}{dr}=-\frac{4\pi\gamma\rho}{r^2}\int_0^r(1-\theta')r'^2dr',$$

since $-\dfrac{dV}{dr}$ is the attraction intensity. Hence

$$c^2r^2\frac{d\theta}{dr} = (1-\theta)r^3 - 3(1-\theta)\int_0^r \theta' r'^2 dr',$$

$$\therefore \quad c^2\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = 3r^2(1-2\theta) - r^2\frac{d\theta}{dr}, \tag{25}$$

neglecting a term of the second order in θ.

To integrate this, let $r = c\xi$ where ξ is very small, since c is very great compared with r. Hence

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 3(1-2\theta)\xi^2 - \xi^2\frac{d\theta}{d\xi}.$$

Now this equation is unaltered if we change the sign of ξ; therefore θ is a function of ξ^2, i.e. of $\dfrac{r^2}{c^2}$, and we may, therefore, assume

$$\theta = m + n\frac{r^2}{c^2},$$

where m and n are undetermined constants. Substituting this in (25), we get $m+n=\frac{1}{2}$, or $n=\frac{1}{2}$, since m is necessarily very small.

Hence, since $\nabla^2\phi \equiv \theta$, where ϕ is the strain potential,

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = mr^2 + \frac{1}{2}\frac{r^4}{c^2},$$

$$\therefore \quad \frac{d\phi}{dr} = \frac{1}{3}mr + \frac{1}{10}\frac{r^3}{c^2},$$

the constant of integration being rejected since, if it existed, the radial displacement $\dfrac{d\phi}{dr}$ would be ∞ at the centre.

Now the principal stresses, P, Q, Q, at any point are given by the equations

$$\left.\begin{aligned} P &= \lambda\theta + 2\mu\frac{d^2\phi}{dr^2}, \\ Q &= \lambda\theta + 2\mu\frac{1}{r}\frac{d\phi}{dr}; \end{aligned}\right\} \tag{a}$$

and expressing the fact that the stress vanishes at the surface of the sphere, we have

$$m = -\frac{3}{10}\cdot\frac{5\lambda+6\mu}{3\lambda+2\mu}\cdot\frac{R^2}{c^2},$$

which gives θ as in (24), and also gives

$$\phi = \frac{1}{40}\frac{r^4}{c^2} - \frac{1}{20}\frac{R^2r^2}{c^2}\cdot\frac{5\lambda+6\mu}{3\lambda+2\mu},$$

from which, of course, the strain displacements follow by differentiation.

5. Determine the strain and stress at any point in a Planetary Crust of uniform thickness and density, surrounding a uniform spherical nucleus.

Let ρ be the density of the crust, a its outer and b its inner radius, and $(n+1)\rho$ the density of the nucleus.

In this case also the displacement at every point is radial and the strain pure.

Hence, if V is the attraction potential at any point,

$$(\lambda+2\mu)\frac{d\theta}{dr}=-\rho\frac{dV}{dr}.$$

But $-\frac{dV}{dr}$ is the attraction intensity, and is therefore

$$\tfrac{4}{3}\pi\gamma\rho\left(r+\frac{nb^3}{r^2}\right).$$

Hence
$$c^2\frac{d\theta}{dr}=r+\frac{nb^3}{r^2},$$

with the same meaning of c^2 as in the last example. If ϕ is the strain potential, $\nabla^2\phi=\theta$; hence

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)=A-\frac{nb^3}{c^2r}+\frac{r^2}{2c^2},$$

$$\therefore\quad \frac{d\phi}{dr}=\frac{r^3}{10c^2}-\frac{nb^3}{2c^2}+\tfrac{1}{3}Ar+\frac{B}{r^2},$$

where A and B are constants. Now the radial stress is $\lambda\theta+2\mu\frac{d^2\phi}{dr^2}$, and if the outer and inner surfaces of the shell are subject to pressure intensities p, p', we have

$$-p=(\lambda+\tfrac{2}{3}\mu)A-\frac{4\mu}{a^3}B+\frac{a^2}{2c^2}\left(\lambda-2n\lambda\frac{b^3}{a^3}+\tfrac{6}{5}\mu\right),$$

$$-p'=(\lambda+\tfrac{2}{3}\mu)A-\frac{4\mu}{b^3}B+\frac{b^2}{2c^2}\left(\lambda-2n\lambda+\tfrac{6}{5}\mu\right),$$

from which A and B are known.

6. In the case of a spherical envelope of small thickness subject to uniform intensities of pressure inside and outside, if T is the tearing stress per unit length perpendicular to a meridian, show from elementary principles that

$$T=\tfrac{1}{2}P\cdot r,$$

where r is the radius and P the excess of outward intensity of pressure.

Deduce this result from (14), example 3.

7. A spherical shell of copper, whose internal radius is 1 decimètre and thickness $\frac{1}{2}$ centimètre, is filled with a gas at an intensity of pressure of 20 atmospheres, the outside being subject to atmospheric pressure; find the radial displacement of a point on the inner surface, being given the following data:—

Modulus of compression for copper $=16{\cdot}84\times10^{11}$ dynes per sq. cm.; modulus of shear $=4{\cdot}47\times10^{11}$; 1 atmosphere $=1{\cdot}014\times10^{6}$ dynes per sq. cm.

Ans. The displacement is $\cdot010146$ millimètres.

396.] **Constants of Elasticity.** Abandoning now the special case of an isotropic body, let us consider the values of the stress components N_1, N_2, ... produced in any body by a strain whose components are a, b, c, $2s_1$, $2s_2$, $2s_3$. Assuming the stress components to be each a linear function of the components of strain, it is clear that if any two states of strain were superposed, their corresponding components of stress would be superposed, and there would, therefore, be a superposition and independence of stress and strain effects in such a body; so that, for example, two separate causes of equal states of stress would, if superposed, produce both double the stress and double the strain. M. de Saint-Venant justifies the assumption of this linear relation in general (see p. 40 of his edition of Clebsch). If the linear relation did not hold, there would be no superposition of effects due to two or more identical causes, of such a nature that, while they all act, each produces the same effect as if the other causes were non-existent. As we have in view the luminiferous ether and the possible propagation of gravitation by means of stress in an ethereal medium, we shall assume this linearity, as perfectly justified by the observed independence and superposition of luminous and gravitative effects.

Thus we shall have the stress components expressed by equations of the form

$$N_1 = n_1 a + p_3 b + p_2 c + t_1 s_1 + t_1' s_2 + t_1'' s_3, \tag{1}$$

where n_1, p_3, ... are constant quantities, each of the nature of force per unit area, depending on the nature of the strained medium. Thus N_1 involves 6 constants of elasticity, and each of the other five components of stress involves also 6 constants, so that, apparently, the stress components at any point depend on 36 constants of elasticity.

But the stress components have been proved (Art. 384) to be related in every medium without exception in such a manner that

$$N_1 da + N_2 db + N_3 dc + 2T_1 ds_1 + 2T_2 ds_2 + 2T_3 ds_3 \equiv d\phi, \tag{2}$$

where ϕ is a function—evidently a homogeneous quadratic function—of the strain components. Hence

$$N_1 = \frac{d\phi}{da}, \quad N_2 = \frac{d\phi}{db}, \dots \quad 2T_1 = \frac{d\phi}{ds_1}. \tag{3}$$

Hence $$\frac{dN_1}{db} \equiv \frac{dN_2}{da}, \dots \quad \frac{dN_1}{ds_1} \equiv \frac{d(2T_1)}{da}, \tag{4}$$

i.e. the coefficient of b in N_1 is the same as that of a in N_2, &c.; so that we have 15 identities among the 36 coefficients.

Hence, *for any medium whatever, the principle of the conservation of energy shows that the greatest number of independent coefficients of elasticity is* 21.

We have, therefore, the following table:

$$N_1 = n_1\ a + p_3 b + p_2 c + t_1 s_1 + t_1' s_2 + t_1'' s_3, \tag{5}$$
$$N_2 = p_3\ a + n_2 b + p_1 c + t_2 s_1 + t_2' s_2 + t_2'' s_3, \tag{6}$$
$$N_3 = p_2\ a + p_1 b + n_3 c + t_3 s_1 + t_3' s_2 + t_3'' s_3, \tag{7}$$
$$T_1 = t_1\ a + t_2 b + t_3 c + v_1 s_1 + q_3 s_2 + q_2 s_3, \tag{8}$$
$$T_2 = t_1'\ a + t_2' b + t_3' c + q_3 s_1 + v_2 s_2 + q_1 s_3, \tag{9}$$
$$T_3 = t_1'' a + t_2'' b + t_3'' c + q_2 s_1 + q_1 s_2 + v_3 s_3. \tag{10}$$

In the view of English physicists, no further reduction below these 21 can be effected in the coefficients, unless the medium possesses some one or more kinds of structural symmetry. M. de Saint-Venant, however, strongly maintains that, without any such structural symmetry, a further reduction to 15 coefficients is always possible; and he bases this reduction on the fact (denied by Green) that the action between any two molecules is a function solely of the distance between these two molecules, depending in no way on the presence of neighbouring molecules.* The consideration of this further reduction we postpone for a moment, and we proceed to find the simplifications introduced by the supposition of various kinds of symmetry of structure in the body.

Symmetry with respect to a plane. Suppose that all along a line perpendicular to a plane xy the structure of the body is constant; then the constants in (5),... will be each a function of x and y only (not involving z).

Now, in general, if we produce the axis of z backwards through the origin, and take this production as the axis of z, retaining the axes of x and y, the new co-ordinates and displacements at a given point P will be expressed in terms of their old values by the equations

$$x' = x,\quad y' = y,\quad z' = -z,\quad u' = u,\quad v' = v,\quad w' = -w,$$

so that c remains unaltered, and s_1 and s_2 both change sign.

---

* See Saint-Venant's Clebsch, p. 65. M. de Saint-Venant is strong in his ridicule of the attempted generality which makes the action between a given pair of molecules dependent on the proximity of other molecules.

Also, from the general formulæ of transformation (Art. 379) or otherwise, we know that N_3 remains unaltered, while T_1 and T_2 change sign.

Introducing these changes into (5), since N_1 remains unaltered, while s_1 and s_2 change sign, it follows that t_1 and t_1' must change sign when $-z$ is put for z. But, by the structural symmetry supposed, neither t_1 nor t_1' is a function of z. Hence they must both vanish.

In this way, considering all the other equations, we have the following conditions for structural symmetry with respect to the plane xy:

$$t_1 = t_1' = t_2 = t_2' = t_3 = t_3' = q_2 = q_1 = 0, \tag{11}$$

so that the values for this case are

$$N_1 = n_1 a + p_3 b + p_2 c + t_1'' s_3, \tag{12}$$
$$N_2 = p_3 a + n_2 b + p_1 c + t_2'' s_3, \tag{13}$$
$$N_3 = p_2 a + p_1 b + n_3 c + t_3'' s_3, \tag{14}$$
$$T_1 = v_1 s_1 + q_3 s_2, \tag{15}$$
$$T_2 = q_3 s_1 + v_2 s_2, \tag{16}$$
$$T_3 = t_1'' a + t_2'' b + t_3'' c + v_3 s_3. \tag{17}$$

Two planes of Symmetry. Let the structure be also constant along every line perpendicular to the plane xz, while varying from one such line to another. In this case the constants are all functions of x and z only, and therefore (from combination with the previous symmetry) of x only. Producing the axis of y backwards, we have a, b, c, s_2 unaltered, and s_1, s_3 altered. Hence

$$t_1'' = t_2'' = t_3'' = q_3 = 0, \tag{18}$$

and the equations now become

$$\left.\begin{aligned} N_1 &= n_1 a + p_3 b + p_2 c; & T_1 &= v_1 s_1, \\ N_2 &= p_3 a + n_2 b + p_1 c; & T_2 &= v_2 s_2, \\ N_3 &= p_2 a + p_1 b + n_3 c; & T_3 &= v_3 s_3. \end{aligned}\right\} \tag{19}$$

Now these equations show evidently that there must be structural symmetry with respect to the third co-ordinate plane, yz. Hence if a medium is structurally symmetrical with regard to two rectangular planes, it is also structurally symmetrical with respect to a third, and its constants of elasticity are 9 in number.

Structural Symmetry round an axis. Suppose the constants to be the same at all points of a cylinder having the axis of z for its axis, while varying from one such cylinder to another. They

are then merely functions of ζ and not of z (Art. 329), and if the planes zx and zy are rotated through any angle, ϕ, the values of the constants at a given point P remain unaltered, while the stress and strain components N_1', ... a', ... with reference to the new axes are easily expressed in terms of N_1, ... a, ... with reference to the old axes.

Let ϕ be a very small angle, so that the new co-ordinates, x', y', of P are given by the equations $x' = x + \phi y$, $y' = y - \phi x$, and the direction-cosines of the axes of x', y', z with reference to the old axes are

$$(1, \phi, 0), \quad (-\phi, 1, 0), \quad (0, 0, 1).$$

Hence we have

$$\begin{array}{lll} a' = a + 2\phi s_3; & b' = b - 2\phi s_3; & c' = c; \\ s_1' = s_1 - \phi s_2; & s_2' = s_2 + \phi s_1; & s_3' = (b-a)\phi + s_3; \\ N_1' = N_1 + 2\phi T_3; & N_2' = N_2 - 2\phi T_3; & N_3' = N_3; \\ T_1' = T_1 - \phi T_2; & T_2' = T_2 + \phi T_1; & T_3' = (N_2 - N_1)\phi + T_3. \end{array}$$

Expressing that these hold for all values of ϕ, we find that all the t's must vanish, as likewise all the q's, and the remaining equations are

$$\nu_1 = \nu_2; \quad p_1 = p_2; \quad \nu_3 = n_1 - p_3, \quad \nu_3 = n_2 - p_3; \qquad (20)$$

which give $n_1 = n_2$.

Hence for symmetry of structure round the axis of z,

$$\left.\begin{array}{ll} N_1 = na + (n-\nu')b + pc; & T_1 = \nu s_1, \\ N_2 = (n-\nu')a + nb + pc; & T_2 = \nu s_2, \\ N_3 = pa + pb + n'c \quad ; & T_3 = \nu' s_3, \end{array}\right\} \qquad (21)$$

which show that 5 distinct constants exist in this case.

Structural Symmetry round two lines. If, in addition, there is symmetry of structure round the axis of y, the group of equations (21) show, without fresh calculation, that we must have $\nu' = \nu$; $p = n - \nu'$; so that we have

$$\left.\begin{array}{ll} N_1 = (n-\nu)\theta + \nu a; & T_1 = \nu s_1, \\ N_2 = (n-\nu)\theta + \nu b; & T_2 = \nu s_2, \\ N_3 = (n-\nu)\theta + \nu c; & T_3 = \nu s_3, \end{array}\right\} \qquad (22)$$

which show that there is structural symmetry round all lines— i.e. there is complete isotropy; and these are identical with the values (p. 435) hitherto so often used for an isotropic medium, $n-\nu$ being λ and $\nu = 2\mu$.

M. de Saint-Venant's further reduction. The 21 coefficients which result from the sole assumption of the principle of con-

servation of energy, or existence of a Potential function, are, for all media, reduced by Saint-Venant to 15 by the following reasoning. Consider any molecule, P, and a neighbouring molecule, Q. The force exerted between them depends solely on the change in the distance between them effected by strain and acts in the line joining them. Hence the x-component of the force between them caused by a stretch, b, parallel to the axis of y is equal to the y-component of the force between them caused by a shear s_3 if $2s_3 = b$.

For if in the first strain Δr is the change in the length PQ, and if the force between the molecules $= A.\Delta r$, where A is a constant, we have the force $= \frac{A}{r}\eta^2 b$, since (p. 379) we have $\Delta\eta = b\eta$, and $\Delta r = \frac{\eta}{r}\Delta\eta$. The x-component of this is $\frac{Ab}{r^2}\xi\eta^2$.

In the second strain $\Delta\xi = s_3\eta$, $\Delta\eta = s_3\xi$, therefore if $2s_3 = b$, $\Delta r = \frac{b}{r}\xi\eta$, and the y-component of their force $= \frac{Ab}{r^2}\xi\eta^2$, as before.

Considering all the molecules now in any element plane at P, since this relation holds between each of them and all their surrounding molecules, we see that the x-component of stress on this plane produced by a stretch, b, parallel to y is equal to the y-component of the stress on the plane produced by the shear $2s_3$; in other words—*if P, Q, R are the components of the stress on any element plane whatever, the coefficient of b in P is equal to the coefficient of $2s_3$ in Q.* Similarly, the coefficient of a in $Q =$ coefficient of $2s_3$ in P; coefficient of c in $P =$ coefficient of $2s_2$ in R; coefficient of a in $R =$ coefficient of $2s_2$ in P; coefficient of b in $R =$ coefficient of $2s_1$ in Q; and coefficient of c in $Q =$ coefficient of $2s_1$ in R.

Expressing that these equalities hold whatever be the direction-cosines of the element plane, we have only 15 distinct coefficients in the general equations (5) ... (10).

Applying these principles to the case of an isotropic medium, as expressed by (22), we have simply

$$n - v = \tfrac{1}{2}v,$$

or, if 2μ is used for v, in accordance with Lamé's notation,

$$\lambda = \mu,$$

the result which M. de Saint-Venant maintains for all hard fine-grained isotropic bodies.

397.] **Propagation of Gravitation.** Clerk Maxwell, in Chap. V, Vol. i, of his *Electricity and Magnetism*, has given a remarkable theorem, the object of which is to show that all actions exerted between two material systems, M_1, M_2, according to the law of inverse square of distance, can be produced by a distribution of stress over closed surfaces in the medium, and he has assigned the components of this stress at every point.

We proceed to explain and to examine this theory.

Let the two influencing systems be M_1 and M_2 in Fig. 288, p. 334, and let any closed surface whatever, S (not that represented in the figure), be drawn having M_1 inside it and M_2 outside it.

Now consider the x-component of attraction produced on M_1 by M_2. Let P be any point in M_1; let ρ be the density at P, and V_2 the Potential produced at P by M_2. Then the x-component of force on the element, $\rho\, d\Omega$, of mass at P is $\rho \dfrac{dV_2}{dx}\, d\Omega$. Hence if X be the total x-component on M_1,

$$X = \int \rho \frac{dV_2}{dx}\, d\Omega. \tag{1}$$

Let V_1 be the Potential produced at P by M_1; then obviously $\int \rho \dfrac{dV_1}{dx} d\Omega = 0$, because this integral is the resultant x-component of force, produced on M_1 by its own attraction. Hence we can write

$$X = \int \rho \frac{d(V_1 + V_2)}{dx}\, d\Omega. \tag{2}$$

Again, $\rho = -\dfrac{1}{4\pi\gamma}\nabla^2 V_1$; and since P is outside the mass M_2, we have $\nabla^2 V_2 = 0$; therefore we can write

$$\rho = -\frac{1}{4\pi\gamma}\nabla^2 (V_1 + V_2).$$

Let $\phi = V_1 + V_2$, and we have

$$X = -\frac{1}{4\pi\gamma}\iiint \frac{d\phi}{dx}\nabla^2\phi \,.\, dx\,dy\,dz, \tag{3}$$

with two exactly similar values of Y and Z, the other components of force exerted by M_2 on M_1.

Let ϕ_1, ϕ_2, ϕ_3 stand for $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$. Then

$$\frac{d\phi}{dx}\nabla^2\phi \equiv \tfrac{1}{2}\frac{d(\phi_1^2-\phi_2^2-\phi_3^2)}{dx} + \frac{d(\phi_1\phi_2)}{dy} + \frac{d(\phi_1\phi_3)}{dz}, \tag{4}$$

with similar values of $\frac{d\phi}{dy}\nabla^2\phi$ and $\frac{d\phi}{dz}\nabla^2\phi$. Hence if

$$\left.\begin{aligned} N_1 &= -\frac{1}{8\pi\gamma}(\phi_1^2-\phi_2^2-\phi_3^2); & T_1 &= -\frac{1}{4\pi\gamma}\phi_2\phi_3, \\ N_2 &= -\frac{1}{8\pi\gamma}(\phi_2^2-\phi_3^2-\phi_1^2); & T_2 &= -\frac{1}{4\pi\gamma}\phi_3\phi_1, \\ N_3 &= -\frac{1}{8\pi\gamma}(\phi_3^2-\phi_1^2-\phi_2^2); & T_3 &= -\frac{1}{4\pi\gamma}\phi_1\phi_2, \end{aligned}\right\} \tag{5}$$

we shall have

$$X = \int\left(\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right)d\Omega, \tag{6}$$

$$Y = \int\left(\frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz}\right)d\Omega, \tag{7}$$

$$Z = \int\left(\frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz}\right)d\Omega; \tag{8}$$

the integrations being extended through the whole volume of the closed surface S (which completely surrounds M_1 and excludes M_2), and not being limited merely to the volume of M_1, because at every point of space devoid of matter $\nabla^2\phi = 0$, and (4) shows that at all such points the three multipliers of $d\Omega$ under the integral signs vanish.

Hence at all points of the medium (ether) outside the attracting matter we have

$$\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \text{ \&c.}, \tag{9}$$

which are the well-known equations of equilibrium of a strained medium whose components of stress are N_1, &c. At all points inside M_1 or M_2 the right-hand sides of equations (9) are not zero, but

$$\frac{d\phi}{dx}\nabla^2\phi, \quad \frac{d\phi}{dy}\Delta^2\phi, \quad \frac{d\phi}{dz}\nabla^2\phi.$$

Thus the components of attraction could be produced by means of a strain in the intervening medium, this strain extending, of course, to all parts—even infinitely distant—of the medium, the stress being defined in (5); and, of course, the same result could

be produced by merely applying the proper stress to all portions of the arbitrary closed surface S, for this surface stress would transmit the proper stress to each point of the enclosed medium.

Moreover such a distribution of stress over S would produce the same moment round any axis as that produced by the attraction of M_2 on M_1. For, let L be the total moment of force round the axis of x. Then

$$L = \int \rho \left(y \frac{dV_2}{dz} - z \frac{dV_2}{dy} \right) d\Omega = \frac{1}{4\pi\gamma} \int \left(z \frac{d\phi}{dy} - y \frac{d\phi}{dz} \right) \nabla^2 \phi \, d\Omega$$

$$= \frac{1}{4\pi\gamma} \iiint \left\{ z \left(\frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} \right) - y \left(\frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} \right) \right\} dx\,dy\,dz$$

$$= \frac{1}{4\pi\gamma} \int \{ z (lT_3 + mN_2 + nT_1) - y (lT_2 + mT_1 + nN_3) \} \, dS; \qquad (10)$$

l, m, n being the direction-cosines of the normal at any point of S, and the triple integrals being replaced by the surface integrals, the values of N_1, T_1, ... being assumed to be continuous.

Now the right-hand side of (10) is the x-moment of the surface stress on the arbitrary surface S. Therefore, &c.

Let us now seek the principal axes of this stress. If the axes of x, y, z are in the directions of these axes, we shall have $T_1 = T_2 = T_3 = 0$; i.e. we must have some two of the three quantities $\phi_1, \phi_2, \phi_3 = 0$. Suppose that $\phi_2 = \phi_3 = 0$; that is, ϕ_1 is the resultant attraction-intensity at P, the point considered either in M_1 or in the medium between the bodies; or, in other words, the axis of x is at once the direction of the resultant force at P and the direction of one principal axis of the stress of the medium at P. If, now, A, B, C are the principal stress intensities at P, we have by (5)

$$A = -\frac{R^2}{8\pi\gamma}, \quad B = \frac{R^2}{8\pi\gamma}, \quad C = \frac{R^2}{8\pi\gamma}; \qquad (11)$$

R being the resultant force-intensity at P. These show that the three principal stresses are equal—that along the line of force being a pressure, and the other two being tensions. The stress of the medium at every point consists, therefore, of *pressure along the line of resultant force, accompanied by equal tension in all directions at right angles to it.*

In the case of electrical action between two systems, we shall see that the stress at each point of the intervening medium will

be the exact reverse of this—viz., tension along the line of force accompanied by pressure of equal intensity in all directions round it.

This is the celebrated Maxwellian stress, which Faraday had *qualitatively* described (Faraday's *Experimental Researches*, §§ 1297, &c.) by merely saying that the intervening medium seemed to experience at every point tension along the line of electric force accompanied by pressure in all directions round it. Observe that Faraday did not enunciate any definite ratio between these intensities of stress.

Now the method by which Clerk Maxwell arrives at the components of the stress shows that his stress components (5) are not the only ones that will produce the gravitative effects of the one system on the other. For, if

$$n_1, n_2, n_3, t_1, t_2, t_3$$

are any six functions of the co-ordinates of a point, which for all positions of the point satisfy the equations

$$\frac{dn_1}{dx} + \frac{dt_3}{dy} + \frac{dt_2}{dz} = 0, \text{ \&c.,} \tag{12}$$

it is clear that the values of $\dfrac{d\phi}{dx}\nabla^2\phi$, &c. in (4), and therefore of X, Y, Z, will be the same as before, if we take the components of stress at every point of the medium to be

$$\left.\begin{array}{r} -\dfrac{1}{8\pi\gamma}(\phi_1^2-\phi_2^2-\phi_3^2)+n_1, \text{ \&c.,} \\ -\dfrac{1}{4\pi\gamma}\phi_2\phi_3+t_1, \text{ \&c.,} \end{array}\right\} \tag{13}$$

and the principal components of this stress will not be related to each other and to the line of force as they are in the Maxwellian stress.

Again, if for any given medium we are given the components of stress at every point we are also given the components of strain, *on the supposition that they are connected with each other by 6 linear equations*, as in p. 464. The components of stress cannot be assumed as any random functions of the co-ordinates; for they must be such as to satisfy the differential equations (1), p. 411. Nor, again, can we assume any six functions of co-ordinates for the components $(a, b, c, 2s_1, 2s_2, 2s_3)$ of strain;

because $\quad 2s_3 = \dfrac{du}{dy} + \dfrac{dv}{dx}, \quad \therefore\ 2\dfrac{ds_3}{dy} = \dfrac{d^2u}{dy^2} + \dfrac{db}{dx};$

therefore $$\frac{d^2 a}{dy^2} + \frac{d^2 b}{dx^2} \equiv 2\frac{d^2 s_3}{dx\,dy}. \tag{14}$$

Similarly, $$\frac{d^2 c}{dx^2} + \frac{d^2 a}{dz^2} \equiv 2\frac{d^2 s_2}{dz\,dx}, \tag{15}$$

$$\frac{d^2 b}{dz^2} + \frac{d^2 c}{dy^2} \equiv 2\frac{d^2 s_1}{dy\,dz}. \tag{16}$$

These identities must be satisfied by any functions which can possibly represent the strain components of any medium whatever.

Now, if we suppose the equations connecting the stress and strain components of the ether to be linear, and to involve only 21 distinct coefficients (as required by the principle of conservation of energy), we may express them in the forms (5) ... (10), p. 464, by simply interchanging a and N_1, b and N_2, s_1 and T_1, &c., in these equations, thus:

$$a = m_1 N_1 + \pi_3 N_2 + \pi_2 N_3 + \tau_1 T_1 + \tau_1' T_2 + \tau_1'' T_3 \tag{17}$$

with five others, the same equality of coefficients holding here as in the inverse equations.

Hence for all values of x, y, z, and all forms of Potential function ϕ, the Maxwellian components (5) of stress when substituted in (17) and the five similar equations, must satisfy the equations (14), (15), (16), if gravitation is propagated by the Maxwellian stress, and the stress and strain of the ether are connected, as they are, for any known body.

Now we find very easily on trial that these conditions are not fulfilled; and therefore the conclusion is that—either gravitation is not propagated by the Maxwellian stress, or the ether is not of the nature of a solid body*.

But, instead of assuming the components of stress at any point of the medium, without reference to the nature of this medium, let us assume that the medium is homogeneous and isotropic, with only the two constants of elasticity, λ and μ, and determine the stress at each point from the general equations of equilibrium of such a medium.

* Clerk Maxwell, in his Article on Gravitation in the *Encyclopædia Britannica*, seems to regard the ether as a homogeneous isotropic body (probably incompressible); for he gives a table of its constants (rigidity, i.e. modulus of shear, density, &c.) exactly such as is usually given for a homogeneous isotropic body. Indeed, at the end of the Article, he explicitly calls it a 'vast homogeneous expanse of isotropic matter.'

Let P be any point inside the solid body which is subject to any gravitative action, whether its own or that due to the attractions of other bodies; let ρ be the density of the body at P, and X the x-component of force at P per unit mass. Then we have to determine N_1, N_2, ... so as to satisfy the equation

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}=\rho X, \tag{18}$$

and two similar. Substituting the strains, we have to find u, v, w, so as to satisfy

$$(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u=\rho X, \tag{19}$$

and two similar, the right-hand sides of (18) and (19) being zero at each point in the medium outside the body.

Equation (19) can always be easily satisfied; for if X is due to any number of bodies, $M, M', \ldots$ producing Potentials $V, V', \ldots$ at P, let dm be an element of mass of M, and r its distance from P; let dm' be an element of M', and r' its distance from P; and so on. Then take the function

$$\psi=\int r\,dm+\int r'\,dm'+\ldots, \tag{20}$$

the integrations extending, respectively, through M, M', ...; and it is easily found that

$$u=\frac{\rho}{2(\lambda+2\mu)}\cdot\frac{d\psi}{dx}, \tag{21}$$

with exactly similar values of v and w, will satisfy (19) and the two similar equations. Thus we have found the components of displacement at P inside the body; and to these values may, of course, be added any values satisfying

$$(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u=0, \tag{22}$$

and the two similar equations, provided that these added terms do not violate any necessary physical condition—they must not, for instance, be such as to make the displacement infinite at any point. The components of displacement in the external ether must be found from (22) and its two analogues, regard being had to the necessary physical conditions—as, for instance, that they must vanish at infinity.

Let us, as a very simple example, take the case of a single homogeneous sphere subject to its own gravitation. P being any point inside, at distance r from the centre, we have

$$X=-\tfrac{4}{3}\pi\gamma\rho x,$$

where γ is, as usual, the constant of gravitation. Then either by (20), or by inspection, we see that $u = Ar^2x$ will satisfy (19), so that

$$u = -\frac{2\pi\gamma\rho^2}{15(\lambda+2\mu)}\cdot r^2x = -Kr^2x, \tag{23}$$

with similar values of v and w. Of course in these equations the λ and μ are supposed to belong to the ether; if they belong to the body, the values of u, v, w express the displacements of a particle of the body.

There is, of course, a strain potential, whose value can be deduced as in p. 455; and we have for points inside the sphere

$$\frac{d\phi}{dr} = -Kr^3. \tag{24}$$

To the value in (23) might be added any term of the form Bx, where B is a constant, since this satisfies (22).

Such terms, however, do not assist in contributing to the gravitative action on any element of the body, since they contribute nothing to the expression

$$\left(\frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz}\right) dx\,dy\,dz, \tag{25}$$

which is the resultant x-stress on the element of volume, and it is this which is to be equal to $\rho X\,dx\,dy\,dz$, the gravitative action on the element.

In the ether outside the sphere the resultant displacement is, of course, radial, or in the line joining the point to the centre of the sphere; so that, as in p. 450, (22) gives $d\theta = 0$; and since in this case $\theta = \nabla^2\phi$, where ϕ is the strain potential of the external ether, $\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$, $\therefore \frac{d\phi}{dr} = \frac{C}{r^2}$, where C is a constant.

Now assuming the radial displacement to be the same for a point just inside as just outside the sphere, we have from (24)

$$\frac{C}{a^2} = -Ka^3, \quad \therefore\ C = -Ka^5.$$

Hence at every external point the resultant displacement is given by

$$\frac{d\phi}{dr} = -K\frac{a^5}{r^2}, \tag{26}$$

from which $\qquad u = -K\dfrac{a^5x}{r^3}$, &c.

Also, by (*a*), p. 461, if A and B are the principal stresses at an inside point,

$$A = -(5\lambda + 6\mu)Kr^2, \tag{27}$$

$$B = -(5\lambda + 2\mu)Kr^2, \tag{28}$$

the principal stress obviously having the radius vector, OP, for one principal axis. It therefore consists of pressure all over a conical frustum, as represented in Fig. 310, the intensity over the ends being greater than that over the lateral surface.

At an external point, Q, we have

$$\left.\begin{aligned} A &= 4\mu K\frac{a^5}{r^3}, & (29)\\ B &= -2\mu K\frac{a^5}{r^3}, & (30)\end{aligned}\right\}$$

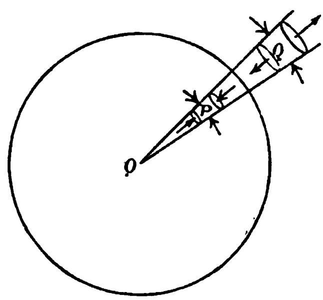

Fig. 310.

which show that the stress consists of tension on the ends and uniform pressure all over the lateral surface of a small conical frustum.

Hence it appears that *the stress of the ether is discontinuous at the surface of the body*, the radial component changing from pressure to tension.

Let us pass to the case of a uniform spherical shell whose outer radius is b and inner a. Here $X = -\frac{4}{3}\pi\gamma\rho\left(x - \frac{a^3 x}{r^3}\right)$, and the equations for the strain components are

$$(\lambda + \mu)\frac{d\theta}{dx} + \mu\nabla^2 u = -\tfrac{4}{3}\pi\gamma\rho^2\left(x - \frac{a^3 x}{r^3}\right), \tag{31}$$

and two similar.

The displacements being all radial, these equations are all included in the equation

$$(\lambda + 2\mu)\,d\theta = -\tfrac{4}{3}\pi\gamma\rho^2 \,.\, d\left[\tfrac{1}{2}r^2 + \frac{a^3}{r}\right], \tag{32}$$

$$\therefore\ \theta = -10K\left(\tfrac{1}{2}r^2 + \frac{a^3}{r}\right) + A, \tag{33}$$

where A is a constant. This is the value of θ at points in the substance of the shell.

For points within the cavity and for points in the ether outside the shell, the right-hand side of (32) must be put equal

to zero, so that θ is constant for such points. Since $\theta = \nabla^2\phi$, where ϕ is the strain potential, (33) gives

$$\frac{d\phi}{dr} = -K(r^3 + 5a^3) + \tfrac{1}{3}Ar + \frac{B}{r^2}, \tag{34}$$

where B is a constant. If K were put equal to zero, we should get the values of $\frac{d\phi}{dr}$ belonging to the cavity and to the ether outside the shell; and these are, respectively,

$$Cr \text{ and } \frac{C'}{r^2}, \tag{35}$$

where C and C' are constants.

Now if we assume continuity of displacement from the cavity to the substance of the shell, we put $r = a$ in (34) and in the first expression in (35); and equating these, we obtain C. Similarly putting $r = b$ in (34) and equating the result to $\frac{C'}{b^2}$, we obtain C'.

But it is obvious that the terms in A and B in (34) may be rejected, because, as they give displacements satisfying the equation

$$(\lambda + \mu)\frac{d\theta}{dx} + \mu\nabla^2 u = 0,$$

and two similar, they contribute nothing to the gravitative action. Hence, then, the values of $\frac{d\phi}{dr}$ are

$-6Ka^2r$ inside the cavity,

$-K(r^3 + 5a^3)$ in the substance of the shell,

$-K(b^3 + 5a^3)\frac{b^2}{r^2}$ in the external ether.

Inside the substance of the shell, if A and B are the principal stress components, we have

$$A = -(5\lambda + 6\mu)Kr^2 - 10\lambda K\frac{a^3}{r}, \tag{36}$$

$$B = -(5\lambda + 2\mu)Kr^2 - 10(\lambda + \mu)K\frac{a^3}{r}. \tag{37}$$

Inside the cavity there is uniform hydrostatic pressure equal to $-6(3\lambda + 2\mu)Ka^2$, while in the external ether we have tension and pressure of the types (29), (30), merely replacing a^5 in those equations by $(5a^3 + b^3)b^2$. There is therefore discontinuity of stress at both bounding surfaces, as before.

Take, as a final example, the case of two homogeneous spheres the radius and density of one being a and ρ, those of the other b and σ, and the co-ordinates of the centre, B, of the second with reference to the centre, A, of the first being (a, β, γ). Taking P inside the first, we have

$$(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2 u=-\tfrac{4}{3}\pi\gamma\rho\left(\rho x+\sigma b^3\frac{x-a}{r'^3}\right), \tag{38}$$

where $r'=PB$. From this we have easily

$$u=-K\left(r^2x-5\frac{\sigma}{\rho}b^3\frac{x-a}{r'}\right), \tag{39}$$

similar values holding for v and w.

The expression (39) indicates that the displacement of the ether at P consists of two components, viz. a motion along PA towards A, and a motion BP from B. This latter motion is equal to $5K\frac{\sigma}{\rho}b^3$, which is the same for all points inside the sphere A and the same whatever be the distance between the spheres A and B—a result which seems absurd. But to get rid of this absurdity, we observe that any constant may be added to the value (39), so that we might take, if $AB=c$,

$$u=-K\left\{r^2x-5\frac{\sigma}{\rho}b^3\left(\frac{x-a}{r'}+\frac{a}{c}\right)\right\}, \tag{40}$$

the added term indicating a motion of translation of the ether inside the sphere A parallel to the line AB. Thus when AB is very great, the term in u depending on the sphere B is infinitesimal.

It may, however, be remarked that as it is with differential coefficients of u, v, w we have to deal in calculating stress, we need not take the trouble of thus correcting their values.

In all these cases in which the stress is discontinuous, the gravitative effect is due solely to the stress of the ether surface which is immediately inside that of the body, as is easily seen thus. Multiply both sides of (18) by $dx\,dy\,dz$ and integrate through the volume of any surface S surrounding the body considered. Then we shall have, if P, Q, R are the components of stress on the element dS of this surface,

$$\int(lP+mQ+nR)\,dS=\iiint\rho X\,dx\,dy\,dz, \tag{41}$$

(l, m, n being the direction-cosines of the normal to the surface,) provided that N_1, T_2, T_3 experience no abrupt changes of value

in any part of the space enclosed by S. But if at a certain surface, U, within S the components experience abrupt changes from points within U to points without it, the triple integration of the left-hand side of (18) must be taken first through the volume of a surface immediately within U, and then through the volume enclosed between a surface immediately outside U and the given surface S. Denoting by (U_i) the surface-integral taken over the surface just inside U, and by (U_e) its value taken over the surface just outside U, we have

$$\int (lP+mQ+nR)\,dS-(U_e)+(U_i)=\int \rho X d\Omega, \tag{42}$$

where $d\Omega$ is the volume element, $dx\,dy\,dz$, of the body considered.

But it is obvious that $\int (lP+mQ+nR)\,dS-(U_e)=0$, because of the equilibrium of the ether outside the body; therefore the whole x-force of gravitation on the body is equal to the surface-integral (U_i).

Miscellaneous Examples.

1. In an irrotational strain which is unaccompanied by cubical compression if the strain Potential is $\dfrac{Ax^2+B(y^2+z^2)}{r^5}$, where r is a radius vector measured from a fixed origin, prove that the lines of flow are plane curves of the form

$$r^2=\pm c^2\sin^2\theta\cos\theta.$$

(Mr. Cockshott, Tripos, 1875.)

Since the cubical dilatation is zero, $\dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz}=0$; and since the strain is irrotational, $u=\dfrac{d\phi}{dx}$, &c. Hence

$$\nabla^2\phi=0,$$

or ϕ is a solid spherical harmonic. The given form makes ϕ a harmonic of degree -3. Now if U is a solid harmonic of degree $-(i+1)$, we know that $r^{2i+1}U$ is a solid harmonic of degree i. Hence $Ax^2+B(y^2+z^2)$ is a harmonic of degree 2. Expressing that ∇^2 of this expression $=0$, we have $A+2B=0$.

Hence, expressing ϕ in polar co-ordinates with the axis of x as polar axis, we have

$$\phi=\frac{B}{r^3}(1-3\mu^2).$$

Now the component displacements along and perpendicular to the radius vector being $\dfrac{d\phi}{dr}$ and $\dfrac{1}{r}\dfrac{d\phi}{d\theta}$, the differential equation of a line of flow is

$$\frac{dr}{\dfrac{d\phi}{dr}}=\frac{r\,d\theta}{\dfrac{1}{r}\dfrac{d\phi}{d\theta}}.$$

Substituting in this the value of ϕ above and integrating, we have the equation of the line of flow as given. (Greenhill's Solutions of the Senate-House Problems, 1875.)

2. A spherical shell whose inner radius is a and outer b is subject to internal and external pressures of intensities p and ϖ, respectively; show that the total amount of work done in the small strain is

$$\frac{2\pi}{b^3-a^3}\left\{\frac{(pa^3-\varpi b^3)^2}{3k}+\frac{a^3b^3}{4\mu}(p-\varpi)^2\right\},$$

where k and μ are the moduli of compression and shear.

3. Find the intensity of pressure at any point inside a globe of homogeneous incompressible fluid held together by the gravitation of its parts alone; and thence deduce the attraction between two hemispheres of a uniform solid globe. (Part of a problem by Prof. Tait, Tripos, 1875.)

The intensity of pressure at any distance, r, from the centre is given by the equation $p = \frac{2}{3}\pi\gamma\rho^2(a^2-r^2)$, where $a =$ radius of globe. Integrating the pressure over any diametral plane section, we obtain a result which must be equal to the attraction between the two hemispheres. This is $\frac{1}{3}\pi^2\gamma\rho^2 a^4$.

4. If the density at any point of a solid sphere (centre A, radius a) varies inversely as its distance from a given external point, B, find the Potential at any given external point, P.

Ans. $$\frac{4k\pi a^3}{cr}\left\{\frac{1}{3}+\frac{L_1}{3.5}\frac{a^2}{cr}+\frac{L_2}{5.7}\frac{a^4}{c^2r^2}+\ldots+\frac{L_i}{2i+1.2i+3}\frac{a^{2i}}{c^ir^i}+\ldots\right\},$$

where $r = AP$, $c = AB$, L_1, L_2,... are the Laplacians for the points P and B (functions merely of cos $\angle PAB$), A being origin, and k a constant.

[Let Q be any point in the sphere, $AQ = R$, $QP = D$, $QB = r'$. Then the Potential $= k\int\frac{R^2dR\,d\mu'\,d\phi'}{Dr'}$. Expand $\frac{1}{D}$ and $\frac{1}{r'}$ in Laplacians thus:

$$\frac{1}{D}=\frac{1}{r}\left\{1+l_1\frac{R}{r}+l_2\frac{R^2}{r^2}+\ldots+l_i\frac{R^i}{r^i}+\ldots\right\},$$

$$\frac{1}{r'}=\frac{1}{c}\left\{1+\lambda_1\frac{R}{c}+\lambda_2\frac{R^2}{c^2}+\ldots+\lambda_i\frac{R^i}{c^i}+\ldots\right\}.$$

Multiply the two together, and observe that in the double integration in μ' and ϕ' all terms except those of the type $l_i\lambda_i$ vanish, while $\iint l_i\lambda_i d\mu' d\phi' = \frac{4\pi}{2i+1}L_i$. Observe also that the function is reciprocal for the points B and P—as is *à priori* evident. If P coincides with B, put $L_1 = L_2 = \ldots = 1$.]

5. Find the strains and stresses in a cylindrical pipe of uniform thickness, which is subject to given internal and external pressure.

Proceed exactly as in Ex. 3, p. 455.

CHAPTER XIX.

ELECTROSTATICS.

398.] **Quantity of Electricity.** The reader is supposed to be familiar with the elementary facts of Electricity as described in any of the current works on Physics.

Our theory of Attraction has hitherto been concerned with distributions of attracting *matter*, and those cases in which this matter was supposed to form a thin shell are intimately related to the branch of the subject which we are now about to discuss. Hitherto, however, except when discussing Magnetic Shells, we have not been obliged to postulate distinctions of *sign* between matter and matter, or to deal with *repulsion* instead of *attraction*. This distinction must be made here also—i. e. we shall assume that there is *positive*, and that there is also *negative*, electricity—the distinction indicating nothing more than a difference of behaviour among electrified bodies, some of them attracting, while others repel, one and the same given electrified body.

For the quantitative (mathematical) treatment of the phenomena of Electricity it is not necessary that we should know the precise nature of Electricity itself; it suffices that we know the laws which regulate its fundamental manifestations. We may rest satisfied with regarding electrification as merely a *state* of a body—as, for example, some re-arrangement of its molecules involving some kind of strain within the body and impressed by the body on the medium (air or other) surrounding it. But it is at the outset necessary to obtain some measure of this state as to *quantity*.

The first thing to be noted with regard to electrification is that it justifies a distinction as to plus and minus, this distinction being a wholly conventional mode of representing such

a phenomenon as the following. Let a piece of glass be rubbed against a piece of resin, and another piece of glass rubbed against another piece of resin. Then it is found that the electricities of the two pieces of glass repel each other, as do also those of the two pieces of resin; but the electricity of either piece of glass attracts that of either piece of resin.

The electricity developed on the glass is called positive, and is denoted by the sign +, while that on the resin is called negative, and denoted by the sign −.

Now if we imagine two very small equal flat pieces of glass (each about the size of a pin-head, suppose) to be rubbed in exactly the same way by a piece of resin, so that we can assume them to have equal electrifications, and to be then placed with a distance of 1 centimètre between them; they will repel each other with a force which can be measured (conceivably) in dynes. If the medium between the pieces of glass is air, and their force of repulsion is exactly 1 dyne, each piece of glass is said to have a *unit charge* of electricity. The total quantity on the two together is two units, and thus we get an idea of a charge consisting of any number of units of electricity.

If we imagine a body of very small dimensions to be charged with e units of positive electricity, and to be placed at a distance of r centimètres from another very small body which has a charge of e' units, the medium between them being air, the force of repulsion exerted by either on the other is

$$\frac{e \cdot e'}{r^2} \text{ dynes,} \tag{a}$$

as Coulomb proved by well-known experiments with his Torsion Balance.

In the attraction of matter we should obtain exactly the same expression for the attraction of two condensed spherical particles with a distance of r centimètres between their centres by making γ equal to unity in the expression (a), p. 236, i.e. by choosing the unit quantity of matter as indicated in p. 275.

Formally, then, *the electrostatic unit of quantity is that charge which, supposed to be condensed at a point, acts on an equal charge condensed at another point distant* 1 *centimètre from the first, with a force of* 1 *dyne, the intervening medium being air.*

Hitherto, unfortunately, no simple name has been devised for this electrostatic unit of charge. We shall often use the ab-

breviation* 'e.s. unit' to signify the C.G.S. electrostatic unit of quantity.

399.] **Influence of the Medium.** From the experiments of Faraday results the fundamental fact that the force of repulsion between two given charges is not the same when the intervening medium is sulphur, or any other insulator, as when this medium is air. Thus, if the two given charges are e and e' units (defined as before with reference to air), the force of repulsion between them is given by the expression

$$\frac{ee'}{Kr^2}, \tag{β}$$

when instead of air the separating medium is any other insulator, K being a constant depending on the medium, and called its *specific inductive capacity.*

Our definition of the electrostatic unit of quantity implies, of course, that K is unity for air. It has the same value for all gases as for air, and a greater value for most known solid and liquid insulators.

If a charge of e absolute units is condensed at a point in a medium of uniform specific inductive capacity, K, which is of practically infinite extent, the amount of work, in ergs, done by the repulsive force of this charge in removing a charge of one unit from a distance r centimètres to infinity is

$$\int_r^\infty \frac{e}{Kr^2}dr.$$

If V stands for this amount of work,

$$V = \frac{e}{Kr}\cdot \tag{γ}$$

And, generally, in such a medium if there is any distribution of electricity whose amount at any point, A, is de electrostatic units, and if P is any point in the field, the amount of work done by the forces of the field in removing a unit charge from P to any position in which the forces are insensible is $\frac{1}{K}\Sigma\frac{de}{PA}$; or denoting PA by r, and the amount of work by V (the Potential at P),

$$V = \frac{1}{K}\int\frac{de}{r}, \tag{δ}$$

the integration being extended all over the attracting system.

In the case of *attractive* forces (i.e. when two elements of the same sign attract each other) V at any point is the work done *by* the forces of the field in bringing a + unit to the point from infinity, whereas in Electrostatics V at any point is the work done *against* the forces of the field in this motion.

Again, if X, Y, Z are the components of the repulsive force per unit charge at P in the positive senses of the axes of co-ordinates,

$$X = -\frac{dV}{dx}, \qquad Y = -\frac{dV}{dy}, \qquad Z = -\frac{dV}{dz}, \tag{ϵ}$$

(x, y, z) being the co-ordinates of the point P.

Instead of the term 'force per unit charge' we shall for the future use the term *electromotive intensity*, which is adopted by Clerk Maxwell.

The nature of the medium will modify the value of the *surface-integral of normal electromotive intensity* (Art. 324) over any closed surface described in the medium. Describe any closed surface surrounding a point A at which a charge of de units is condensed; let P be any point on the surface, and let $PA = r$. Then, with the notation of Art. 316, the surface-integral of the electromotive intensity due to the charge de is

$$\frac{de}{K}\int\frac{\cos\psi}{r^2}\,dS, \quad \text{or} \quad \frac{4\pi\, de}{K};$$

and if e_i is the total charge inside the surface, we have

$$K\int N\,dS = 4\pi e_i. \tag{ζ}$$

Just as in Art. 324, the surface-integral of normal electromotive intensity due to any *external* charge is zero; and if a charge e_0 is distributed *on* the surface, as an infinitely thin layer,

$$K\int n\,dS = 2\pi e_0, \tag{η}$$

where n is the normal electromotive intensity at a point strictly *on* the surface. Inasmuch as, in this case, the small electrified element of surface at the point does not contribute to n (see Art. 322) $n = N - \frac{2\pi\sigma}{K}$, where σ is (next Article) the surface-density at the point, so that (η) could be deduced from (ζ).

400.] **Volume-density, surface-density.** If at any point P, a very small element, $d\Omega$, of volume is taken—this being measured in cubic centimètres—and if within this volume there is included a charge of de electrostatic units, the ratio $\frac{de}{d\Omega}$ is

called the *volume-density*, ρ, of the charge at P. The volume-density at P is thus the charge per unit volume at P.

The ratio $\frac{de}{d\Omega}$ may be infinite, as in the following case. Suppose the charge to be distributed as an *infinitely* thin layer on a surface on which P is any point. Then if on a very small element, dS, of area of the surface at P there are σdS units of charge where σ is a *finite* quantity, the volume-density at P is infinite, because if dn is a small elementary length at P along the normal to the surface, $d\Omega = dS \, . \, dn$, so that

$$\frac{de}{d\Omega} = \frac{\sigma}{dn},$$

which $= \infty$ when dn is infinitely diminished.

When the charge is thus distributed as an infinitely thin layer on a surface, the *surface-density* at P is the limiting value of the ratio of the charge on any small element of area of the surface at P to this element of area; that is, it is the charge per unit area—so many electrostatic units per square centimètre.

401.] **Conductors and Insulators.** A perfect conductor is a body in which electricity cannot be in equilibrium while there exists electromotive intensity at any point within the substance of the body. If electrification consists in some kind of strain, a conductor is thus a body which is incapable of supporting the electrical strain. In such a body this strain instantly disappears, and can be renewed only by a fresh application of the process of electrification.

A perfect insulator (called also a *dielectric*) is a body which *can* support electrical strain, or continue to experience electrical force, at any point indefinitely.

Faraday pictured a dielectric medium as consisting of an immense number of small perfectly conducting bodies separated from each other by a substance through which electricity cannot flow—as, for instance, an immense number of small shot imbedded in a cake of sulphur or shellac; and on the view that electricity is a double fluid, 'positive' and 'negative,' when electric force is exerted at any point of a dielectric its effect is assumed to be a separation of this double fluid in all the little shot particles at the point, whereby the positive fluid is pushed to one end of each shot and the negative drawn to the opposite end, the line joining the two poles thus formed in each shot being that along which

the electrical force acts at the point considered, and the amount of 'neutral fluid' separated into + and − being proportional to the electromotive intensity at this point.

A conductor may be either a solid metallic body or a metallic sheet of any thickness, forming either a closed or an unclosed* surface. In any case there is no volume-density at any point in the substance of the conductor, no matter how, or to what extent, the conductor is electrified. For, since there is no electrical force at any point in its substance, V is constant throughout its substance, so that at each point $\nabla^2 V = 0$, therefore $\rho = 0$.

Hence the electrification exists only at the points of contact of the metal with the air, or other dielectric medium, surrounding it; or, as it is usually expressed, the electricity resides wholly on the surface of the conductor. If the conductor is in the form of a sheet, the bounding surface consists, of course, of both faces and the edges; if the sheet forms a closed shell, the electricity may reside, generally, on the outer and the inner surfaces, but never in the substance between these.

Since no force can be exerted inside a conducting substance, it follows from (β), Art. 399, that $K = \infty$ for such a substance; that is, *the specific inductive capacity of a perfectly conducting substance is* ∞.

402.] **Poisson's Equation.** Take the case of a uniform medium of specific inductive capacity, K, and apply the equation (ζ) which expresses that the surface-integral of normal force is equal to $4\pi \times$ included charge, to the elementary parallelopiped $dx\,dy\,dz$. If ρ is the volume-density at the position of this element, we have obviously

$$K\left(\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2}\right) + 4\pi\rho = 0. \tag{1}$$

If K varies from point to point in the medium, the result will be different.

On the first face $dy\,dz$ (that nearest to the origin) the outward normal force is $K\dfrac{dV}{dx}\,dy\,dz$; and on the opposite face the force is

$$\left[-K\frac{dV}{dx} - \frac{d}{dx}\left(K\frac{dV}{dx}\right)dx\right]dy\,dz.$$

---

* This expression is not strictly correct, because a metallic sheet is in reality a closed surface, formed by both faces and the edges.

Hence Poisson's equation becomes

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) + 4\pi\rho = 0. \tag{2}$$

In special cases this equation may be more usefully expressed in polar co-ordinates. Thus, supposing the medium to be symmetrical round a centre, V and K will be functions of r alone. Then we may either transform (2), or make use of the fact which it expresses. Describe a cone of small conical angle ω, and cut off a frustum by spheres of radii r and $r+dr$, each having the vertex of the cone for centre. Then take the surface-integral of normal force over this frustum. Its lateral surface contributes nothing. The end near the origin contributes $K\,\dfrac{dV}{dr}\cdot\omega r^2$, and the opposite end gives

$$-K\frac{dV}{dr}\cdot\omega r^2 - \frac{d}{dr}\left(K\frac{dV}{dr}\cdot\omega r^2\right)dr,$$

while the included charge is $\rho\,.\,\omega r^2 dr$.

Hence Poisson's equation is

$$\frac{d}{dr}\left(Kr^2\frac{dV}{dr}\right) + 4\pi\rho r^2 = 0. \tag{3}$$

If the dielectric is symmetrical round an *axis*, Poisson's equation may be used in the following form. Let ζ be the perpendicular drawn from any point to the axis; then, taking the normal flux of electromotive intensity over the cylindrical element of volume, since K and V are functions of ζ only, we have

$$\frac{d}{d\zeta}\left[K\zeta\frac{dV}{d\zeta}\right] + 4\pi\rho\zeta = 0. \tag{4}$$

403.] **Equation for V at an Electrified Surface.** In addition to the volume equation (2) for V, which holds at any point of a dielectric, it is necessary to see what happens to the differential coefficients of V at the surface of separation of two dielectric media. We shall, for generality, suppose that this surface is electrified, by friction or otherwise, and that σ is the surface-density of the charge at any point.

Consider a very small element, dS, of the surface, and let P and Q (Fig. 278, p. 262) be two points on the normal to it, P being in the one medium (K_1) and Q being in the other (K_2). Regard the force-intensity at P as due to the electrified element dS and the remainder of the field. Let this latter produce at P

an electromotive intensity having components n and t, respectively, along the normal and in the tangent plane. The small plate dS contributes no tangential component, while it gives a normal component equal to $2\pi\sigma$. Hence the components of force at P are $n+2\pi\sigma$ and t. Evidently at Q the components are $n-2\pi\sigma$ and t. Also the normal force measured from the surface towards P is $-K_1\frac{dV}{dn_1}$, where dn_1 is an element of normal drawn into the medium in which P is; and that of Q is $-K_2\frac{dV}{dn_2}$. Therefore

$$-K_1\frac{dV}{dn_1} = n+2\pi\sigma,$$

$$-K_2\frac{dV}{dn_2} = -(n-2\pi\sigma),$$

$$\therefore\quad K_1\frac{dV}{dn_1}+K_2\frac{dV}{dn_2}+4\pi\sigma = 0, \qquad (a)$$

which is the equation connecting the normal variations of V in the two media at their surface of separation.

From this it is obvious that if a line of force meets obliquely a non-conducting surface which is electrified, *the line will be refracted in passing through the surface.* For, the tangential component of force suffers no change, while the normal component does, in the passage through the surface. At the surface of separation of a conductor and a dielectric, if σ is the surface-density at any point of the conductor, the surface equation is simply

$$K\frac{dV}{dn}+4\pi\sigma = 0, \qquad (\beta)$$

since no force exists in the substance of the conductor.

404.] **Principle of Superposition.** The additive property of the Potential, which has been already noted (Art. 326), is one which must always be kept in view in Electrostatics. In virtue of this principle, if in any electrical field we have a system of charged bodies, which we may denote by A, and another system of charged bodies, which we denote by B, and if we wish to determine the total Potential, or the resultant force, at any point due to the combined actions of A and B—or their resultant inductive (Art. 406) effect on any conductor in the field—we may consider the effect of A alone and the effect of B alone, and combine these

—by addition if it is the total Potential that we seek; by vector composition if it is the resultant force.

405.] **Potential of the Earth.** Defining the Potential at any point, P, in space as $\int \frac{de}{r}$, where r is the distance between P and any point in the Universe at which there is a charge de, it is clear that the integration would take in all the electrified bodies in the Universe. This integration would give us the *Absolute Potential* at P (on the supposition that the electrified portion of the Universe does not extend to infinity). What the value of this integration is for points on the Earth it is impossible to say, and we are not practically concerned with it. The Potential of the Earth is taken as *zero*—since we are never concerned with anything but *differences* of Potential—and any body which is put in metallic connection with the Earth has the zero potential of the Earth. It is necessary to explain this statement.

Every electrified body is, of course, connected with the Earth by some means; and even if it were connected by a wire with a slab of marble or of ebonite it would not be 'connected with the Earth' in the sense in which this vague expression is intended to be used—i.e. it is not at zero Potential. When a body is connected with the Earth, as a matter of fact it is always metallically connected with a gas or water-pipe—that is, with a conductor of vastly greater size than the charged body itself, this conductor consisting of a whole system of connected gas and water-pipes, moist earth, streams, etc. Thus the charged body shares its charge with this huge conductor, and (as will appear more clearly when we treat of the capacity of a conductor) its Potential becomes sensibly equal to that of the Earth at the place.

406.] **Induction.** A conductor can be electrified in either of two ways—viz. either by touching it with an electrified body, or by bringing it near such a body without touching it. In the latter case it is said to be electrified by *induction*, and this process may be exemplified as follows. Take a mass of metal of any shape, either completely solid or forming a shell; suspend it from a fixed point by a silk thread or any other insulator. Take also a small glass ball suspended by a silk thread, and electrified by friction, so that it has a certain positive charge. If the suspended glass ball is brought tolerably close to the conductor, it will be

found that the surface of the latter is no longer in its neutral state, the portion of this surface in the neighbourhood of the glass ball being negatively, while the more remote portion of the surface is positively, electrified. The *total* amount of electrification on the conductor is zero, the positive and negative amounts being equal in quantity, but neither of them equal in quantity to the charge on the ball if the conductor is solid or if it is hollow and the ball is suspended *outside* it. If the conductor is hollow, and if the glass ball is suspended *within* it (a small aperture being made in the conductor to allow of this suspension), it will be found that the outer surface of the conductor is at every point positively electrified, while the inner is everywhere negatively electrified, the amounts of these charges being equal and *each equal to the charge on the surrounded ball.*

Thus, whenever electrification is produced by induction, the amount of positive electricity produced is equal to that of negative; and this remains true whatever be the mode in which electrification is produced. If, for example, electrification is produced by the friction of ebonite and catskin, the ebonite and the catskin have charges of opposite signs and equal amounts; and if by the friction of a piece of glass, and a piece of resin, the + charge on the glass and the − charge on the resin are equal in amounts. All passes exactly *as if* electrification really consisted in the decomposition of a perfectly neutral fluid—but then this hypothesis is only one among many that might be devised for explaining the fact.

In the case in which the glass ball is suspended inside the hollow conductor, we may enquire whether the quantity of the charge on the inner, or on the outer, surface depends on the position of the ball inside. Now both by experiment and by theory we can prove that the quantity of neither charge depends on this position. Experimentally it is shown thus. Suppose that the glass ball was electrified by friction with a piece of resin. Suspend *both* the glass and the resin inside in any positions whatever, and we find no trace of electrification at any point on the outer surface. The glass and the resin contained equal charges, and since their effects on the outer surface destroy each other completely, the amount and the law of distribution of the charge on the *outer* surface of the conductor are both independent of the position of the surrounded charge. The surface-

density of the charge induced on the *inner* surface at any point will depend on the position of the included charge; and when both the glass and the resin are inside the conductor, the inner surface will be electrified, its total charge being zero, so that the surface-density is + in one part and − in the other part of this inner surface.

We shall presently deduce these results from theory.

If instead of an electrified *glass* ball we had used an electrified brass (or other metallic) ball, the results would have been quite the same; but in this case if we allow the brass ball to touch the inner surface, the results will be that—

1. The brass ball and the *inner* surface of the conductor will both lose their charges completely.

2. The outer surface of the conductor will be charged with an amount exactly equal to that which the brass ball had, and of the same sign—in other words, the charge of the brass ball is simply transferred to the outer surface of the conductor.

This complete transference of the charge from the brass ball to the conductor could not have been effected by suspending the latter *outside* the conductor and then allowing the two to touch. If this is done, the charge is divided between the ball and the conductor in a certain ratio depending on their sizes and shapes; but if the surface of the conductor is very large in comparison with that of the ball, very little of the charge will remain on the latter.

We have here substituted a metallic for a glass ball, because when the metallic ball touches the inside of the conductor, the flow of its charge takes place instantly, whereas if the ball were of glass and this were a *perfect* insulator, its charge could never wholly combine with that on the inner surface of the conductor.

It is important, then, to observe that if any system of charged bodies is completely surrounded by a conductor, the charge induced on the outer surface of the latter is always the same in amount and sign as the sum of the charges on the surrounded bodies, and this charge is accompanied by an equal and opposite charge on the inner surface; whereas if the charged bodies are not completely surrounded by the conductor, the amounts separated on it are *not* each equal in quantity to the sum of the inducing charges. Moreover, neither the amount nor the law of distribution of the charge induced on the outer surface depends

on the positions of the included charges; nor the *amount* of the charge induced on the inner surface, but the law of distribution of this charge does. We may therefore move the internal charges about in any manner without producing any effect on the outer charge on the conductor; we thus simply produce changes in the distribution, or law of surface-density, of the charge on the inner surface.

407.] **Fundamental Properties of a Conductor.** Suppose a conductor placed in any electrical field. Before it was brought into the field let it have received any charge; then when it is brought into the field, its charge will modify the distribution of electricity on every other conductor, and its own distribution will also be modified by the induction of these bodies.

Now *the surface of this conductor is an equipotential surface for the whole electrical system in the field, its own electricity included.* For, since there is no force at any point in the substance of the conductor, the Potential due to all the existing charges must be constant throughout its mass. The same is true for every other conductor in the field. Hence at every point on a conductor the line of force is perpendicular to the surface.

Again, *if the conductor does not include within itself any charged body, the Potential is the same at all points inside, and equal to that on the surface*—just as if the conductor were a solid mass of metal, instead of a shell.

For, let A be the Potential of the conductor, and P any point inside, at which the Potential $= B$, and suppose $B < A$. Drawing an infinite number of rays from P, in all directions, to meet the conductor, it must be possible to find on each ray a point at which the Potential has the value C which is intermediate to B and A. These points form a closed surface round P. The result follows then by Art. 324, since at every point of this closed surface N, which is $\frac{dV}{dn}$, is a positive quantity, and the surface-integral cannot vanish. Hence the supposition that the Potential varies inside is untenable. Therefore when there is no included charge, whatever there may be outside, the whole space inside the conductor is at the same electrical level; and in this case there is *no charge at any point on the inner surface of the conductor*, as will be proved in the next Article.

The inability of any external electrified bodies to produce the

slightest *force* in the interior of a closed conductor is shown experimentally in a most striking manner by *Faraday's house.* (See Faraday's *Experimental Researches*, § 1174.)

If a conductor surrounds any charged bodies and has also any charged bodies outside it, the amount of the charge induced on the inner surface is equal (with opposite sign) to the algebraic sum of the charges surrounded by the conductor.

Let A and B (Fig. 311) be the outer and inner surfaces of a conductor; let S be any closed surface drawn in the body of the conductor; and let the finely dotted lines at the outside of the outer and the inside of the inner surface represent the induced charges on these surfaces.

Take the surface-integral of normal electromotive intensity over S. This is zero, because the force is zero at each point on S. Hence $e_i = 0$ (Art. 399); but e_i is the sum of the surrounded charges and the charge on B; therefore, &c.

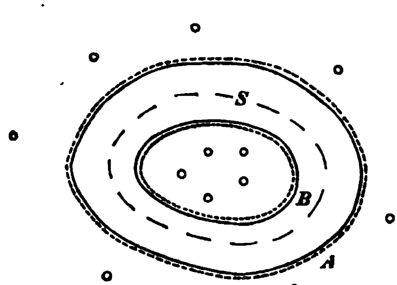

Fig. 311.

In this case we may usefully consider the effects of the external charges, E_e, and the internal charges, E_i, separately. By Art. 407 it appears that E_e produces a certain + and − distribution wholly on the external surface, A, and no effect whatever in the substance of the conductor, so that no part of the charge on B is due to the external charges.

The charge on the inner surface, B, is due wholly to the surrounded charges, E_i, and the charge on the outer surface, A, is partly due to E_i, this part being equal to E_i. Thus the total charge on A is due partly to the external and partly to the internal charges, while the charge on B is due wholly to the internal charges.

Again, *lines of force do not exist in the substance of a conductor;* they cease at its surface of contact with the surrounding, or included, dielectric medium. Defining the positive sense along a line of force as that in which the force acts, since $\sigma = \dfrac{KN}{4\pi}$,

at a point either on the outer or on the inner surface at which the surface-density is +, the line of force starts from the surface of the conductor into the medium; and at a point at which σ is −, the line of force comes into the surface from the medium. But lines of force in the space surrounded by the conductor (when they exist—as they do when there are charges inside) are in no sense to be regarded as continuations of the lines of force starting from, or entering, the exterior surface. We may express this characteristic of a conducting substance otherwise, thus—*no electrostatic action can be propagated through a conducting substance* (solid or liquid); such actions are propagated only through insulating media; whence they were called *dielectrics* by Faraday.

The charge on either surface of contact of a hollow conductor with the dielectric—whether this charge be due to the induction of the other charged bodies in the field, or to contact originally with a charged body, or to both—is sometimes measured with reference to *unit tubes of force* (see Art. 340).

It is manifest that if we fill the whole dielectric space in the field (honey-comb fashion) with unit tubes, the number of such tubes (fractions included) which start *from* the surface is equal to 4π times the total algebraic charge on the surface. This is nothing more than a re-statement of the fundamental result

$$\sigma = \frac{N}{4\pi}, \text{ or } \quad \sigma dS = \frac{1}{4\pi}\int N dS^*.$$

408.] **Inner and Outer Surface-Densities at any point on a Conductor.** Let AB and CD (Fig. 312) be portions of the outer and inner surfaces of a closed conductor; over the contour of any small element of area, dS, at P draw normals, thus forming a tube of force (represented in the figure by the normals at P and Q); produce these normals to meet the inner surface, and through the

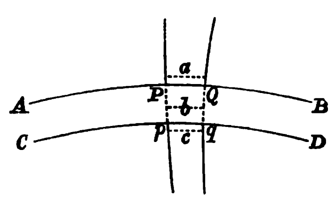

Fig. 312.

* The way most in vogue with electricians for expressing the charge on one surface of a conductor is the following. Imagine all the field filled with *lines* of force; then the number of these that intersect the surface in the positive direction is a measure of the charge on it. A very inconvenient measure, truly. Not only is this mode of speaking unjustifiable, but it is mathematically impossible to attach the slightest logical meaning to it.

inner surface towards the interior. The normals drawn outwards at P, Q, ..., are lines of force; the portions Pp, Qq, ... of them inside the substance of the conductor cannot be regarded as lines of force, because no force exists within this substance; the normals drawn inwards at p, q, ... are lines of force if the Potential is not constant inside, i. e. if there are charged bodies inside the conductor—as we now suppose to be the case.

Draw any normal section, a, of the external tube indefinitely close to PQ; any normal section, c, of the internal tube indefinitely close to pq; and any normal section, b, within the substance.

Let σ be the surface-density at P, and let N be the electromotive intensity at any point on the section a; then take the surface-integral of normal force over the closed tubular surface terminated by the sections a and b. Normal force exists only on the section a, and if dS' is the area of this section, we have by Art. 399,

$$K.N.dS' = 4\pi\sigma.dS,$$

$$\therefore \quad K.N = 4\pi\sigma, \tag{1}$$

(K being the specific inductive capacity of the external medium,) since in the limit $\dfrac{dS'}{dS} = 1$.

If V is the Potential of the conductor, and dn is an element of the outward-drawn normal at P, we have

$$N = -\frac{dV}{dn};$$

hence

$$\sigma = -\frac{K}{4\pi}\frac{dV}{dn}. \tag{2}$$

Let σ' be the surface-density at p, K' the specific inductive capacity of the internal medium, dn' an element of inward-drawn normal; then integrating over the tube terminated by the sections b and c, we have

$$\sigma' = -\frac{K'}{4\pi}\frac{dV}{dn'}. \tag{3}$$

From (3) we have at once the result mentioned (see p. 491) in the last Article; for if V does not vary towards the interior, $\dfrac{dV}{dn'} = 0$, therefore $\sigma' = 0$ at every point.

409.] **Fundamental Theorem.** *On a conductor removed from*

the influence of all electrified bodies a charge of given amount can be distributed in only one way.

Suppose a charge of amount e given to the conductor by contact with a charged body which is then removed, and let σ be the surface-density at any point, P. If possible, let the charge e be spread over the surface in another manner so that σ' is the surface-density at P. Let the first distribution produce a Potential A on the conductor, and the second a Potential A'. Reverse the second distribution in sign and superpose it on the first; this superposition gives a new state of equilibrium with surface-density $\sigma-\sigma'$ at P and a Potential $A-A'$ on the conductor, with a total charge equal to zero. This is impossible. For, let C be any magnitude between $A-A'$ and zero. Then since at all points at infinity the Potential due to the charge is zero, it is possible to surround the conductor with a closed surface, S, at every point of which the Potential is C. Taking $U = V$ in Green's equation, apply this equation to the surface S and the included space. Thus we have

$$\int V\nabla^2 V . d\Omega = C\int \frac{dV}{dn} dS - \int R^2 d\Omega, \tag{1}$$

where R is the resultant force at any internal point.

Now $\nabla^2 V . d\Omega$ is zero at all points inside S, except points on the surface of the given conductor, and it is easy to see that at P, if dS is an element of area of the surface of the conductor, $\nabla^2 V . d\Omega = -4\pi(\sigma-\sigma')\, dS$. Hence (1) becomes

$$-4\pi(A-A')\int(\sigma-\sigma')\, dS = C\int \frac{dV}{dn} dS - \int R^2 d\Omega. \tag{2}$$

But the left-hand integral is zero, since the total charge on the conductor is zero. Again, by Art. 399, the first integral on the right-hand side is zero; hence the remaining integral $= 0$, i.e.

$$R = 0,$$

at *every* point inside S, and therefore at every point, P, on the conductor. But (Art. 408) $\sigma - \sigma' = \dfrac{R}{4\pi}$, therefore $\sigma = \sigma'$ at every point, and the two distributions are identical.

This very important result may be thus stated—*if we know any one possible mode of distributing a given charge on a given conductor which is removed from the influence of all electrified bodies, we know the only one possible.*

Thus, for instance, if a sphere of radius a centimètres receives a charge of e units, one (and therefore the only) mode of distribution consists in spreading the charge uniformly over the surface, so that at each point

$$\sigma = \frac{e}{4\pi a^2}.$$

In the same way it can be proved that in a system of insulated conductors placed in given positions, if the total charge on each of them is zero, the only possible distribution is one in which each conductor is in its natural state.

For, if possible, let there be a distribution in which the potentials on the conductors are, in order of descending magnitudes, $V_1, V_2, V_3, \ldots$ Then it is evidently possible to describe round the conductor whose potential is V_1 a closed surface which will not meet any of the other conductors and on which the potential has a constant value, a, $< V_1$, and $> V_2$. Applying equation (1) to this surface and its enclosed volume, we have

$$\int R^2 d\Omega = a\int \frac{dV}{dn} dS + 4\pi V_1 \int \rho d\Omega.$$

Now $\int \rho d\Omega = 0$, by hypothesis; therefore, as before, $R = 0$, and the first conductor is in its natural state. Proceed to the second, &c.

If each conductor, instead of having zero charge, has a charge of given amount, *there is only one law of distribution on each conductor*, the relative positions of all the conductors being supposed fixed. For, if there be a second possible distribution, reverse it and superpose it on the first; then each conductor has zero charge, and is, by what has just been proved, in its natural state at every point.

This result will be useful when we deal with *Capacities*.

410.] **Free Charge on a Conductor.** If an insulated conductor is touched by an electrified body which is then removed so far as to produce no influence on the conductor, the portion of the charge which has been taken up by the conductor spreads over its outer surface after the manner of a thin layer of gravitating matter with thickness varying from point to point—or with variable surface-density. This layer is, of course, in equilibrium under its own repulsive forces; it produces constant potential all over the conductor and all through its interior.

It is called a *free charge*. Since it is self-equilibrating, it would be more expressively called an *idiostatic* charge.

Let its amount be e electrostatic units. Then, from what has been proved, it cannot, while self-equilibrating, be spread over the surface in any other way. Let v be the Potential which it produces on and through the conductor, and σ its surface-density at any point P. Let another charge also equal to e be given to the conductor, so that the total charge on it $= 2e$; then the Potential $= 2v$, and the surface-density at $P = 2\sigma$. Hence if the charge on the conductor is made ne, the Potential $= nv$, and the surface-density at $P = n\sigma$.

Thus, then, if the total charge on an insulated conductor removed from the influence of all charged bodies is E, and the Potential on it is V, the ratio $\frac{E}{V}$ is constant for all values of E. This ratio is, for example, equal to the number of e. s. units (Art. 398) which must be given to the conductor in order to raise its Potential from zero to *one erg per e. s. unit*, or it is the reciprocal of the Potential which will be produced by imparting to the conductor a charge of one e. s. unit.

A free, or idiostatic, charge is contrasted with an induced, or 'bound', charge. If an electrified body is brought near the outside of a conductor, a charge will be induced on the outer surface of the conductor, with + and − surface-densities in different portions. But this induced charge is not self-equilibrating; it disappears the moment the inducing charged body is withdrawn, and it was kept in equilibrium partly by its own attractive and repulsive forces and partly by those of the inducing body.

A 'bound' and an idiostatic charge may both simultaneously exist on the same conductor, by superposition. Thus, if the conductor had received an idiostatic charge previously to the approach of the inducing body, we might in imagination completely separate the two charges and regard the idiostatic charge as existing all the time, with the other simply superposed.

From what has been said (Art. 406) about the effect of charges surrounded by a hollow conductor, it may readily be conjectured that the charge which they induce on the outer surface (A, Fig. 311) of the conductor is idiostatic; and that this is so we shall presently prove.

The law of surface-density, according to which an idiostatic charge distributes itself over a conductor, is the same as the law of thickness of a uniform shell of attracting matter spread over the surface so as to produce no attraction at any internal point (p. 257); and therefore for an ellipsoidal conductor the surface-density at any point is directly proportional to the central perpendicular on the tangent plane at the point.

411.] **Capacity of a Conductor.** The constant ratio which, as has been proved, any idiostatic charge on a conductor removed from the influence of all charged bodies and from the presence of other conductors, bears to the potential produced on the conductor by the charge, is called the *capacity* of the conductor. Thus, if C is the capacity of a conductor whose charge is E and Potential V,

$$E = CV. \qquad (1)$$

The value of C depends on two things—the figure of the conductor, and the medium in which it is placed.

If C is the capacity of the conductor when it includes and is surrounded by air, and if the air is replaced by a uniform medium of specific inductive capacity K, the new capacity will be KC.

For, if P is any point on the conductor, and Q any other point on it at which the surface-density is σ, and $PQ = r$, we have, when air is the medium (Art. 399),

$$V_a = \int \frac{\sigma\, dS}{r},$$

and when the other medium replaces air,

$$V_m = \frac{1}{K} \int \frac{\sigma\, dS}{r},$$

the Potentials of the conductor in air and in the medium being denoted by V_a and V_m. Hence

$$V_a = K \,.\, V_m.$$

But $E = CV_a$, therefore $E = KC \,.\, V_m$, and E is the same in both cases. Hence generally

$$E = KC \,.\, V. \qquad (2)$$

The electrical capacity of a conductor is analogous to the capacity of a vessel for water. Thus, suppose a number of cylindrical vessels placed side by side on the table and let a mark be made on each of them, the marks being all at the same height. Let each be filled with water up to this height.

Then the volume of water which must be poured into any cylinder is greater the greater the area of its base; a very broad cylinder may require several litres of water, while one of very small capacity (a narrow tube) will be filled to the required height by a few drops. The height or level of the water is the analogue of V, the electrical level of the conductor, and the volume of the water is analogous to E, the electrical charge.

Again, if a conductor of capacity C_1 at Potential V_1 is connected by a wire with one of capacity C_2 at Potential V_2, both will assume a common Potential equal to

$$\frac{C_1 V_1 + C_2 V_2}{C_1 + C_2}.$$

For, the total charge to be redistributed is $C_1 V_1 + C_2 V_2$, and the capacity of the compound conductor is $C_1 + C_2$.

Hence if C_1 is vastly greater than C_2, the new Potential of both conductors is simply V_1. This is the theory of the connection of any conductor with the Earth (Art. 405). Analogy with water level: if a glass tube, filled to any height, is connected with a lake, the level of the lake is unaltered, and the level of the water in the tube becomes equal to that of the lake.

The capacity of a sphere for an idiostatic charge is obviously equal to its radius; for, if E is the charge, the Potential at the centre is $\frac{E}{a}$, $\therefore E = aV$, therefore capacity in absolute electrostatic measure = number of centimètres in the radius.

Into the discussion of the relations between the Potentials and charges of a system of conductors occupying given positions and influencing each other, we do not enter. [See Clerk Maxwell, vol. 1, pp. 100, &c.]

Examples.

1. A conductor placed in air is subject to the action of any electrified bodies; if σ is the surface-density at any point on the conductor, prove that the force exerted on the electricity of the conductor per unit area at the point is

$$2\pi\sigma^2.$$

The electromotive intensity at the point, P, considered is normal to the conductor. This force may be considered as due to the action of a very small element, dS, of area at P (forming a small uniform plate) and the remainder of the field; the first part is (Art. 318) $2\pi\sigma$; and since the force just inside P in the substance of the conductor is zero,

the electromotive intensity due to the remainder of the field is $2\pi\sigma$; just as in Art. 322. Hence the force produced on the quantity σdS by the remainder is $2\pi\sigma\times\sigma dS$, or $2\pi\sigma^2$ dynes per square centimètre if σ is measured in e.s. units per square centimètre.

This quantity, $2\pi\sigma^2$, is what Sir W. Thomson calls the *electric diminution of air pressure on the surface* (*Papers on Electrostatics and Magnetism*, p. 254), for the following reason:—each element of surface of an electrified soap-bubble being repelled by the force $2\pi\sigma^2$ per unit of surface, the bubble expands, just as it would do if the air pressure diminished, and when discharged it contracts. Hence the electric diminution of air pressure at any point of a conductor is

$$2\pi\sigma^2, \text{ or } \frac{N^2}{8\pi}, \tag{a}$$

N being the electromotive intensity produced by the whole field in the air just outside the conductor at the point.

Moreover, this force per unit area, $\frac{N^2}{8\pi}$, acts *from* the conductor towards the dielectric *whether the surface-density at the point is positive or negative.*

2. If a tube of force starts from the surface of one electrified conductor and meets the surface of another conductor, the charges on the portions of surface intercepted on the two conductors by the tube are equal and opposite.

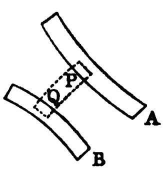

Fig. 313.

Let A and B (Fig. 313) be portions of the two conductors, and let a tube of force, of any cross section, great or small, start from any portion of A and terminate in B. Produce the tube into the substance of each conductor, and close its ends. Now since there is no force at any point on either end, the theorem of Art. 324 gives the result that the total charge inside it is zero—which proves the proposition.

If the surfaces A and B are very close to each other at P and Q, the surface-densities at P and Q are equal and opposite, since if we make the tube very narrow, the two elements of area which it intercepts are equal.

[Observe that though a tube of force may be produced into the substance of any conducting body, it there loses its function as a tube of force—no lines of force existing in the substance.]

3. Two parallel metallic plates, whose areas are very great compared with the distance between them, are kept at given Potentials, V and V'; find the charges on their adjacent surfaces.

The Potentials being measured in ergs per e.s. unit, let h centimètres be the distance between the plates, and K the specific inductive capacity of the insulator between them. Then, except near their edges, the surface-density on each will be constant; there

will be a uniform field of force between them; the value of $\frac{dV}{dn}$ at any point on the first, measured outwards towards the second, will be practically $\frac{V'-V}{h}$, and therefore the surface-density on the face of the first opposite to the second will be

$$\frac{K}{4\pi}\cdot\frac{V-V'}{h}, \tag{1}$$

so that the bound charge, E, on this plate is given by the equation

$$E=\frac{KS(V-V')}{4\pi h}, \tag{2}$$

where S is its area in square centimètres, an equal and opposite charge existing on the opposite face of the other, if their areas are equal.

This case is approximately that of a Leyden Jar, the two plates being cylinders of tinfoil, having glass as a dielectric between them.

It is also the case of an Absolute Electrometer, consisting of two large plates very close to each other, with air between them.

In this case, if a portion of one plate consists of a moveable area, or trapdoor, of area s square centimètres, since the force per unit area experienced by the surface of either plate (Ex. 1) is $2\pi\sigma^2$, the total force experienced by the trapdoor is $2\pi\sigma^2.s$ dynes, or

$$\frac{s(V-V')^2}{8\pi h^2}. \tag{3}$$

The quantity E given by (2) is only the charge on that face of the plate which is adjacent to the other plate; in addition, there is a charge on the other face of each, the quantity of which is very small compared with E.

Thus the capacity of a system of two parallel plates is

$$\frac{KS}{4\pi h},$$

so that if we wish to obtain a very large capacity, it will be advisable to use plates with a very large surface, separated by a very small distance, and to fill the space between them with a dielectric of very high specific inductive capacity.

The quantity of electricity at a given Potential, V, which can in this way be accumulated on a plate is vastly greater than the quantity, at the same Potential, which could be accumulated on it if it were merely connected with the source at Potential V—such as one pole of a given battery—without having the second plate close to it, because (see Example 6) the capacity of the system of two plates is vastly greater than that of one of them for an idiostatic charge. Thus, if the separating medium is air and each plate is a circle of radius 1 decimètre, the distance between them being 1 mm., the capacity of the system is 250 in electrostatic measure; while the capacity of either plate for an idiostatic charge is only $\frac{20}{\pi}$.

An electric 'vessel' of this kind is called an *Accumulator*. The Accumulators (called also Condensers) actually in use consist of hundreds or thousands of sheets of tinfoil, $a_1, a_2, a_3, a_4, \ldots$, each a few square decimètres in area, placed parallel to each other and each separated from the previous and following one by sheets of thin paraffined paper, the whole being pressed into a compact mass. The plates $(a_1, a_3, a_5, \ldots)$ are then all metallically connected together at a point, or pole, A, and the intermediate plates $(a_2, a_4, a_6, \ldots)$ are also metallically connected together at a point, or pole, B, and the Accumulator is filled by connecting the poles A and B with the poles of a battery or other source.

4. Find the work done in the discharge of a Leyden Jar, or of the system of parallel plates in last example.

If in any electrified system V is the Potential at any point, and de the element of charge at this point, the work of the forces of the system in completely destroying all electrification is (Art. 331) $\frac{1}{2}\int Vde$, the integration extending all through the system; and the work done in transforming it from one state to another is $\frac{1}{2}\int Vde$ in the first state $-\frac{1}{2}\int Vde$ in the second state.

In the present case $\frac{1}{2}\int Vde$ in the first state is $\frac{1}{2}VE - \frac{1}{2}V'E$, or $\frac{1}{2}(V-V')E$, or $\dfrac{KS(V-V')^2}{8\pi h}$.

The value of the integral in the second state is zero, because when the plates are connected, the contrary charges combine so that each plate is in its natural state. Hence the work is

$$\frac{KS(V-V')^2}{8\pi h}.$$

This work of dissipating the electrification would be equal to the work done by the operator in charging the Jar or the system of plates (by friction of glass plates against rubbers, for example) if none of the work of charging succeeded in passing into heat (of axles against bearings, &c.)—by the Principle of the Conservation of Energy.

5. A condenser is formed of two concentric spherical conductors of radii a and c, separated by two dielectrics bounded by a concentric sphere of radius b. Prove that if in one dielectric $K = \dfrac{\mu}{r^2}$ and in the other $K' = \dfrac{\mu'}{r^2}$, the capacity of the system is

$$\frac{\mu\mu'}{\mu'(b-a)+\mu(c-b)}.$$

(Mathematical Tripos, 1885.)

Employing Poisson's equation (3), p. 486, since $\rho = 0$ at each point in the dielectrics, we have

$$\frac{dV}{dr} = A,$$

where A is a constant which may be different for the two media.

Let V_1 be the Potential of the inner conductor, a, and σ its surface-density. Then by (β), p. 487, we have at this surface

$$\frac{\mu}{a^2}\frac{dV}{dr} = -4\pi\sigma, \quad \therefore\ A = -\frac{4\pi a^2\sigma}{\mu} = -\frac{Q}{\mu}, \tag{1}$$

if Q is the whole charge on the sphere. Integrating, we have

$$V = V_1 - \frac{Q}{\mu}(r-a). \tag{2}$$

Similarly, if σ' is the surface-density on the conductor c,

$$-\frac{\mu'}{c^2}\frac{dV}{dr} = -4\pi\sigma', \tag{3}$$

$$\therefore\ V = V_2 - \frac{Q'}{\mu'}(c-r), \tag{4}$$

where Q' is the whole charge on this sphere.

At the surface of contact of the two dielectrics we have from (a), p. 487,

$$-\frac{\mu}{b^2}\frac{dV}{dr} + \frac{\mu'}{b^2}\left(\frac{dV}{dr}\right)' = 0, \tag{5}$$

where $\frac{dV}{dr}$ is given by (1), and $\left(\frac{dV}{dr}\right)'$ by (3). This gives

$$Q + Q' = 0. \tag{6}$$

Since V is always continuous through space, the values (2) and (4) are equal when $r = b$, that is, at the surface of separation of the dielectrics.

Hence
$$V_1 - V_2 = Q\left(\frac{c-b}{\mu'} + \frac{b-a}{\mu}\right),$$

which gives for the capacity, $\frac{Q}{V_1 - V_2}$, the value in the problem.

The solution is equally simple if we assume K and K' to be any functions of r.

6. To find the capacity of a circular plate for an idiostatic charge.

Considering first an ellipsoidal conductor, $\sigma = kp = \frac{Ep}{4\pi abc}$, E being the charge on the conductor.

When this conductor becomes a plate, $c = 0$, and we must find the limiting value of $\frac{p}{c}$ for each point on the surface.

Now $\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{1}{c^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$; therefore

$$\frac{c}{p} = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

If $a = b$ the plate is circular, and if r is the distance of any point on it from the centre,

$$\frac{p}{c} = \frac{a}{\sqrt{a^2 - r^2}}; \quad \therefore\ \sigma = \frac{E}{4\pi a\sqrt{a^2 - r^2}}.$$

The Potential of the plate is the same at all points, therefore it will suffice to find its value of the centre. This is obviously

$$4\pi\int_0^a \sigma\, dr,$$

including both surfaces of the plate, for both are identically charged.

Hence $$V=\frac{\pi E}{2a}, \quad \therefore \quad C=\frac{2a}{\pi}.$$

The result could have been deduced from the Potential of an Ellipsoidal Shell. If the semi-axes, in descending order of magnitude, are a, b, c, the Potential, V, at the centre (and therefore on the surface) is given by the equation

$$V=E\int_0^{\omega}\frac{d\theta}{\sqrt{a^2-c^2-(a^2-b^2)\sin^2\theta}},$$

where $\omega=\sin^{-1}\dfrac{\sqrt{a^2-c^2}}{a}$.

The capacity of the ellipsoid is the reciprocal of the coefficient of E.

For an elliptical plate, put $c=0$, and if C is its capacity, we have

$$\frac{1}{C}=\frac{1}{a}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}},$$

e being the excentricity of the ellipse.

We may observe that if the ellipsoid is of revolution round the axis a, and if $l=\sqrt{a^2-c^2}=$ distance of centre from focus of generating ellipse,

$$V=\frac{E}{l}\log_e\frac{a+l}{a-l},$$

which is precisely the Potential due to a uniform bar stretching between the foci (see p. 295, and next Article).

7. A sphere receives a charge E, and it is surrounded by a spherical shell to which a charge E' is given; find the Potentials on the two surfaces.

Let a and b be the radii of the sphere and shell. Inside the sphere (whether it is hollow or solid) the Potential is constant (Art. 407), and therefore equal to its value at the centre. The electrified inner sphere will act inductively on the shell, generating a charge $=-E$ on the inner, and a charge $=E$ on the outer, surface of the shell. Hence if the surfaces are separated by air, the Potential, V, of the whole field at the centre is given by

$$V=\frac{E}{a}+\frac{E'}{b}. \tag{1}$$

Now take any point on the shell. The Potential of the sphere at this point is (p. 298) $\frac{E}{b}$, and the Potential due to the shell itself is $\frac{E'}{b}$. Hence if V' is the Potential of the shell,

$$V' = \frac{E+E'}{b}. \tag{2}$$

Of course if it is the Potentials that are given—the inner sphere being connected with one pole, and the outer with the other pole of a battery—and the charges to be determined, these charges satisfy (1) and (2), and are deduced from these equations.

We have
$$E = \frac{ab}{b-a}(V-V')$$
for the charge on the inner, so that this charge can be made very large by diminishing $b-a$. Thus, the condenser might be made by coating the inside and outside of a very thin indiarubber ball with some conducting substance, and then blowing the ball out to a great size; or the inside might be filled with acidulated water and the outside coated with a conducting layer; and the capacity would be $K\frac{ab}{b-a}$, where K is the specific inductive capacity of the indiarubber.

The tubes of force between the two surfaces exist in the separating dielectric and cease on the outer surface of the inner sphere and the inner surface of the outer; and, in accordance with Example 2, the quantities at the ends of these tubes are equal and opposite. From the outer surface of the outer sphere new tubes of force proceed outwards into the external air; and from the inner surface of the inner no tubes of force proceed in any direction—illustrating the principle that no lines of force exist in the substance of any conducting body.

8. Find the capacity of a very long cylindrical conductor for an idiostatic charge.

Except near the ends, the surface-density of the charge will be sensibly constant. Let it be σ. Also the Potential is constant everywhere inside (whether the cylinder is solid or hollow) and therefore equal to its value at the middle point, O, of the axis.

Take this point as origin, and consider the electrified ring cut off from the cylinder by a plane at a distance x and one at a distance $x+dx$ from the centre, both being perpendicular to the axis. If r is the radius of the cylinder the Potential of this ring at O is $\frac{2\pi\rho\sigma\,dx}{\sqrt{r^2+x^2}}$; therefore if l is the length of the cylinder,

$$V = 4\pi\sigma r \log_e \frac{l}{r},$$

l being assumed to be very much greater than r.

The total charge, E, on the cylinder is $2\pi rl\sigma$; hence

$$V = \frac{2E}{l}\log_e\frac{l}{r}, \quad \therefore \quad C = \frac{l}{2\log_e\frac{l}{r}}.$$

Hence if r is exceedingly small in comparison with l, C is practically zero. The cylinder in this case is simply a wire; so that when a wire is used to connect two electrified conductors, we may neglect the portion of the charge which is taken by the wire.

Consider the case in which a very long cylindrical conductor (as, for instance, a wire or strand of wires) is surrounded by a dielectric (as gutta-percha) which also forms a cylinder, this dielectric, again, being surrounded by another conductor (the ocean). This is obviously the case of an electric cable, in which the outer conductor is always at zero Potential.

It is required to find the capacity of this system for a statical charge, the inner conductor being kept at Potential V_1 and the outer at V_2.

We may find V at any point in the dielectric by the equation in cylindrical co-ordinates (Art. 329) which is simply

$$\frac{1}{\zeta}\frac{dV}{d\zeta} = m = \text{constant};$$

$$\therefore \quad V = m\log\zeta + m';$$

and since $V = V_1$ when $\zeta = r$, and $V = V_2$ when $\zeta = R =$ radius of outer cylinder, m and m' are found. Hence

$$V = V_2 + \frac{V_1 - V_2}{\log_e\frac{R}{r}}\cdot\log_e\frac{R}{\zeta}.$$

At the surface of the inner $\sigma = -\dfrac{K}{4\pi}\dfrac{dV}{d\zeta}$, when $\zeta = r$; therefore

$$\sigma = \frac{K(V_1 - V_2)}{4\pi r\log_e\frac{R}{r}}.$$

The charge on a length l of this cylinder is, then,

$$\frac{Kl}{2\log_e\frac{R}{r}}(V_1 - V_2).$$

9. Calculate the surface-tension of an electrified soap-bubble.

When a membrane is acted on by forces of any kind, there will be along every line traced on the membrane a tendency of the two portions separated by this line to tear away from each other; in other words, one of these portions exercises on the other a set of internal forces along the line of separation.

In the neighbourhood of any point P of the membrane (Fig. 314) consider a very small rectangular portion, $ABCD$, of the membrane isolated from the remainder. Then there will be forces exerted on its sides at their middle points, m, m', n, n', by the removed portion. These forces will, if the rectangle $ABCD$ is chosen at random, be oblique to its sides; but it is always possible to choose the rectangle at P so that these forces are at right angles to the sides on which they act. Suppose this done. The amount of force exerted on AB is, of course, proportional to the length AB; so that if t_1 is the amount exerted on AB per unit of length, the force at m in the sense $m'm$ is $t_1 \times AB$. Similarly, if t_2 is the force per unit of length on AD, the force on AD is $t_2 \times AD$. The quantities t_1 and t_2 are the *surface-tensions* at P perpendicular to AB and AD.

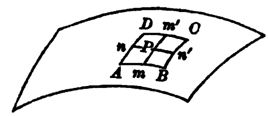

Fig. 314.

For the equilibrium of the rectangle resolve forces along the normal to its plane at P. Then, if r_1 and r_2 are the radii of curvature of the curves mm' and nn', and N the amount of external normal force exerted at P per unit area, we have

$$N.AB \times AD = t_1.AB\frac{mm'}{r_1} + t_2.AD\frac{nn'}{r_2},$$

or

$$N = \frac{t_1}{r_1} + \frac{t_2}{r_2}.$$

In a soap-bubble t_1 and t_2 are evidently equal, and this equation becomes

$$N = \frac{2t}{r},$$

where t is its surface-tension and r its radius.

Now in an electrified bubble N consists of two parts—one an excess of air pressure inside over the air pressure outside, and the other the repulsion of the electricity on itself (Ex. 1). Denote by p the intensity of the *excess* of air pressure and by σ the electrical density at P, and we have

$$p + 2\pi\sigma^2 = \frac{2t}{r}.$$

10. A spherical soap-bubble is electrified in such a manner that the internal pressure remains constant; find the relation between the densities of electrification when its volume has become n and m times its original value.

(Mr. Orchard, in the *Educational Times.*)

The external pressure presumably remaining constant, there will be a constant excess of pressure, p. Equate the work done by this pressure in enlarging the volume to the potential work of the electrification. Now if v is the volume of the bubble at any instant, the work done by the pressure in altering the volume by dv is pdv. Hence if $\Omega =$ original volume, the work done in the first electrification must be

$$p(n-1)\Omega.$$

But if V is the potential of this electrification and Q the charge, the energy of the electrification is (p. 286) $\frac{1}{2}VQ$; and evidently $V=\dfrac{Q}{r}$, if r is the radius of the bubble. Also if σ is the density,

$$Q=4\pi r^2\sigma;$$

$$\therefore\ p(n-1)\Omega=8\pi^2 r^3\sigma^2.$$

But $$\tfrac{4}{3}\pi r^3=n\Omega;$$

$$\therefore\ p(n-1)=6n\pi\sigma^2.$$

Similarly, $$p(m-1)=6m\pi\sigma'^2,$$

if σ' is the density of the second electrification,

$$\therefore\ \frac{\sigma}{\sigma'}=\sqrt{\frac{m(n-1)}{n(m-1)}}.$$

11. A spherical soap-bubble is electrified in such a manner that it is just in equilibrium when the pressures of the external and internal air are equal. Calculate the surface-tension in terms of the potential. (Mr. Orchard, *Educational Times.*)

Ans. $t=\dfrac{V^2}{16\pi r}$.

12. Two spherical soap-bubbles are caused to unite into a single spherical one. Show that a diminution of surface takes place, and calculate the charge of electricity which must be given to the single bubble in order to draw out the film to its former superficial extent. (Mr. Greenhill, *Tripos*, 1875.)

Let T be the surface-tension, in dynes per cm., of the water-air surface at the temperature existing during the process (see p. 177); let ϖ be the intensity of atmospheric pressure in dynes per square cm.; let a and b be the radii of the bubbles in cm.; p and p' the intensities of pressure of the air inside them; P the intensity of air pressure inside the compound bubble, and x its radius.

Then we have

$$p-\varpi=\frac{2T}{a};\ p'-\varpi=\frac{2T}{b};\ P-\varpi=\frac{2T}{x}.$$

Also, assuming no change of temperature, p changes to $p\dfrac{a^3}{x^3}$ and p' to $p'\dfrac{b^3}{x^3}$, in the compound bubble; and $P=$ the sum of these.

Hence $Px^3=pa^3+p'b^3$; so that we have

$$\varpi(x^3-a^3-b^3)+2T(x^2-a^2-b^2)=0, \qquad (a)$$

to determine the radius of the compound bubble.

Now ϖ is vastly greater than T; for, an intensity of pressure denoted by a barometer of the normal height, 760 mm., is $1{\cdot}014\times10^6$ dynes per sq. cm., while T for the temperature 20°C. is about 81 dynes per linear cm. Hence the value of x which annuls the coefficient

of ϖ in (a) must be very nearly the value which satifies the equation. Assume, therefore,

$$x=(a^3+b^3)^{\frac{1}{3}}+m,$$

where m is a small quantity. From (a) we get the approximate value

$$m=\frac{2T}{3\varpi}\left\{\frac{a^2+b^2}{(a^3+b^3)^{\frac{2}{3}}}-1\right\}, \qquad (\beta)$$

which is obviously a positive quantity. Hence $x^3>a^3+b^3$, that is, the volume of the compound bubble is greater than the sum of the volumes of the constituents. Also we find easily that $x^2<a^2+b^2$, which shows that the reverse is the case for the surfaces.

Now let a charge with surface-density σ e. s. units per sq. cm. be given to the compound surface so as to make $x^2=a^2+b^2$; then, if P' is the enclosed air pressure intensity, $P'+2\pi\sigma^2-\varpi=\frac{2T}{x}$, and we have

$$(\varpi-2\pi\sigma^2)x^3=\varpi(a^3+b^3),$$

which gives σ, since x is given to be $\sqrt{a^2+b^2}$.

13. When conductors A_1, A_2,... of given shapes occupy given relative positions, and their Potentials V_1, V_2,... are given, their charges Q_1, Q_2,... are determined from a system of linear equations,

$$Q_1=q_{11}V_1+q_{12}V_2+q_{13}V_3+\ldots$$
$$Q_2=q_{21}V_1+q_{22}V_2+q_{23}V_3+\ldots.$$

Apply the theorem of Gauss (p. 287) to prove that $q_{21}=q_{12}$, or, generally, $q_{mn}=q_{nm}$.

Since the coefficients q_{11}, q_{12},... depend merely on the shapes and relative positions, we may consider the case in which A_1 is connected with a source at a given Potential, P, all the others being connected with earth, i.e. at zero Potential. Then the charges will be given by the equations

$$Q_1=q_{11}P;\ \ Q_2=q_{21}P;\ \ Q_3=q_{31}P\ldots.$$

Denote this distribution by (a).

Now reduce them all to the natural state, and produce another state, (a'), by raising A_2 to the Potential P, and connecting all the others to earth. The charges now are

$$Q_1'=q_{12}P;\ \ Q_2'=q_{22}P;\ \ Q_3'=q_{32}P\ldots.$$

Now apply Gauss's equation $\Sigma mV'=\Sigma m'V$ to these two states or systems, (a) and (a'). The left-hand side of this equation is the product of each charge in (a) and the Potential at its position produced by (a'). It is, then, simply $Q_2 . P$, or

$$q_{21} . P^2.$$

The right-hand side is simply $Q_1' . P$, or

$$q_{12} . P^2.$$

Hence $q_{21}=q_{12}$. Similarly for the other q's with two distinct suffixes, which are called the coefficients of induction of the given system.

This simple proof is due to M. Bertrand (see Mascart and Joubert's *Leçons sur l'Électricité et le Magnétisme*, p. 56).

14. If a given charge, e, is held at a point P outside a given conductor which has no free charge, and which may be insulated or not, and if P' is any other point outside the conductor, prove that the Potential at P' produced by the induced charge on the conductor is the same as that which would be produced at P by the conductor if the charge e were placed at P'.

This is also, as shown by M. Bertrand, a simple result of the equation of Gauss.

Suppose the conductor connected with Earth, e being at P. Let Q be the charge induced on the conductor, V the Potential at P, and $G_{pp'}$ that at P'. Denote this system by (a). Remove e to P', let Q' be the new induced charge on the conductor, V' the Potential at P', and $G_{p'p}$ that at P. Denote this state by (a').

Now apply the equation $\Sigma m V' = \Sigma m' V$ to the two states (a) and (a'). The left-hand side is simply $e \times G_{p'p}$, while the right is $e \times G_{pp'}$. Hence

$$G_{pp'} = G_{p'p}.$$

The function $G_{pp'}$ is known as Green's Function.

Exactly the same proof applies if the conductor is insulated. In this case its total charge is zero.

15. Show that an idiostatic charge on a conductor must be of the same sign at all points.

16. Several Accumulators ('Condensers'), $A_1, A_2, \ldots A_n$, of capacities $c_1, c_2, \ldots c_n$ are placed in series; a pole of A_1 and a pole of A_n are connected with the poles of a battery which are at Potentials V and V'; find the charge in any given Accumulator of the series.

Let (V, x_1) be the Potentials of the poles of A_1; (x_1, x_2) those of the poles of A_2; and so on. Then

$$\frac{V-x_1}{\frac{1}{c_1}} = \frac{x_1-x_2}{\frac{1}{c_2}} = \ldots = \frac{x_{n-1}-V'}{\frac{1}{c_n}};$$

each of which is, therefore, equal to

$$\frac{V-V'}{\Sigma\frac{1}{c}}, \text{ or } k, \text{ suppose.}$$

Then the difference of Potential of the poles of A_r is $\frac{k}{c_r}$, and the charge is the same in all, viz. k.

412.] **Case of Green's Equation.** Let M and M' (Fig. 315) be any system of masses gravitating according to the law of the inverse square of distance (the case in which they are electrical distributions is, of course, included); let V be their Potential at

any point, P; and in Green's Equation choose for U the function $\frac{1}{r}$, where r is the distance of P from any fixed point, O. Now draw any closed surface, S, surrounding M', and apply equation (β), p. 334, to the volume enclosed by this surface, the point O being any point *outside* this surface. The points P included in the integration being all internal to S, or on its surface, r is never zero, therefore $\nabla^2 \frac{1}{r} = 0$.

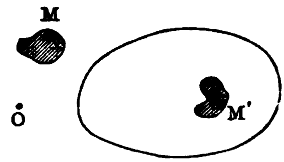

Fig. 315.

Also $\nabla^2 V = -4\pi\rho$, where ρ is the volume-density at P, which will be zero except for points inside the mass M'. Thus the equation becomes

$$-4\pi\int\frac{\rho\, d\Omega}{r} = \int\frac{1}{r}\frac{dV}{dn}\, dS - \int V\frac{d\frac{1}{r}}{dn}\, dS. \qquad (1)$$

Now $\int\frac{\rho\, d\Omega}{r}$ is the Potential at O due to the mass M', and this we shall denote by $V_e^{(i)}$. Also

$$\frac{d\frac{1}{r}}{dn} dS = -\frac{\cos\theta}{r^2}\, dS,$$

where θ is the angle made with the normal to the surface at any point by the line joining this point to O; and this expression is therefore equal to $d\omega$ (Art. 316) where $d\omega$ is the conical angle subtended at O by dS.

If the matter is self-repulsive, instead of attractive—i.e. if the force between two elements of the same sign is repulsive—the normal force-intensity at the position of dS due to the whole system M, M', is $-\frac{dV}{dn}$, so that (1) gives

$$V_e^{(i)} = \frac{1}{4\pi}\int\left(N - \frac{V\cos\theta}{r}\right)\frac{dS}{r}. \qquad (2)$$

Now the right-hand side of (2) is the Potential at O due to a layer of attracting matter spread over S with surface-density equal to

$$\frac{1}{4\pi}\left(N - \frac{V\cos\theta}{r}\right). \qquad (a)$$

Hence *the Potential at any point, O, due to M' may be produced*

by a layer spread over ANY *surface surrounding M' and having O outside it*; so that if a layer is spread over the surface equal, but with opposite sign, to (a), the attraction of this layer would annul that of M' at all points outside the surface.

A case of special simplicity and importance occurs when S is an equipotential, or level, surface of the system M, M'. In this case V comes outside the integral, and (2) becomes

$$V_e^{(i)} = \frac{1}{4\pi}\int N dS - V\int d\omega$$

$$= \frac{1}{4\pi}\int N dS, \tag{3}$$

so that the density of the layer at any point is

$$\frac{N}{4\pi}. \tag{4}$$

Denote the Potential produced by this layer at any external point, O, by ϕ_e; then $\quad V_e^{(i)} = \phi_e. \qquad (5)$

If the matter is self-attractive, instead of repulsive, $N = +\dfrac{dV}{dn}$, and the layer in question has surface-density equal to $-\dfrac{N}{4\pi}$.

If M' is any system of electrical charges, we see that, so far as effect outside any closed surface whatever, S, surrounding M' is concerned, M' may be spread as a surface-charge over the given surface, with the law of surface-density given by (a). But if this layer were actually produced, and all the other electricity of the field removed, it would not, in general, be self-equilibrating, unless the surface S is a non-conducting surface.

Secondly, suppose O to be internal to the surface S. Then, if P actually coincided with O, $\nabla^2\frac{1}{r}$ would not vanish; but we shall in this case exclude the point O by applying Green's equation to the region included between S and an indefinitely small sphere of radius, c, surrounding O and having O for centre (see p. 334).

Thus we have

$$-4\pi\int\frac{\rho d\Omega}{r} = \int\frac{1}{r}\frac{dV}{dn}dS - \int V\frac{d\frac{1}{r}}{dn}dS$$

$$-\int\frac{1}{c}\frac{dV}{dn}dS' + \int V\frac{d\frac{1}{r}}{dn}dS', \tag{6}$$

in which the last two integrals refer to the surface of the small sphere round O. Now the third integral in this equation vanishes, because $dS' = c^2 d\mu d\phi$, μ and ϕ being the polar angular co-ordinates of any point on the surface of the sphere; so that this integral is $-c\int\frac{dV}{dn}d\mu d\phi$, which obviously is infinitesimal. In the last integral V may be taken outside, since it is practically constant at all points on the infinitely small sphere and equal to its value at O—which we shall denote by V_i—this Potential being due to both M and M'. Also

$$\int\frac{d\frac{1}{r}}{dn}dS' = -4\pi.$$

Hence (6) becomes (for self-repulsive matter)

$$-4\pi V_i^{(i)} = -\int\frac{NdS}{r} + \int\frac{V\cos\theta}{r^2}dS - 4\pi V_i, \tag{7}$$

in which $V_i^{(i)}$ denotes the Potential at the internal point O due to the internal mass, M'. But if $V_i^{(e)}$ denotes the Potential at O due to the external mass, M, we have $V_i = V_i^{(e)} + V_i^{(i)}$; therefore

$$V_i^{(e)} = \frac{1}{4\pi}\int\left(\frac{V\cos\theta}{r} - N\right)\frac{dS}{r}. \tag{8}$$

Let S be a level surface of the systems M, M', and A the Potential on it; then

$$V_i^{(e)} = A - \frac{1}{4\pi}\int\frac{NdS}{r}; \tag{9}$$

which shows that if a layer with surface-density equal to $\frac{N}{4\pi}$ at each point is spread over the surface, and ϕ_i is the Potential produced by it at any internal point, O,

$$V_i^{(e)} = A - \phi_i. \tag{10}$$

Equation (5) gives the result that, so far as all points external to the given level surface are concerned, the internal mass, M', may be replaced by the layer on the surface; and, of course, if the surface-density of this layer were reversed at every point, we should obtain a layer whose action at all external points would exactly annul that of the internal mass.

Equation (10) shows that this layer destroys the effect of the external mass, M, at all points inside the level surface, since

$\frac{d}{dx}V_i^{(e)} = -\frac{d\phi}{dx}$, in whatever direction x is measured; and, moreover, if there are no external masses, this layer produces a constant Potential inside, and therefore no force at any internal point; for in case $V_i^{(e)} = 0$, therefore $\phi_i = A$.

The quantity of this superficial layer is equal to that of the internal mass, since (Art. 324) $\int N dS = 4\pi M'$, for repulsive forces, and $\int N dS = -4\pi M'$ for attractive forces.

Thus, for example, consider the case of a thin uniform bar (p. 296). The level surfaces are a system of confocal ellipsoids of revolution, and $N = -\frac{2\gamma k\rho}{y}\sin\frac{\psi}{2}$, where $\psi = APB$, and y the perpendicular from P on AB; and the bar may be spread over any one of them with surface-density $= \frac{\gamma k \rho}{2\pi y}\sin\frac{\psi}{2}$, with the result that at all points outside this surface the attraction of the layer is the same as that of the bar, while it produces constant Potential throughout the interior. When, as in this instance, there is no external mass, the layer on the surface follows exactly the law of an idiostatic distribution of electricity on the surface. Hence, as the equipotential surface of the bar is an ellipsoid, and as we know (Art. 410) that in an idiostatic layer the surface-density at any point is directly proportional to the central perpendicular on the tangent plane at the point, it follows from the uniqueness of the law of idiostatic distribution Art. 409) that $\frac{\gamma k\rho}{2\pi y}\sin\frac{\psi}{2}$ must be proportional to this perpendicular. The truth of this is easily verified, and we find this expression equal to

$$\frac{\gamma c k \rho}{2\pi a b^2}\cdot p,$$

where p is the central perpendicular, a and b are the semi-axes of the ellipse, and $c = \sqrt{a^2 - b^2}$.

413.] **Application to Conductors.** The results in the last Article have a direct and important application to the case of conductors. We shall now speak of charges of electricity instead of 'masses.' Take the case of a single hollow conductor, and let Fig. 311, p. 492, represent the inner and outer surface charges on the conductor, and also the external and included charges.

Then the surface S in the substance of the conductor is a level surface for these four electrical distributions. At every point on S we have $N = 0$, so that no layer is to be spread over S, and (5) of last Article shows that at every point outside S, the included charges together with the inner surface charge produce a constant zero Potential; or, in other words, this system produces no force anywhere outside the conductor.

Also from (10) we see that the Potential everywhere inside due to the external charges together with the outer surface charge is constant and equal to that on the conductor; consequently this system produces no force anywhere inside.

In other words, the internal charges together with the inner surface charge on the conductor form a self-equilibrating system producing no external effect whatever; and the external charges together with the outer surface charge form a self-equilibrating system producing no internal *force*, the only internal effect which they produce being to establish a uniform Potential throughout the whole interior, equal to that of the conductor—so that if the conductor were kept constantly at zero Potential (by an earth connection) this electrical system, however it were varied, would produce no internal effect whatever (either of force or of Potential). Thus we have proved the results referred to in Art. 410.

A conductor is thus shown to act as an *electrical screen* for everything inside it from the action of external charges; and this is why delicate instruments, such as electrometers, are, in accurate experiments, surrounded by cages of wire-gauze, which are practically closed surfaces. These cages are usually connected with earth in order to prevent external electrifications, whether permanent or transitory (such as are due to accidental rubbings of insulators or other bodies), from even uniformly altering the Potential of the protected instruments*.

414]. **Theory of Electric Images.** *Given an insulated conductor and an external electrified system, to determine the surface-density of the induced charge at any point of the conductor.*

* From the preceding theory it will be seen that for the complete protection of such instruments from external influence it is necessary that a *conducting* surface should immediately separate them from the external bodies which are, or may be accidentally, electrified. Hence it is a mistake to surround them, as is often done, by glass, which is a non-conductor.

Let M (Fig. 315), represent the given electrified system, and then let the closed surface represented be that of the insulated conductor. Denote it by S. Now if the charge on M is E, the total charge on M and S is E; and, with the positions of M and S fixed, there is only one way in which this charge can be distributed (Art. 409); so that it will suffice to find *any* manner in which the charge E can be distributed so as to be in equilibrium.

But one method is the following: determine any internal electrified system, M', which with M would make S a level surface, with any constant Potential, A, suppose; then spread the charge, E', of the system, M', over the surface according to the law $\sigma = \frac{N}{4\pi}$, where N is the electromotive intensity due to E and E' at any point of S. This surface charge together with E will produce the constant Potential A on S and throughout its interior, since their joint Potential inside is $V_i^{(e)} + \phi_i$, and, as before proved,

$$V_i^{(e)} + \phi_i = A,$$

and the amount of this surface charge is E'.

Hence this surface charge together with E would not produce force causing any *further* charge given to S to move along S, so that if this further charge was idiostatic, it would still be in equilibrium. Let, then, this further charge be equal to $-E'$, so that the total charge on $S = 0$, and we shall have succeeded in distributing the total amount E as required.

This solution of the problem requires, therefore, two distinct things—

1. The determination of the auxiliary system E';

2. The determination of the manner of distributing an idiostatic charge on the conductor.

Each of these subsidiary problems is, in general, one of great difficulty, and they have been solved in only some simple cases.

The system M' is called the electrical *image* of M in the given surface S, and this *method of images* is due to Sir W. Thomson. Usually M' is determined so that S shall be a surface of zero potential.

If C is the capacity of S for an idiostatic charge, the Potential of the conductor will be finally $\frac{E}{C}$, or $A + \frac{E}{C}$, according as the

image system, M', is determined so that S is a surface of Potential zero or of Potential A.

The image of a given electrical system external to a conductor may be briefly defined as an internal system, such that for the two systems together the conductor is a surface of constant Potential.

For clearness, we give formally a statement of the method of images, showing the precise mode and order of thought in the process.

Given a closed conductor in presence of any electrical system, M, either inside or outside it; to determine the law of distribution of the induced charge. Regard, at first, the surface, S, of the conductor as a mere geometrical surface—not a metallic one; find any electrical system, M', at the side opposite to that at which M is placed, in such a way that S would be an equipotential surface for M and M' together; calculate N, the normal electromotive intensity at any point on the surface (due to M and M'); finally, dispense with the system M', make S a metal surface, and cover it with a charge having at each point the surface-density $\frac{N}{4\pi}$.

An idiostatic charge may have to be superposed on this to satisfy any condition as to the total charge or the Potential which the conductor may have had assigned to it originally.

415.] **Combination with Inversion.** If in any case we have deduced a law of distribution of charge on any surface or surfaces by the method of images, we can immediately deduce from this another possible distribution by the method of inversion (see Art. 334).

Assuming the thin shells treated of in Art. 334 to become surface electrifications, the products $\rho\tau$ and $\rho'\tau'$ become the given and the derived surface-densities, σ and σ'; and therefore

$$\sigma' = \frac{k^3}{r^3}\sigma.$$

It is necessary to point out, however, that in applying the method of inversion to an electrified conductor there is introduced, by the peculiar nature of a conductor, a condition of which the general theory of inversion for fixed masses takes no cognisance—the condition, namely, that not only is the given surface M (Fig. 283), p. 311) one of constant Potential (being a conductor), but the derived surface, M', if it is to be the surface of a conductor,

must also be one of constant Potential. Now in the general theory the inverse of a surface of constant Potential in the old distribution will not be a surface of constant Potential in the new distribution; for if the point A (p. 311) is any point on M, the value of D will vary (unless M is a sphere and O its centre), so that if V is constant, V' will not be constant. However, if the old distribution is such as to make M a surface of *zero* Potential, the inverse surface will be equipotential, also with zero Potential. Hence a conductor at zero Potential, under the influence of any distribution, will always invert into a conductor at zero Potential, under the influence of the inverse distribution.

If M is a conductor at any Potential, c, the inverse, M', will be a surface, the Potential at any point, Q', of which is $\frac{kc}{OQ'}$, which is the Potential at Q' due to a charge kc placed at O, the origin of inversion. Hence, if we place a charge $-kc$ at O, in addition to the inverse of the whole original field, the inverse of the given conductor becomes a conductor at Potential zero.

Sometimes it is desirable to choose the origin of inversion at the position of an electrified point, at which there is a finite charge e. The inverse point, at which there is a new charge in the new system, is then at infinity, and the new charge, e', is of infinite amount, because $e' = \frac{k}{r} e$ where $r = 0$. But it is easy to show that this infinite and infinitely distant charge produces a *finite Potential* of constant value at all points in the new distribution. For in Fig. 283, p. 311, let there be in the old distribution a charge e at a point B indefinitely close to O; then $e' = \frac{k}{OB} e$, and the Potential of e' at O is

$$\frac{e'}{OB'}, \quad \text{or} \quad \frac{k}{OB \cdot OB'} \cdot e, \quad \text{or} \quad \frac{e}{k},$$

which is finite, and is the Potential produced by e' at all points (not infinitely distant) in the new distribution, since it evidently produces the same Potential at all such points as at O. Hence, if the original conductor, M, was at zero Potential, and we invert from any point at which there is a finite charge, the Potential of the new conductor due to everything in the new distribution, omitting the infinitely distant charge, is $-\frac{e}{k}$.

Examples of Images and Inversion.

1. A given charge is condensed at a point outside an insulated uncharged spherical conductor; find the surface-density at any point of the conductor and its Potential.

Let P (Fig. 277, p. 257) be the position of the given charge e; then the sphere can be made a surface of zero Potential for e and for a charge $-e'$ placed at the inverse point, P', if the distances, r, r', of any point on the sphere from P and P' are connected by the equation

$$\frac{e}{r}-\frac{e'}{r'}=0,$$

which gives

$$e'=\frac{a}{D}\cdot e.$$

To find N, or $-\frac{dV}{dn}$, at Q, we may either take the value of V at any point, due to $(e, -e')$, viz.

$$V=\frac{e}{r}-\frac{e'}{r'},$$

and differentiate it; or imagine a $+$unit at Q and find the resultant of a force $\frac{e}{r^2}$ acting on it from P to Q and a force $\frac{e'}{r'^2}$ from Q to P'. Thus $N=-\frac{(D^2-a^2)e}{ar^3}$ (measured outwards); therefore the surface-density of the first charge (that which would be induced on the conductor if, in presence of e, it were connected with earth) is $-\frac{(D^2-a^2)e}{4\pi ar^3}$. On this we have to superpose an idiostatic charge of amount e'; and this is, of course, a uniformly distributed layer, with surface-density $\frac{e}{4\pi aD}$. Hence, finally,

$$\sigma=\frac{e}{4\pi a}\left(\frac{1}{D}-\frac{D^2-a^2}{r^3}\right),$$

$$V=\frac{e}{D},$$

where V is the Potential of the conductor.

The Potential at any point in space outside the conductor is that due to a charge e at P, a charge $-e'$ at P', and the idiostatic charge on the conductor. It is therefore $\frac{e}{r}-\frac{e'}{r'}+\frac{e'}{r''}$, where r, r', r'' are the distances of the point considered from P, P', and O.

The case of the spherical shell discussed in example 8, p. 301 is therefore that of a spherical conductor connected with earth and influenced by a charge fixed at an external point P.

If the influencing charge is internal, at P', the problem is solved in exactly the same way; the image is a charge $\frac{D}{a}e'$, or $\frac{a}{D'}e$, at P.

2. Find the distribution induced on an infinite plane conducting surface by a charge condensed at a given point.

The plane may be regarded as a closed surface—closed by a portion of surface at infinity.

Let P be the inducing point. Then for zero Potential on the plane the image of P is a point equidistant from the plane on the opposite side, and on the perpendicular from P; also the charge at P' must $= -e$, if e is that at P.

Since the capacity of an infinite plane is infinitely great, the surface-density of the idiostatic charge is infinitely small at each point, so that the distribution is simply that of a negative charge (if e is $+$) on the face next P with surface-density

$$\sigma = -\frac{p \cdot e}{2\pi r^3},$$

where p is the perpendicular from P on the plane.

It is easy to verify that the total amount of this charge is $-e$.

This problem can be at once deduced from Example 1 by inversion. The inverse of a sphere from any point on it is a plane. Take then the sphere in last example at zero Potential; in other words, take a charge e at P, and a layer with surface-density $= -\frac{D^2-a^2}{4\pi a r^3} \cdot e$ on the sphere, and invert the whole from the extremity A of the diameter through P. The charge to be placed at the inverse of P is $\frac{k}{AP} \cdot e$ (from the equation $\frac{dm'}{dm} = \frac{k}{r}$, Art. 334), and the surface-density at any point on the plane is inversely as the cube of its distance from this inverse point.

3. Two infinite plane conducting surfaces terminate in a common edge and intersect at right angles; an electrified point is placed anywhere between them; find the induced surface-density at any point on either plane.

Regard the planes as mere geometrical surfaces.

Through the inducing point, P, draw a plane at right angles to both planes cutting them in two lines Ox, Oy, the point O being on the edge of intersection of the two given planes. To determine the image-system draw a perpendicular from P on Ox and take the point A on this perpendicular at the opposite side of the plane Ox at a distance equal to that of P from Ox.

Then if e is the charge at P, a charge $-e$ at A combined with e at P would make the plane Ox a surface of zero Potential; but these charges would not make Oy a surface of zero Potential. Take the image, B, of P in Oy, and also the image, C, of B in Ox. The point C is also the image of A in Oy.

Then, if at A, B, C we place charges $-e$, $-e$, e, these together with e at P will make both planes surfaces of zero Potential, since the Potential at any point whatever, Q, due to this system would be

$$\frac{e}{r}-\frac{e}{r'}-\frac{e}{r''}+\frac{e}{r'''}, \tag{1}$$

where r, r', r'', r''' are the distances of Q from P, A, B, C.

The two given planes may be regarded as a surface closed by a surface at infinity; P is inside this surface, and the image-system outside it.

Then if α and β are the perpendiculars from P on Ox and Oy, respectively, we have for any point, Q, on Ox

$$N=2\alpha e\left(\frac{1}{QP^3}-\frac{1}{QC^3}\right); \tag{2}$$

and for any point, Q, on Oy

$$N=2\beta e\left(\frac{1}{QP^3}-\frac{1}{QC^3}\right). \tag{3}$$

Now regard the planes as metallic, dispense with the image-charges at A, B, C, and we shall have an induced charge with surface-density $\frac{N}{\sigma}$ at each point.

From this we can derive a new distribution by inversion from any point. Suppose that we invert the whole system from P. Then the inverse of the plane Ox is a sphere whose centre, a, is on PA, this centre being the inverse of A; the inverse of the plane Oy is a sphere whose centre, b, is the inverse of B; the inverse of C is the point, c, in which the line ab cuts PC, these lines being perpendicular to each other; the two spheres intersect at right angles, and PC passes through their intersection. If their radii are a and b,

$$\alpha=\frac{k^2}{2a}, \quad \beta=\frac{k^2}{2b};$$

and if e_1, e_2, e_3 are the charges to be placed at a, b, c, respectively,

$$e_1=-\frac{a}{k}e; \quad e_2=-\frac{b}{k}e; \quad e_3=\frac{ab}{k\sqrt{a^2+b^2}}e.$$

Again, if Q is any point on Ox, and q the corresponding point on the sphere (a), the surface-density, σ', at q is, from (1), equal to

$$\frac{\alpha e}{2\pi}\left(\frac{1}{PQ^3}-\frac{1}{CQ^3}\right)\frac{k^3}{Pq^3},$$

which, when expressed entirely in terms of the new distribution, is $-\frac{e_1}{4\pi a^2}\left(1-\frac{Pc^3}{cq^3}\right)$. But b, c are obviously inverse points with respect to the sphere (a), and a, c are inverse with respect to (b), so that $cq=\frac{a}{\sqrt{a^2+b^2}}.\,bq$, and we have

$$\sigma=-\frac{e_2}{4\pi ab}\left(1-\frac{b^3}{r_2^3}\right),$$

for the density at any point, q, on the sphere (a), where r_2 denotes the distance of this point from b, the centre of the other sphere.

A similar expression gives the density at any point on the other sphere.

We may, of course, reverse all signs in the new distribution, and take e_1 and e_2 plus while e is minus.

It is merely for the purpose of exemplifying the process of inversion that we have thus solved this case of two orthogonal spheres. Obviously the problem could have been solved much more simply as a case of Example 1. For, since b and c are inverse points with respect to the sphere (a), proper charges placed at them, together with any charge whatever at a, the centre of this sphere, will make the surface equipotential. Similarly, proper charges at a and c, with any charge at b, will make the other surface equipotential. Thus, let these unknown charges be x, y, z at a, b, c, respectively; let c denote the distance ab. Then (Ex. 1) $z = -\frac{b}{c}x$, and also $z = -\frac{a}{c}y$, therefore $x : y : z = a : b : -\frac{a \cdot b}{c}$; and the Potential of both spheres is $\frac{x}{a}$, as we see by calculating it for the point P.

Dispensing with the image-system, at a, b, c, make the surfaces metallic, all at the Potential $\frac{e}{k}$ (due to the infinitely distant charge); then the charge, with its law of distribution, which must be applied in order to produce this Potential, is that which we have just determined.

The total of the internal charges $= -\left(a + b - \frac{ab}{\sqrt{a^2+b^2}}\right)\frac{e}{k}$, and they produce a Potential $-\frac{e}{k}$ on the compound surface, so that the capacity of such a conductor is $a + b - \frac{ab}{\sqrt{a^2+b^2}}$.

For an exhaustive treatment of this problem see Clerk Maxwell's *Electricity and Magnetism*, Art. 168.

4. In the space between two uncharged insulated and concentric spheres, A and B, is placed a charge e at a point P; determine the surface-density at every point.

Let d be the distance between P and the common centre, O; let a and b be the radii of the spheres; draw the line OP and produce it indefinitely. The spheres may be regarded as forming a closed surface in the space enclosed by which the charge e is placed; and this double surface, A, B, can be made one of zero Potential by placing proper charges at a succession of inverse points derived from P. Thus, take the inverse, a_1, of P with respect to A. Then the charge e at P together with a numerically greater charge, $-\frac{a}{d}e$, at a_1 would make A a surface of zero Potential, but not B. However, if we take the inverse, a_2, of a_1 with respect to B, and place at

a_2 a charge equal to $-\dfrac{b}{Oa_1}\times$ charge at a_1, this, together with the charge at a_1, would make B a surface of zero Potential. Take, again, the inverse, a_3, of a_2 with respect to A; and so on.

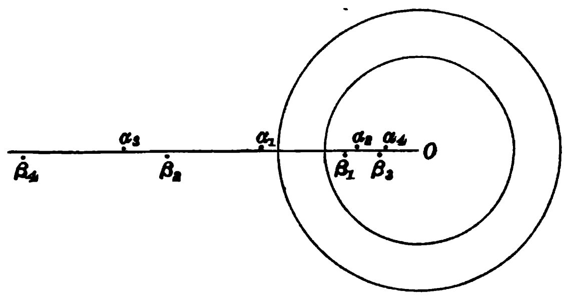

Fig. 316.

Again, start with β_1, the inverse of P with respect to B, and take β_2, the inverse of β_1 with respect to A; and so on, *ad infin.*

For clearness of figure we have drawn the a images slightly above the line OP, and the β images slightly below it.

We shall form a table of the distances of the images and the charges to be placed at them; thus:

Points	Distances	Charges	Points	Distances	Charges
P	d	e			
a_1	$\dfrac{a^2}{d}$	$-\dfrac{a}{d}e$	β_1	$\dfrac{b^2}{d}$	$-\dfrac{b}{d}e$
a_2	$\dfrac{b^2}{a^2}d$	$\dfrac{b}{a}e$	β_2	$\dfrac{a^2}{b^2}d$	$\dfrac{a}{b}e$
a_3	$\dfrac{a^4}{b^2d}$	$-\dfrac{a^2}{bd}e$	β_3	$\dfrac{b^4}{a^2d}$	$-\dfrac{b^2}{ad}e$
a_4	$\dfrac{b^4}{a^4}d$	$\dfrac{b^2}{a^2}e.$	β_4	$\dfrac{a^4}{b^4}d$	$\dfrac{a^2}{b^2}e$

The law of continuation *ad infin.* is obvious.

To get the surface-density at any point, Q, on A, group the points in pairs thus:

$(P,\ a_1),\ (a_2,\ a_3),\ (a_4,\ a_5)\dots(a_{2n},\ a_{2n+1}),\dots;\quad (\beta_1,\ \beta_2),\ (\beta_3,\ \beta_4),\dots.$

If r is the distance QP, the first pair give a surface-density

equal to $\dfrac{a^2-d^2}{r^3}\cdot\dfrac{e}{4\pi a}$: therefore if r_{2n} is the distance of Q from a_{2n}, e_{2n} = charge at a_{2n}, and d_{2n} = distance of a_{2n} from O, we have the surface-density at Q due to the general pair of a points equal to

$$\frac{a^2-d^2_{2n}}{r^3_{2n}}\cdot\frac{e_2}{4\pi a}.$$

But $d_{2n}=\left(\dfrac{b}{a}\right)^{2n}.d$, and $e_{2n}=\left(\dfrac{b}{a}\right)^{n}.e$; therefore the surface-density arising from the whole set of a points is

$$\frac{e}{4\pi a}\sum_0^\infty\frac{a^2-\left(\frac{b}{a}\right)^{4n}.d^2}{r^3_{2n}}\cdot\left(\frac{b}{a}\right)^n,$$

n receiving all integer values from 0 to ∞.

Similarly, the surface-density at Q arising from the β points is

$$-\frac{e}{4\pi d}\sum_0^\infty\frac{a^2-\left(\frac{b}{a}\right)^{4n}\cdot\frac{a^4}{d^2}}{r^3_{2n-1}}\cdot\left(\frac{b}{a}\right)^n.$$

The total charge on the sphere B is equal to the sum of the charges at $a_2, a_4, \ldots, \beta_1, \beta_3, \ldots$, which is

$$-\frac{a-d}{a-b}\cdot\frac{b}{d}\,e;$$

and of course $-e$ = charge on A + charge on B.

If O is at infinity, the spheres become two infinite parallel planes with a charge between them at P; and if p and q are the perpendiculars from P on A and B, the charge on the plate B is $-\dfrac{p}{h}e$, and that on A is $-\dfrac{q}{h}e$, where h is the distance between the plates.

5. Find the law of distribution of an idiostatic charge on a conductor generated by the revolution of the limaçon

$$r=m+n\cos\theta$$

about its axis, n being $<m$.

Invert the distribution on a prolate ellipsoid of revolution (p. 514) from one of its foci.

NOTES.

A.

THE EQUATION OF CAPILLARITY.

M. Resal (*Physique Mathématique*, p. 22) gives a very simple proof of the fundamental equation (p. 180) for the surface of separation of two fluids, on the supposition that the density of each is constant in all layers adjacent to this surface—a supposition which is rejected by M. Mathieu.

Let AB (Fig. 259, p. 134) be the surface of separation of two fluids F, F', the former being at the upper and the latter at the lower side of AB in the figure. Draw the tangent plane, Am, at the position of any particle, A, of the surface, and consider separately the action of the *meniscus* of fluid contained between AB and this plane.

Imagine the upper fluid, F, to be prolonged down to the tangent plane, and then subtract the effect of the meniscus, regarded as consisting of fluid F. Denote the meniscus of fluid F by (μ). Similarly, the action of F' on the particle at A may be considered to be due conjointly to that portion of F' which lies below the tangent plane and to the meniscus (μ') of fluid F'. Now, obviously, the forces exerted at A by the fluids F and F', supposed terminated by the tangent plane, are normal to the plane. Also the resultant force exerted at A by the external forces (if any) and the fluids F, F' as they actually exist is normal. Hence the resultant force due to the external forces, the positive meniscus (μ'), and the negative meniscus (μ) must be normal; i.e. the Potential due to these must be constant at all points A on the surface.

It is assumed that the capillary forces are exerted only between molecules whose distance is less than an extremely small length, ϵ, called the 'radius of spherical activity.' In the figure let Am be $< \epsilon$, and let a small element, dS, of area at m on the tangent plane be drawn by taking two planes through the normal An including an indefinitely small angle, $d\theta$, then, describing a cylinder round An with radius $Am = r$, and a concentric cylinder with radius $r + dr$. Thus the element $dS = rdrd\theta$, and the little prism standing on dS and included between the surface AB and the tangent plane, has for volume $Bm \times rdrd\theta$.

If R is the radius of curvature of the section of the surface AB made by the normal plane drawn through An, we have

$$2R \,.\, Bm = Am^2,$$

$$\therefore \quad Bm = \frac{r^2}{2R}.$$

Now all the particles in the prism Bm may be considered as distant by r from A. Let $\phi(r)$ express the Potential at A due to unit mass at m, the form of ϕ being quite unknown. Then the Potential produced at A by the prism Bm is

$$\frac{d\theta}{2R}\cdot r^3\phi(r)\,dr. \tag{1}$$

Integrating this from $r=0$ to $r=\epsilon$, and keeping θ constant, we obtain the Potential due to all those particles of the meniscus which are included between the two close normal planes defined by the azimuth θ. Now $\int_0^\epsilon r^3\phi(r)\,dr$ is a constant depending solely on the nature of the fluid F, and not on the position of A.

Denote this constant by C. Then the Potential at A due to the whole meniscus (μ) is

$$\frac{C}{2}\int_0^{2\pi}\frac{d\theta}{R}. \tag{2}$$

But if R_1 and R_2 are the principal radii of curvature of the surface at A, and θ is measured from one principal section,

$$\frac{1}{R}=\frac{\cos^2\theta}{R_1}+\frac{\sin^2\theta}{R_2}, \tag{3}$$

and the Potential (2) is obviously $\frac{1}{2}\pi C\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$.

The meniscus (μ') gives a similar term, so that if V is the Potential of the external forces at A, the equation of equilibrium is

$$\tfrac{1}{2}\pi(C'-C)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+V=\text{constant}. \tag{4}$$

This equation is deduced without assuming unchanged density near the surface by M. Mathieu (*Théorie de la Capillarité*, Chap. I) by means of the Principle of Virtual Work.

See also M. Mathieu's treatise (p. 65) for the proof of the fundamental theorem that the three surface-tensions at any point of meeting of the surfaces of three contiguous fluids satisfy the conditions of equilibrium of three forces—a very simple result, almost invariably assumed as obvious, or based on some such 'proof' as this: since there is equilibrium, the three tensions must be proportional each to the sine of the angle between the other two.

Now, inasmuch as there is no *one* particle which is acted upon by these three tensions, this 'proof' has no relevancy whatever; and the theorem is, as M. Mathieu justly says, 'admis sans raisons suffisantes dans les Ouvrages de Physique.' His own demonstration proceeds by the Principle of Virtual Work.

B.

Potential of a Homogeneous Ellipsoid.

The following investigation of the Potential produced at an external point, P, by a homogeneous solid ellipsoid has been given by Colonel A. R. Clarke (see the *Phil. Mag.*, December, 1877).

Take the principal axes of the ellipsoid as axes of co-ordinates; let x, y, z be the co-ordinates of P; let Q be any point inside the mass at which an element dm is taken; let x', y', z' be the co-ordinates of Q; O the centre of the ellipsoid, $OP = R$, $OQ = r$, and $\psi = \cos POQ$. Then, ρ being the mass per unit volume of the body,

$$V = \gamma\rho\int\frac{dm}{(R^2 - 2Rr.\psi + r^2)^{\frac{1}{2}}} \qquad (1)$$

$$= \frac{\gamma\rho}{R}\int\left(1 + P_1\frac{r}{R} + P_2\frac{r^2}{R^2} + P_3\frac{r^3}{R^3} + \ldots\right)dm, \qquad (2)$$

where $P_1, P_2, \ldots$ are the Legendrian coefficients, as in (γ), p. 349, or at p. 358, with ψ written instead of μ.

But since $P_1, P_3, P_5, \ldots$ are each of the form $\psi f(\psi^2)$, the terms in them vanish because of the complete symmetry of the figure, so that

$$V = \frac{\gamma\rho}{R}\int\left(1 + P_2\frac{r^2}{R^2} + P_4\frac{r^4}{R^4} + P_6\frac{r^6}{R^6} + \ldots\right)dm. \qquad (3)$$

Now $\psi = \dfrac{xx' + yy' + zz'}{rR} = \dfrac{lx' + my' + nz'}{r} = \dfrac{\Delta}{r}$, suppose, l, m, n being the direction-cosines of OP. Hence from the values of the Legendrians, p. 358, we have

$$P_2r^2 = \tfrac{1}{2}(3\Delta^2 - r^2); \quad P_4r^4 = \tfrac{1}{8}(35\Delta^4 - 30r^2\Delta^2 + 3r^4);$$
$$P_6r^6 = \tfrac{1}{16}(231\Delta^6 - 315r^2\Delta^4 + 105r^4\Delta^2 - 5r^6).$$

The results of performing the integrations in (3) as far as $\int P_6r^6dm$ are very remarkable.

Thus, it will be found that if a, b, c are the semi-axes and Ω the whole volume of the ellipsoid, and if we put

$$b^2 - c^2 = d_1^2; \quad c^2 - a^2 = d_2^2; \quad a^2 - b^2 = d_3^2,$$

and also denote by L_2 the value of the Legendrian P_2 when l is put for μ; by M_2 the value of P_2 when m is put for μ; and by N_2 the value of P_2 when n is put for μ; with similar meanings of L_4, M_4, N_4 with reference to P_4, &c., we shall have

$$\int P_2 r^2 dm = -\frac{\Omega}{15}\{L_2(d_2^2-d_3^2)+M_2(d_3^2-d_1^2)+N_2(d_1^2-d_2^2)\},$$

$$\int P_4 r^4 dm = -\frac{3\Omega}{35}\{L_4 d_2^2 d_3^2+M_4 d_3^2 d_1^2+N_4 d_1^2 d_2^2\},$$

$$\int P_6 r^6 dm = \frac{\Omega}{42}\{L_6 d_2^2 d_3^2(d_2^2-d_3^2)+M_6 d_3^2 d_1^2(d_3^2-d_1^2)$$
$$+N_6 d_1^2 d_2^2(d_1^2-d_2^2)+K d_1^2 d_2^2 d_3^2\},$$

where in the last $K=\frac{231}{16}(l^2-m^2)(m^2-n^2)(n^2-l^2)$.

In these expressions for the terms in (3) the sequence is, as Colonel Clarke observes, remarkable, 'and suggests the idea that possibly an expression might be obtained for the general term.'

INDEX TO VOL. II.

(The references are to the pages.)

END OF VOL. II.

SELECT LIST OF STANDARD WORKS

PRINTED AT

The Clarendon Press, Oxford.

1. DICTIONARIES.

A New English Dictionary on Historical Principles, founded mainly on the materials collected by the Philological Society. Imperial 4to. In Parts, price 12*s*. 6*d*. each.

Vol. I (A and B), half morocco, 2*l*. 12*s*. 6*d*.

Part IV, Section 2, C—CASS, beginning Vol. II, price 5*s*.

Part V, CASS—CLIVY, price 12*s*. 6*d*.

Edited by James A. H. Murray, LL.D.

Vol. III (E, F, and G), Part I, edited by Henry Bradley. *In the Press.*

An Etymological Dictionary of the English Language, arranged on an Historical Basis. By W. W. Skeat, Litt.D. *Second Edition.* 4to. 2*l*. 4*s*.

An Anglo-Saxon Dictionary, based on the MS. collections of the late Joseph Bosworth, D.D. Edited and enlarged by Prof. T. N. Toller, M.A., Owens College, Manchester. Parts I-III. A-SAR. 4to. stiff covers. 15*s*. each. Part IV (completing the Work) *in the Press.*

An Icelandic-English Dictionary, based on the MS. collections of the late Richard Cleasby. Enlarged and completed by G. Vigfússon, M.A. With an Introduction, and Life of Richard Cleasby, by G. Webbe Dasent, D.C.L. 4to. 3*l*. 7*s*.

A Greek-English Lexicon, by H. G. Liddell, D.D., and Robert Scott, D.D. *Seventh Edition, Revised and Augmented throughout.* 4to. 1*l*. 16*s*.

An Intermediate Greek-English Lexicon, abridged from the above. Small 4to. 12*s*. 6*d*.

A Latin Dictionary, founded on Andrews' edition of Freund's Latin Dictionary, revised, enlarged, and in great part rewritten by Charlton T. Lewis, Ph.D., and Charles Short, LL.D. 4to. 1*l.* 5*s.*

A School Latin Dictionary. By Charlton T. Lewis, Ph.D. Small 4to. 18*s.*

A Sanskrit-English Dictionary. Etymologically and Philologically arranged, with special reference to Greek, Latin, German, Anglo-Saxon, English, and other cognate Indo-European Languages. By Sir M. Monier-Williams, D.C.L. 4to. 4*l.* 14*s.* 6*d.*

Thesaurus Syriacus: collegerunt Quatremère, Bernstein, Lorsbach, Arnoldi, Agrell, Field, Roediger: edidit R. Payne Smith, S.T.P. Vol. I, containing Fasc. I–V, sm. fol. 5*l.* 5*s.*
Fasc. VI. 1*l.* 1*s.* Fasc. VII. 1*l.* 11*s.* 6*d.* Fasc. VIII. 1*l.* 16*s.*

2. ENGLISH AND ROMAN LAW.

Anson. *Principles of the English Law of Contract, and of Agency in its Relation to Contract.* By Sir W. R. Anson, D.C.L. *Fifth Edition.* 8vo. 10*s.* 6*d.*

—— *Law and Custom of the Constitution.* Part I. Parliament. 8vo. 10*s.* 6*d.*

Bentham. *An Introduction to the Principles of Morals and Legislation.* By Jeremy Bentham. Crown 8vo. 6*s.* 6*d.*

Digby. *An Introduction to the History of the Law of Real Property.* By Kenelm E. Digby, M.A. *Third Edition.* 8vo. 10*s.* 6*d.*

Grueber. *Lex Aquilia.* The Roman Law of Damage to Property: being a Commentary on the Title of the Digest 'Ad Legem Aquiliam' (ix. 2). With an Introduction to the Study of the Corpus Iuris Civilis. By Erwin Grueber, Dr. Jur., M.A. 8vo. 10*s.* 6*d.*

Hall. *International Law.* By W. E. Hall, M.A. *Third Edition.* 8vo. 22*s.* 6*d.*

Holland. *Elements of Jurisprudence.* By T. E. Holland, D.C.L. *Fourth Edition.* 8vo. 10*s.* 6*d.*

—— *The European Concert in the Eastern Question;* a Collection of Treaties and other Public Acts. Edited, with Introductions and Notes, by T. E. Holland, D.C.L. 8vo. 12*s.* 6*d.*

Holland. *Alberici Gentilis De Iure Belli Libri Tres.* Edidit T. E. Holland, I.C.D. Small 4to. half morocco, 21*s.*

Holland. *The Institutes of Justinian,* edited as a recension of the Institutes of Gaius, by T. E. Holland, D.C.L. *Second Edition.* Extra fcap. 8vo. 5*s.*

Holland and Shadwell. *Select Titles from the Digest of Justinian.* By T. E. Holland, D.C.L., and C. L. Shadwell, B.C.L. 8vo. 14s.

Also sold in Parts, in paper covers, as follows:—

Part I. Introductory Titles. 2s. 6d.

Part II. Family Law. 1s.

Part III. Property Law. 2s. 6d.

Part IV. Law of Obligations (No. 1). 3s. 6d.

Part IV. Law of Obligations (No. 2). 4s. 6d.

Markby. *Elements of Law considered with reference to Principles of General Jurisprudence.* By Sir William Markby, D.C.L. *Fourth Edition.* 8vo. 12s. 6d.

Moyle. *Imperatoris Iustiniani Institutionum Libri Quattuor;* with Introductions, Commentary, Excursus and Translation. By J. B. Moyle, D.C.L. *Second Edition.* 2 vols. 8vo. Vol. I. 16s. Vol. II. 6s.

Pollock and Wright. *An Essay on Possession in the Common Law.* By Sir F. Pollock, M.A., and R. S. Wright, B.C.L. 8vo. 8s. 6d.

Poste. *Gaii Institutionum Juris Civilis Commentarii Quattuor;* or, Elements of Roman Law by Gaius. With a Translation and Commentary by Edward Poste, M.A. A New Edition. *Nearly Ready.*

Raleigh. *The English Law of Property.* By Thos. Raleigh, M.A. *In the Press.*

Stokes. *The Anglo-Indian Codes.* By Whitley Stokes, LL.D.

Vol. I. Substantive Law. 8vo. 30s.

Vol. II. Adjective Law. 8vo. 35s.

A Supplement to the above, 1887–1888. Stiff covers, 2s. 6d.

Twiss. *The Law of Nations considered as Independent Political Communities.* By Sir Travers Twiss, D.C.L.

Part I. On the Rights and Duties of Nations in time of Peace. New Edition. 8vo. 15s.

Part II. On the Rights and Duties of Nations in time of War. Second Edition. 8vo. 21s.

3. HISTORY, BIOGRAPHY, ETC.

Baker's Chronicle. *Chronicon Galfridi le Baker de Swynebroke.* Edited with Notes by Edward Maunde Thompson, Hon. LL.D. St. Andrews; Hon. D.C.L. Durham; F.S.A.; Principal Librarian of the British Museum. Small 4to., stiff covers, 18s.; cloth, gilt top, 21s.

Bluntschli. *The Theory of the State.* By J. K. Bluntschli. Translated from the Sixth German Edition. 8vo. half bound, 12s. 6d.

Boswell's *Life of Samuel Johnson, LL.D.* Edited by G. Birkbeck Hill, D.C.L. In six volumes, medium 8vo. With Portraits and Facsimiles. Half bound, 3l. 3s.

Calendar *of the Clarendon State Papers,* preserved in the Bodleian Library. In three volumes. 1869–76.

Vol. I. From 1523 to January 1649. 8vo. 18s.

Vol. II. From 1649 to 1654. 16s.

Vol. III. From 1655 to 1657. 14s.

Calendar *of Charters and Rolls* preserved in the Bodleian Library. 8vo. 1*l.* 11*s.* 6*d.*

Carte's *Life of James Duke of Ormond.* 6 vols. 8vo. 1*l.* 5*s.*

Clarendon's *History of the Rebellion and Civil Wars in England.* Re-edited from a fresh collation of the original MS. in the Bodleian Library, with marginal dates and occasional notes, by W. Dunn Macray, M.A., F.S.A. 6 vols. Crown 8vo. 2*l.* 5*s.*

—— *History of the Rebellion and Civil Wars in England.* To which are subjoined the Notes of Bishop Warburton. 7 vols. medium 8vo. 2*l.* 10*s.*

—— *History of the Rebellion and Civil Wars in England.* Also his Life, written by himself, in which is included a Continuation of his History of the Grand Rebellion. Royal 8vo. 1*l.* 2*s.*

—— *Life, including a Continuation of his History.* 2 vols. medium 8vo. 1*l.* 2*s.*

Clinton's *Fasti Hellenici.* The Civil and Literary Chronology of Greece, from the LVIth to the CXXIIIrd Olympiad. *Third Edition.* 4to. 1*l.* 14*s.* 6*d.*

—— *Fasti Hellenici.* The Civil and Literary Chronology of Greece, from the CXXIVth Olympiad to the Death of Augustus. *Second Edition.* 4to. 1*l.* 12*s.*

—— *Epitome of the Fasti Hellenici.* 8vo. 6*s.* 6*d.*

—— *Fasti Romani.* The Civil and Literary Chronology of Rome and Constantinople, from the Death of Augustus to the Death of Heraclius. 2 vols. 4to. 2*l.* 2*s.*

Clinton's *Epitome of the Fasti Romani.* 8vo. 7*s.*

Earle. *Handbook to the Land-Charters, and other Saxonic Documents.* By John Earle, M.A., Professor of Anglo-Saxon in the University of Oxford. Crown 8vo. 16*s.*

Finlay. *A History of Greece from its Conquest by the Romans to the present time,* B.C. 146 to A.D. 1864. By George Finlay, LL.D. A new Edition, revised throughout, and in part re-written, with considerable additions, by the Author, and edited by H. F. Tozer, M.A. 7 vols. 8vo. 3*l.* 10*s.*

Fortescue. *The Governance of England:* otherwise called The Difference between an Absolute and a Limited Monarchy. By Sir John Fortescue, Kt. A Revised Text. Edited, with Introduction, Notes, etc., by Charles Plummer, M.A. 8vo. half bound, 12*s.* 6*d.*

Freeman. *History of the Norman Conquest of England; its Causes and Results.* By E. A. Freeman, D.C.L. In Six Volumes. 8vo. 5*l.* 9*s.* 6*d.*

—— *The Reign of William Rufus and the Accession of Henry the First.* 2 vols. 8vo. 1*l.* 16*s.*

—— *A Short History of the Norman Conquest of England. Second Edition.* Extra fcap. 8vo. 2*s.* 6*d.*

Gardiner. *The Constitutional Documents of the Puritan Revolution,* 1628–1660. Selected and Edited by Samuel Rawson Gardiner, M.A. Crown 8vo. 9*s.*

Gascoigne's *Theological Dictionary ('Liber Veritatum'):* Selected

Passages, illustrating the Condition of Church and State, 1403–1458. With an Introduction by James E. Thorold Rogers, M.A. Small 4to. 10s. 6d.

George. *Genealogical Tables illustrative of Modern History.* By H. B. George, M.A. *Third Edition.* Small 4to. 12s.

Hodgkin. *Italy and her Invaders.* With Plates and Maps. By T. Hodgkin, D.C.L. Vols. I–IV, A.D. 376–553. 8vo. 3l. 8s.

—— *The Dynasty of Theodosius;* or, Seventy Years' Struggle with the Barbarians. By the same Author. Crown 8vo. 6s.

Hume. *Letters of David Hume to William Strahan.* Edited with Notes, Index, etc., by G. Birkbeck Hill, D.C.L. 8vo. 12s. 6d.

Kitchin. *A History of France.* With Numerous Maps, Plans, and Tables. By G. W. Kitchin, D.D. In three Volumes. *Second Edition.* Crown 8vo. each 10s. 6d.

Vol. I. to 1453. Vol. II. 1453–1624. Vol. III. 1624–1793.

Lucas. *Introduction to a Historical Geography of the British Colonies.* By C. P. Lucas, B.A. With Eight Maps. Crown 8vo. 4s. 6d.

—— *Historical Geography of the Colonies.* Vol. I. By the same Author. With Eleven Maps. Crown 8vo. 5s.

Luttrell's (*Narcissus*) *Diary.* A Brief Historical Relation of State Affairs, 1678–1714. 6 vols. 8vo. 1l. 4s.

Magna Carta, *a careful Reprint.* Edited by W. Stubbs, D.D., Bishop of Oxford. 4to. stitched, 1s.

Metcalfe. *Passio et Miracula Beati Olaui.* Edited from a Twelfth-Century MS. by F. Metcalfe, M.A. Small 4to. 6s.

Oxford. *Manuscript Materials relating to the History of Oxford; contained in the Printed Catalogues of the Bodleian and College Libraries.* By F. Madan, M.A. 8vo. 7s. 6d.

Ranke. *A History of England, principally in the Seventeenth Century.* By L. von Ranke. Translated under the superintendence of G. W. Kitchin, D.D., and C. W. Boase, M.A. 6 vols. 8vo. 3l. 3s.

Rawlinson. *A Manual of Ancient History.* By George Rawlinson, M.A. *Second Edition.* Demy 8vo. 14s.

Ricardo. *Letters of David Ricardo to T. R. Malthus* (1810–1823). Edited by James Bonar, M.A. 8vo. 10s. 6d.

Rogers. *History of Agriculture and Prices in England, A.D. 1259–1793.* By James E. Thorold Rogers, M.A.

Vols. I and II (1259–1400). 8vo. 2l. 2s.

Vols. III and IV (1401–1582). 8vo. 2l. 10s.

Vols. V and VI. (1583–1702). 8vo. 2l. 10s.

—— *First Nine Years of the Bank of England.* 8vo. 8s. 6d.

—— *Protests of the Lords, including those which have been expunged, from 1624 to 1874;* with Historical Introductions. In three volumes. 8vo. 2l. 2s.

Smith's *Wealth of Nations.* A New Edition, with Notes, by J. E. Thorold Rogers, M.A. 2 vols. 8vo. 21s.

Sprigg's *England's Recovery;* being the History of the Army under Sir Thomas Fairfax. 8vo. 6s.

Stubbs. *Select Charters and other Illustrations of English Constitutional History, from the Earliest Times to the Reign of Edward I.* Arranged and edited by W. Stubbs, D.D., Lord Bishop of Oxford. *Fifth Edition.* Crown 8vo. 8s. 6d.

—— *The Constitutional History of England, in its Origin and Development. Library Edition.* 3 vols. Demy 8vo. 2l. 8s.

Also in 3 vols. crown 8vo. price 12s. each.

—— *Seventeen Lectures on the Study of Medieval and Modern History, delivered at Oxford* 1867–1884. Crown 8vo. 8s. 6d.

Stubbs. *Registrum Sacrum Anglicanum.* An attempt to exhibit the course of Episcopal Succession in England. By W. Stubbs, D.D. Small 4to. 8s. 6d.

Wellesley. *A Selection from the Despatches, Treaties, and other Papers of the Marquess Wellesley, K.G., during his Government of India.* Edited by S. J. Owen, M.A. 8vo. 1l. 4s.

Wellington. *A Selection from the Despatches, Treaties, and other Papers relating to India of Field-Marshal the Duke of Wellington, K.G.* Edited by S. J. Owen, M.A. 8vo. 1l. 4s.

Whitelock's *Memorials of English Affairs from* 1625 *to* 1660. 4 vols. 8vo. 1l. 10s.

4. PHILOSOPHY, LOGIC, ETC.

Bacon. *Novum Organum.* Edited, with Introduction, Notes, &c., by T. Fowler, D.D. *Second Edition.* 8vo. 15s.

—— *Novum Organum.* Edited, with English Notes, by G. W. Kitchin, D.D. 8vo. 9s. 6d.

—— *Novum Organum.* Translated by G. W. Kitchin, D.D. 8vo. 9s. 6d.

Berkeley. *The Works of George Berkeley, D.D., formerly Bishop of Cloyne; including many of his writings hitherto unpublished.* With Prefaces, Annotations, and an Account of his Life and Philosophy, by Alexander Campbell Fraser, LL.D. 4 vols. 8vo. 2l. 18s.

The Life, Letters, &c., separately, 16s.

Bosanquet. *Logic; or, the Morphology of Knowledge.* By B. Bosanquet, M.A. 8vo. 21s.

Butler's *Works, with Index to the Analogy.* 2 vols. 8vo. 11s.

Fowler. *The Elements of Deductive Logic, designed mainly for the use of Junior Students in the Universities.* By T. Fowler, D.D. *Ninth Edition,* with a Collection of Examples. Extra fcap. 8vo. 3s. 6d.

—— *The Elements of Inductive Logic, designed mainly for the use of Students in the Universities.* By the same Author. *Fifth Edition.* Extra fcap. 8vo. 6s.

—— *The Principles of Morals* (Introductory Chapters). By T. Fowler, D.D., and J. M. Wilson, B.D. 8vo. boards, 3s. 6d.

Fowler. *The Principles of Morals.* Part II. By T. Fowler, D.D. 8vo. 10s. 6d.

Green. *Prolegomena to Ethics.* By T. H. Green, M.A. Edited by A. C. Bradley, M.A. 8vo. 12s. 6d.

Hegel. *The Logic of Hegel;* translated from the Encyclopaedia of the Philosophical Sciences. With Prolegomena by William Wallace, M.A. 8vo. 14s.

Hume's *Treatise of Human Nature.* Edited, with Analytical Index, by L. A. Selby-Bigge, M.A. Crown 8vo. 9s.

Locke's *Conduct of the Understanding.* Edited by T. Fowler, D.D. *Second Edition.* Extra fcap. 8vo. 2s.

Lotze's *Logic*, in Three Books; of Thought, of Investigation, and of Knowledge. English Translation; Edited by B. Bosanquet, M.A. *Second Edition.* 2 vols. Crown 8vo. 12s.

—— *Metaphysic*, in Three Books; Ontology, Cosmology, and Psychology. English Translation; Edited by B. Bosanquet, M.A. *Second Edition.* 2 vols. Crown 8vo. 12s.

Martineau. *Types of Ethical Theory.* By James Martineau, D.D. *Third Edition.* 2 vols. Crown 8vo. 15s.

—— *A Study of Religion: its Sources and Contents. Second Edition.* 2 vols. Crown 8vo. 15s.

5. PHYSICAL SCIENCE.

Aplin. *The Birds of Oxfordshire.* By O. V. Aplin. Demy 8vo. 10s. 6d.

Chambers. *A Handbook of Descriptive and Practical Astronomy.* By G. F. Chambers, F.R.A.S. *Fourth Edition*, in 3 vols. Demy 8vo.

Vol. I. The Sun, Planets, and Comets. 21s.

Vol. II. Instruments and Practical Astronomy. 21s.

Vol. III. The Starry Heavens. *Immediately.*

De Bary. *Comparative Anatomy of the Vegetative Organs of the Phanerogams and Ferns.* By Dr. A. De Bary. Translated and Annotated by F. O. Bower, M.A., F.L.S., and D. H. Scott, M.A., Ph.D., F.L.S. Royal 8vo., half morocco, 1*l.* 2s. 6d.

—— *Comparative Morphology and Biology of Fungi, Mycetozoa and Bacteria.* By Dr. A. De Bary. Translated by H. E. F. Garnsey, M.A. Revised by Isaac Bayley Balfour, M.A., M.D., F.R.S. Royal 8vo., half morocco, 1*l.* 2s. 6d.

De Bary. *Lectures on Bacteria.* By Dr. A. De Bary. *Second Improved Edition.* Translated by H. E. F. Garnsey, M.A. Revised by Isaac Bayley Balfour, M.A., M.D., F.R.S. Crown 8vo. 6s.

Fisher. *A Class Book of Elementary Chemistry.* By W. W. Fisher, M.A., F.C.S. Crown 8vo. 4s. 6d.

Goebel. *Outlines of Classification and Special Morphology of Plants.* A new Edition of Sachs' Text-Book of Botany, Book II. By Dr. K. Goebel. Translated by H. E. F. Garnsey, M.A. Revised by Isaac Bayley Balfour, M.A., M.D., F.R.S. Royal 8vo., half morocco, 1*l.* 1s.

Cos 8

Sachs. *Lectures on the Physiology of Plants.* By Julius von Sachs. Translated by H. Marshall Ward, M.A., F.L.S. Royal 8vo., half morocco, 1*l.* 11*s.* 6*d.*

—— *A History of Botany.* Translated by H. E. F. Garnsey, M.A. Edited by I. Bayley Balfour, M.A., M.D., F.R.S. Crown 8vo. 10*s.*

Solms-Laubach. *Introduction to Fossil Botany.* By Count H. von Solms-Laubach. Authorised English Translation, by H. E. F. Garnsey, M.A. Edited by Isaac Bayley Balfour, M.A., M.D., F.R.S. *In the Press.*

Annals of Botany. Edited by Isaac Bayley Balfour, M.A., M.D., F.R.S., Sydney H. Vines, D.Sc., F.R.S., and W. G. Farlow, M.D.

Vol. I. Parts I–IV. Royal 8vo., half morocco, gilt top, 1*l.* 16*s.*

Vol. II. Parts V–VIII. 2*l.* 2*s.*
Part IX. 11*s.* 6.
Part X. 13*s.* 6*d.*

Biological Series. (*Translations of Foreign Biological Memoirs.*)

I. *The Physiology of Nerve, of Muscle, and of the Electrical Organ.* Edited by J. Burdon-Sanderson, M.D., F.R.SS. L. & E. Medium 8vo. 1*l.* 1*s.*

II. *The Anatomy of the Frog.* By Dr. Alexander Ecker, Professor in the University of Freiburg. Translated, with numerous Annotations and Additions, by G. Haslam, M.D. Medium 8vo. 21*s.*

III. *Contributions to the History of the Physiology of the Nervous System.* By Professor Conrad Eckhard. Translated by Miss Edith Prance. *In Preparation.*

IV. *Essays upon Heredity and Kindred Biological Problems.* By Dr. August Weismann. Translated and Edited by E. B. Poulton, M.A., Selmar Schönland, Ph.D., and Arthur E. Shipley, M.A. Medium 8vo. 16*s.*

Prestwich. *Geology, Chemical, Physical, and Stratigraphical.* By Joseph Prestwich, M.A., F.R.S. In two Volumes.

Vol. I. Chemical and Physical. Royal 8vo. 1*l.* 5*s.*

Vol. II. Stratigraphical and Physical. With a new Geological Map of Europe. Royal 8vo. 1*l.* 16*s.*

New Geological Map of Europe. In case or on roller. 5*s.*

Rolleston and Jackson. *Forms of Animal Life.* A Manual of Comparative Anatomy, with descriptions of selected types. By George Rolleston, M.D., F.R.S. *Second Edition.* Revised and Enlarged by W. Hatchett Jackson, M.A. Medium 8vo. 1*l.* 16*s.*

Oxford
AT THE CLARENDON PRESS
LONDON: HENRY FROWDE
OXFORD UNIVERSITY PRESS WAREHOUSE, AMEN CORNER, E.C.

$L \quad R \quad p$

$$A = \frac{\Sigma R}{\Sigma R + \Sigma p L} \qquad B \quad \frac{\Sigma p L}{\Sigma R + \Sigma p L}$$

$$\sqrt{A^2 + pB^2}$$